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THE RESTRICTED TANGENT BUNDLE OF SMOOTH CURVES IN GRASSMANNIANS AND CURVES IN FLAG VARIETIES

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ABSTRACT. Let X be a smooth curve of genus $g \geq 2$ over an algebraically closed base field k of any characteristic. Denote by $G(r, \nu)$ the Grassmannian of the rank r quotients of k^ν and by $\mathcal{Q}$ the universal quotient bundle of $G(r, \nu)$. Let us consider degree d embeddings $\varphi : X \to G(r, \nu)$. We prove that, for $d \geq \nu + r(g-1)$ and $(\nu, r, d) \neq (4, 2, 2g+2)$, varying φ we obtain as restricted quotient bundles $\varphi^*(\mathcal{Q})$ points of an open dense subset of the moduli space $M(X; r, d)$ of rank r stable vector bundles on X with degree d. We can extend this result to the flag varieties. For the projective spaces $\mathbf{P}^n$, we obtain that if d is large with respect to g, $d \geq ng + 1$, then degree d embeddings $\varphi : X \to \mathbf{P}^n$ cover a dense open subset of the moduli space $M(X; n, (n+1)d)$ by means of the restricted tangent bundles $\varphi^*(T_{\mathbf{P}^n})$. This fact does not hold for restricted tangent bundles of a Grassmannian $G(r, \nu)$ with $2 \leq r \leq \nu - 2$. However, for a large degree d, we are able to characterize the restricted tangent bundles $\varphi^*(T_{G(r,\nu)})$ of a Grassmannian, obtaining that in general they are stable. For an elliptic curve Y, we show that in characteristic 0 there is a degree d embedding of Y in a Grassmannian with a stable restricted tangent bundle if and only if there is not a numerical restriction to its existence.

Introduction. Let X be a smooth curve of genus $g \geq 2$ over an algebraically closed base field k of any characteristic. Let $M(X; r, d)$ be the irreducible smooth variety parameterizing the stable vector bundles on X with rank $r > 0$ and degree d. We have $\dim M(X; r, d) = r^2(g-1) + 1$. If L is a line bundle on X with degree d, we denote by $M(X; r, L)$ the irreducible smooth subvariety of $M(X; r, d)$ parameterizing the stable rank r vector bundles with determinant L. We have $\dim M(X; r, L) = (r^2 - 1)(g-1)$.

Let us consider degree d embeddings $\varphi : X \to \mathbf{P}^n$, $n \geq 3$. Several authors studied the semi-stability and stability of the restricted tangent

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bundle $\varphi^*(T_{\mathbf{P}^n})$ [**7**], [**8**] and [**4**]. We prove that, for $d \geq ng+1$, varying φ we obtain as restricted tangent bundles $\varphi^*(T_{\mathbf{P}^n})$ points of an open dense subset of the moduli space $M(X;n,(n+1)d)$ of rank n stable vector bundles on X with degree $(n+1)d$ (Theorem 1.5). For a large d, $d \geq ng+n+2$, and in characteristic 0, the above statement on the restricted tangent bundle of $\mathbf{P}^n$ is easy and certainly "well-known to the specialists." That follows from the fact that $T_{\mathbf{P}^n}(-1)$ is the universal rank n quotient bundle on $\mathbf{P}^n$ and, for a large d, a general E in $M(X;n,d)$ is such that $E, E(-P)$ and $E(-P-P')$ are nonspecial for every $P, P' \in X$ ([**16**], [**13**] and [**5**]). To have the result for $d \geq ng+1$ and in any characteristic of k, we use positive elementary transformations of a known restricted tangent bundle and degenerations of curves as in [**4**].

Furthermore, by using elementary transformations, we can prove that in any characteristic of k a general E in $M(X;r,d)$ has either $h^0(X,E)=0$ or $h^1(X,E)=0$ (Lemma 1.2).

Then, if the normalized restricted tangent bundle $\varphi^*(T_{\mathbf{P}^n}(-1))$ is a general vector bundle in $M(X;n,d)$, we have $h^0(X,\varphi^*(T_{\mathbf{P}^n}(-1))) \geq n+1$ and $h^1(X,\varphi^*(T_{\mathbf{P}^n}(-1)))=0$. The Riemann-Roch Theorem implies $d \geq ng+1$. Hence the numerical assumptions in Theorem 1.5 are necessary conditions and, from this point of view, Theorem 1.5 is sharp.

We extend the above result to the universal quotient bundle of a Grassmannian. If $0<r<\nu$, $G(r,\nu)$ denotes the Grassmannian of the rank r quotients of k^ν. Let $\mathcal{Q}$ be the universal rank r quotient bundle of $G(r,\nu)$ and $\mathcal{S}$ the universal rank $\nu-r$ subbundle of $G(r,\nu)$. If ν, r and d are integers with $\nu > r \geq 2$, $d \geq \nu + r(g-1)$ and $(\nu,r,d) \neq (4,2,2g+2)$, we prove that for a general vector bundle Q in $M(X;r,d)$ there exists a nondegenerate degree d embedding $\varphi : X \to G(r,\nu)$ such that $\varphi^*(\mathcal{Q}) \cong Q$ (Theorem 2.1). Note that the numerical assumption $d \geq \nu + r(g-1)$ is sharp to have a nonspecial Q.

We obtain some results concerning the restricted tangent bundle of a Grassmannian to a curve. If d is large with respect to ν, r and g, and L is a general degree d line bundle on X, we prove that for a general vector bundle Q in $M(X,r,L)$ and a general vector bundle S in $M(X,\nu-r,L^\vee)$ a nondegenerate embedding $\varphi : X \to G(r,\nu)$ exists such that $Q \cong \varphi^*(\mathcal{Q})$, $S \cong \varphi^*(\mathcal{S})$ and in particular $\varphi^*(T_{G(r,\nu)}) \cong$

$Q \otimes S^\vee$ (Theorem 2.13).

With the assumption $\operatorname{char} k = 0$, if E is a general stable bundle in $M(X; r, a)$ and F is a general stable bundle in $M(X; s, b)$, then the tensor product $E \otimes F$ is stable. Hence in the above result (Theorem 2.13) with the assumption $\operatorname{char} k = 0$, we can obtain stable restricted tangent bundles $\varphi^*(T_{G(r,\nu)})$ (Remark 2.14).

Note that, if $\nu \geq r + 2 \geq 4$, for any integer d the restricted tangent bundles to degree d embeddings of X into $G(r, \nu)$ cannot cover a dense open subset of $M(X; r(\nu - r), \nu d)$ (Remark 2.15).

In Theorem 3.1 we extend the result on the universal quotient bundle of a Grassmannian restricted to a curve (Theorem 2.1) to (partial) flag varieties.

In the last section we will consider briefly the stability of the restricted tangent bundle of maps from an elliptic curve Y to a Grassmannian $G(r, \nu)$ with $1 \leq r \leq \nu - 1$ and $\nu \geq 4$, showing that in characteristic 0 there is a degree d embedding of Y in $G(r, \nu)$ with a stable restricted tangent bundle if and only if there is not a numerical restriction to its existence (Theorem 4.2).

1. Degenerations and the restricted tangent bundle of n. Let X be a smooth projective curve, P a point of X, E a vector bundle on X and $k(P)$ a skyscraper sheaf with length 1 supported at P. A surjection $\phi : E^\vee \to k(P)$ gives a point K of $\mathbf{P}(E^\vee)$ lying on the fiber of $\mathbf{P}(E^\vee)$ at P. The vector bundle $\ker \phi$ is denoted by $\operatorname{elm}_K^- E^\vee$ and it is called a *negative elementary transformation of* $E^\vee$. The vector bundle $E' := (\ker \phi)^\vee$ is called a *positive elementary transformation of* E, and it is denoted by $\operatorname{elm}_K^+ E$. E' fits in an exact sequence $0 \to E \to E' \to k(P) \to 0$.

Note that, for a general point K of $\mathbf{P}(E^\vee)$, we have $h^0(X, \operatorname{elm}_K^- E^\vee) = \max\{0, h^0(X, E^\vee) - 1\}$ and $h^1(X, \operatorname{elm}_K^+ E) = \max\{0, h^1(X, E) - 1\}$.

With the assumption $\operatorname{char} k = 0$, Sundaram in [**16**] and Laumon in [**13**] proved that a general vector bundle E on a smooth curve X of genus $g \geq 2$ has either $h^0(X, E) = 0$ or $h^1(X, E) = 0$. By using elementary transformations and a result of Hirschowitz (see the following Lemma 1.1) on the deformations of vector bundles contained in [**9**], we will prove in Lemma 1.2 the above statement for a base field

k of any characteristic.

Lemma 1.1 (Hirschowitz). *Let X be a genus g smooth curve over a field k of arbitrary characteristic with $g \geq 2$. Then every vector bundle F on X is a flat limit of a flat family of stable bundles having as determinant the line bundle* $\det(F)$.

Proof. We report the proof contained in [**9**]. Let F and G be two vector bundles on X of rank r with isomorphic determinants. Then a rank r vector bundle F_0 exists, with determinant $\det(F_0) \cong \det(F) \cong \det(G)$ such that both F and G specialize to F_0.

The proof of that statement uses the induction on the rank r. For $r = 1$ the assertion is trivial. Let $r \geq 2$. For an ample line bundle L on X and for a large n, the bundles $F \otimes L^n$ and $G \otimes L^n$ have a section that is nonnull at every point of X (X is a curve). So we obtain two exact sequences $0 \to L^{-n} \to F \to F' \to 0$ and $0 \to L^{-n} \to G \to G' \to 0$ where F' and G' are two vector bundles of rank $r-1$ with $\det(F') \cong \deg(G')$.

By considering the universal extensions of $\mathrm{Ext}^1(F', L^{-n})$ and $\mathrm{Ext}^1 \times (G', L^{-n})$, we have that $L^{-n} \oplus F'$ and $L^{-n} \oplus G'$ are respectively specializations of F and G (see [**12**]). We conclude by the inductive hypothesis.

Now let F be a rank r vector bundle on X and $G \in M(X; r, \det(F))$. Let F_0 be a specialization of both F and G and $E \to S \times X$ a versal deformation of F_0. The scheme S is smooth because $\mathrm{Ext}^2(F_0, F_0) = 0$. Since there is an $s_0 \in S$ such that $E(s_0) = G$, that is stable, and stability is an open property, we have that for a general $s \in S$ the bundle $E(s)$ is stable.

Lemma 1.2. *Fix integers g, r, d with $g \geq 2$, $r > 0$, $d \geq r(g-1)$ and a smooth genus g curve X. Then a general $E \in M(X; r, d)$ has $h^1(X, E) = 0$.*

Proof. Fix a large integer $t > 0$ such that $G \in M(X; r, d + tr)$ exists with $h^1(X, G) = 0$ (e.g., take $t \geq g$). Since $\deg G > r(g-1)$, we have $h^0(X, G) \neq 0$ and then a general negative elementary transformation

G' of G has $h^0(X, G') = h^0(X, G) - 1$ and $h^1(X, G') = 0$. Thus the rank r bundle F obtained from G making rt general negative elementary transformations has degree d and $h^1(X, F) = 0$. Use the semi-continuity theorem and Lemma 1.1.

Let us consider a morphism $f : X \to \mathbf{P}^n$ and set $E := f^*(T_{\mathbf{P}^n}(-1))$. To every line D in $\mathbf{P}^n$ passing through $f(P)$ we associate a positive elementary transformation $\mathrm{elm}_K^+ E$ of E with support at P because the line D corresponds to a one-dimensional subspace of $(T_{\mathbf{P}^n}(-1))_{f(P)}$.

The following proposition, proved in [**4**], interprets, in terms of positive elementary transformations, the flat limit of the restricted tangent bundle of a degeneration of curves in projective spaces considered in [**3**, Proposition 2.1].

Note that the proof made in [**4**] works in arbitrary characteristics of k.

Proposition 1.3. *Fix a smooth projective curve X. Let L_0 be a very ample line bundle on X of degree d_0 with $h^1(X, L_0) = 0$ giving a nondegenerate embedding $h : X \to \mathbf{P}^n$, $n \geq 3$. Consider a point P of X and a line D passing through $h(P)$. Assume that D is not the tangent line to $h(X)$ at $h(P)$. Then we have the following assertions*:

a) (see [**3**]) *The reducible curve $h(X) \cup D$ is a flat limit of a flat family of smooth curves $\{C_t\}$ obtained as images of nondegenerate embeddings $h_t : X \to \mathbf{P}^n$ with $h_t^*(\mathcal{O}_{\mathbf{P}^n}(1)) \cong L_0(P)$.*

b) (see [**4**, Proposition 1.4]) *Set $E := h^*(T_{\mathbf{P}^n}(-1))$ and $E_t := h_t^*(T_{\mathbf{P}^n}(-1))$ where h_t are the above embeddings. Let E' be the bundle obtained from E by the positive elementary transformation associated to the point P and the line D.*

Then E' is the flat limit of the flat family $\{E_t\}$ of spanned rank n vector bundles of degree $d_0 + 1$ with $det(E_t) \cong L_0(P)$.

Lemma 1.4. *Let X be a smooth projective curve of genus $g \geq 2$ and $h : X \to \mathbf{P}^n$ a degree d_0 embedding as in Proposition* 1.3.

Set $l := h^1(X, H^*(T_{\mathbf{P}^n}(-1)))$. For every $d \geq d_0 + l$ there is a Zariski open nonempty subset Ω of the variety $M(X; n, d)$ of rank n degree d stable vector bundles on X such that for every $F \in \Omega$ there is a nondegenerate degree d embedding $\varphi : X \to \mathbf{P}^n$ such that

$F \cong \varphi^*(T_{\mathbf{P}^n}(-1))$.

Proof. It is sufficient to prove the existence of a rank n degree d stable vector bundle F on X and a nondegenerate degree d embedding $\varphi : X \to \mathbf{P}^n$ such that $F \cong \varphi^*(T_{\mathbf{P}^n}(-1))$.

Put $s := d - d_0$. Take s general points $P_1, \dots , P_s$ of X and for each P_i a general line D_i in $\mathbf{P}^n$ passing through $h(P_i)$. Let Y be the union of $h(X)$ and the s lines $D_1, \dots , D_s$. Y is the flat limit of a flat family $\{C_t\}$ of smooth curves in $\mathbf{P}^n$, where C_t is the image of an embedding $h_t : X \to \mathbf{P}^n$ (see part a) of Proposition 1.3). By the second part of Proposition 1.3, the positive elementary transformation G of $E := h^*(T_{\mathbf{P}^n}(-1))$ at $D_1, \dots , D_s$ is the flat limit of the flat family of the vector bundles $G_t := h_t^*(T_{\mathbf{P}^n}(-1))$ with $L := L_0(P_1 + \dots + P_s) \cong \det(G_t)$.

For a general positive elementary transformation E' of E we have $h^1(X, E') = \max\{0, h^1(X, E) - 1\}$. Then the rank n bundle G obtained from E applying $s = d - d_0$ general positive elementary transformations has $h^1(X, G) = 0$.

Let F_0 be a general bundle of the family $\{G_t\}$ and $F \in M(X; n, L)$ a general stable deformation of F_0 with $L \cong \det(F_0)$ (see Lemma 1.1). By semi-continuity we have $h^1(X, F_0) = h^1(X, F) = 0$.

Note that $T_{\mathbf{P}^n}(-1)$ is the universal rank n quotient bundle of $\mathbf{P}^n$. To be spanned and inducing an embedding are open conditions in a family of bundles on X with constant cohomology and moreover the bundle F_0 induces a nondegenerate embedding $X \to \mathbf{P}^n$. So the same assertion holds for the stable vector bundle F.

Theorem 1.5. *Fix integers g, n with $g \geq 2$, $n \geq 3$ and a smooth projective curve X of genus g. Then for every $d \geq ng + 1$ there is a Zariski open nonempty subset Ω of the variety $M(X; n, d)$ of stable vector bundles of rank n and degree d on X such that for every $F \in \Omega$ there is a nondegenerate degree d embedding $\varphi : X \to \mathbf{P}^n$ such that $F \cong \varphi^*(T_{\mathbf{P}^n}(-1))$.*

Proof. By [**4**, Theorem 2.3], if $n > g$, a general line bundle L_0 on X with degree $d_0 = g + n$ induces a linearly normal embedding $h : X \to \mathbf{P}^n$

having $h^0(X, h^*(T_{\mathbf{P}^n}(-1)) = n+1$.

Note that the proof of this result made in [**4**] holds in any characteristic of k.

Then we have $h^1(X, h^*(T_{\mathbf{P}^n}(-1)) = -g - n + 1 + ng$. Applying Lemma 1.4, we obtain the proof for $n > g$.

Now let L_0 be a line bundle on X with degree $2g+1$ inducing a linearly normal embedding $h_0 : X \to \mathbf{P}^{g+1}$. The projection of $C_0 = h_0(X)$ over a $\mathbf{P}^g$ from a general point of $\mathbf{P}^{g+1}$ gives a degree $2g+1$ embedding $h_1 : X \to \mathbf{P}^g$. By considering the sequence of all the general projections over the projective spaces $\mathbf{P}^n$, with $3 \le n \le g$, we obtain degree $2g+1$ embeddings $h_{g+1-n} : X \to \mathbf{P}^n$ with $h^*_{g+1-n}(\mathcal{O}_{\mathbf{P}^n}(1)) \cong L_0$. Put $C_{g+1-n} = h_{g+1-n}(X)$. A general projection of the curve C_{g-n} in $\mathbf{P}^{n+1}$ over a projective space $\mathbf{P}^n$ induces the following commutative diagram whose rows and columns are exact:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_X & = & \mathcal{O}_X & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L_0^\vee & \longrightarrow & \mathcal{O}_X^{\oplus(n+2)} & \longrightarrow & h^*_{g-n}(T_{\mathbf{P}^{n+1}}(-1)) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L_0^\vee & \longrightarrow & \mathcal{O}_X^{\oplus(n+1)} & \longrightarrow & h^*_{g+1-n}(T_{\mathbf{P}^{n}}(-1)) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

We obtain $h^0(h^*_{g+1-n}(T_{\mathbf{P}^n}(-1))) = n+1$ and $h^1(h^*_{g+1-n}(T_{\mathbf{P}^n}(-1))) = -2g+ng$. Applying Lemma 1.4 to curves C_{g+1-n}, we have the assertion also for $3 \le n \le g$.

Remark 1.6. Since for all r, d and $L \in \mathrm{Pic}^d(X)$ a general $F \in M(X; r, L)$ has either $h^0(X, F) = 0$ or $h^1(X, F) = 0$ (Lemma 1.2) in the statement of Theorem 1.5 we have $h^1(X,F)=0$. By the Riemann-Roch Theorem, the conditions $h^1(X,F)=0$ and $h^0(X,F)\geq n+1$ imply $d\geq ng+1$. Hence the numerical assumptions in Theorem 1.5 are necessary conditions and, from this point of view, Theorem 1.5 is sharp.

2. Universal quotient bundles and tangent bundles of Grassmannians restricted to curves. If $0<r<\nu$, $G(r,\nu)$ denotes the Grassmannian of the rank r quotients of k^ν. Let $\mathcal{Q}$ be the universal rank r quotient bundle of $G(r,\nu)$ and $\mathcal{S}$ the universal rank $\nu-r$ subbundle of $G(r,\nu)$. We have the following exact sequence: $0\to\mathcal{S}\to\mathcal{O}_{G(r,\nu)}^{\oplus\nu}\to\mathcal{Q}\to 0$.

We say that a morphism φ from X into a Grassmannian $G(r,\nu)$ is *nondegenerate* if its image is not contained in a Grassmannian $G(r',\nu')$ with $\nu'<\nu$, i.e., neither $\varphi^*(\mathcal{Q})$ nor $\varphi^*(\mathcal{S})$ has $\mathcal{O}_X$ as a component.

A rank r bundle E on X quotient of $\mathcal{O}_X^{\oplus\nu}$ induces a morphism $\varphi:X\to G(r,\nu)$; φ is an embedding if and only if we have $h^0(X,E(-P-P'))<h^0(X,E(-P))$ for every $P,P'\in X$ (allowing $P=P'$).

We denote by X a smooth projective curve of genus $g\geq 2$.

Theorem 2.1. *Let X be a smooth projective curve of genus $g\geq 2$. Fix integers ν,r and d with $\nu>r\geq 2$, $d\geq\nu+r(g-1)$ and $(\nu,r,d)\neq(4,2,2g+2)$. Then there is a nonempty open subset $\mathcal{U}$ of $M(X;r,d)$ such that for every $Q\in\mathcal{U}$ there is a nondegenerate embedding $\varphi:X\to G(r,\nu)$ with $\varphi^*(\mathcal{Q})\cong Q$.*

Proof. Let us consider the case $r\geq 3$. Since $G(r,r+1)=\mathbf{P}^r$, for $\nu=r+1$ the result is contained in Theorem 1.5. If $\nu>r+1$, we also have $d>rg+1$ and by Theorem 1.5 a general bundle F in $M(X;r,d)$ gives a degree d embedding $f:X\to\mathbf{P}^r$ with $f^*(T_{\mathbf{P}^r}(-1))\cong F$; then we have $h^0(X,F(-P-P'))<h^0(X,F(-P))$ for every $P,P'\in X$.

Since F is general, we have $h^1(X,F)=0$ (Lemma 1.2) and $h^0(X,F)=d+r(1-g)$. F is spanned and there is a surjection $\sigma:H^0(X,F)\otimes\mathcal{O}_X\to F$ giving a nondegenerate embedding $h:X\to G(r,d+r-rg)$ with $h^*(\mathcal{Q})\cong F$.

We point out that h is an embedding because the same holds for f.

Since F is stable and $H^0(X, \ker\sigma) = 0$, we have that neither F nor $\ker\sigma$ has $\mathcal{O}_X$ as a component and so h is nondegenerate.

Consider the sequence of all the general projections of $h(X)$ over the Grassmannians $G(r,\nu)$ with $r+1 \le \nu < d+r-rg$. We obtain nondegenerate morphisms $h_\nu : X \to G(r,\nu)$ that are embeddings because $h_{r+1} = f$ and we have $h_\nu^*(\mathcal{Q}) \cong F$.

Now we consider the case $r = 2$. Let M be a spanned nonspecial line bundle on X with degree $g+1$. We have an exact sequence

$$0 \longrightarrow M^\vee \longrightarrow \mathcal{O}_X^{\oplus 2} \longrightarrow M \longrightarrow 0. \tag{1}$$

Suppose $\nu \ge 6$. For $n \ge 3$, consider a nonspecial line bundle L_n on X with $\deg L_n \ge g+n$, giving an embedding $f_n : X \to \mathbf{P}^n$.

Put $F_n := f_n^*(T_{\mathbf{P}^n}(-1))$. We have an exact sequence

$$0 \longrightarrow F_n^\vee \longrightarrow \mathcal{O}_X^{\oplus(n+1)} \longrightarrow L_n \longrightarrow 0. \tag{2}$$

Exact sequences (1) and (2) give the following exact sequence:

$$0 \longrightarrow F_{\nu-3}^\vee \oplus M^\vee \longrightarrow \mathcal{O}_X^{\oplus\nu} \longrightarrow L_{\nu-3} \oplus M \longrightarrow 0. \tag{3}$$

$E := L_{\nu-3} \oplus M$ is a rank 2 vector bundle with degree $d \ge \nu + 2(g-1)$. E is nonspecial and spanned. Moreover, for every $P, P' \in X$, allowing $P = P'$, we have $h^0(X, E(-P-P')) < h^0(X, E(-P))$. Thus the exact sequence (3) gives a nondegenerate embedding $h : X \to G(2,\nu)$ with $h^*(\mathcal{Q}) \cong E$.

On a smooth curve every vector bundle is the flat limit of a flat family of stable vector bundles, and we may also assume that in the flat family the determinant is constant (see Lemma 1.1). Then a general Q in $M(X;2,d)$ is spanned and induces an embedding from X to $G(2,\nu)$.

For $r = 2$ and $\nu = 5$, consider a nonspecial line bundle L_2 on X with $\deg L_2 \ge g+2$ giving a morphism $f : X \to \mathbf{P}^2$ whose image is a nodal curve. We have an exact sequence

$$0 \longrightarrow F_2^\vee \longrightarrow \mathcal{O}_X^{\oplus 3} \longrightarrow L_2 \longrightarrow 0. \tag{4}$$

Now check that, for every pair of points $P, P' \in X$, allowing $P = P'$, such that $h^0(X, L_2(-P-P')) = h^0(X, L_2(-P))$, the line bundle M in (1) satisfies the condition $h^0(X, M(-P-P')) < h^0(X, M(-P))$.

Exact sequences (1) and (4) give the following exact sequence:

$$0 \longrightarrow F_2^\vee \oplus M^\vee \longrightarrow \mathcal{O}_X^{\oplus 5} \longrightarrow L_2 \oplus M \longrightarrow 0. \tag{5}$$

Take $E := L_2 \oplus M$. E is a rank 2 vector bundle with degree $d \geq 2g+3$. We can conclude as above.

To have the assertion for the Grassmannian $G(2,4)$, take a general Q in $M(X;2,d)$, with $d \geq 2g+3$, giving an embedding $h : X \to G(2,5)$ and consider a general projection of $h(X)$ over a Grassmannian $G(2,4)$ (see [**1**, Section 5]).

Remark 2.2. Let us consider the Grassmannian $G(2,4)$ and a smooth projective curve X of genus $g \geq 2$. For every $d \geq 2g+2$ there is a nonempty open subset $\mathcal{U}$ of $M(X;2,d)$ such that for every $Q \in \mathcal{U}$ there is a nondegenerate morphism $\varphi : X \to G(2,4)$ whose image is a nodal curve and $\varphi^*(\mathcal{Q}) \cong Q$.

In fact we can consider two spanned nonspecial line bundles M, M' on X with degree $g+1$. They give two coverings of $\mathbf{P}^1$. We can choose M and M' such that the two sets of ramification points are disjoint. Take $E := M \oplus M'$ and proceed as in the proof of Theorem 2.1.

Remark 2.3. Fix $P \in G(r,\nu)$, P corresponds to a codimension r linear subspace A of K^ν, i.e., P corresponds to an exact sequence of linear spaces $0 \to A \to k^\nu \to B \to 0$ with $\dim B = r$. The fibers of the universal vector bundles $\mathcal{Q}$ and $\mathcal{S}$ at P corresponds to B and A, respectively.

A line D in $G(r,\nu)$ with $P \in D$ is uniquely determined by the choice of linear subspaces A', A'' with $A' \subset A \subset A''$ and with $\dim A'' = \dim A' + 2 = \dim A + 1$. The points of D correspond to the codimension 1 linear subspaces of A'' containing A'.

The exact sequence $0 \to A \to A'' \to k \to 0$ induces an exact sequence $0 \to k \to B \to B'' \to 0$ where B'' is the quotient k^ν / A'', so we have the exact sequence $0 \to (B'')^\vee \to (\mathcal{Q}^\vee)_{|\{P\}} \xrightarrow{\alpha} k(P) \to 0$. Note that α gives a point of $\mathbf{P}(\mathcal{Q}^\vee)$ lying on the fiber at P.

The exact sequence $0 \to A' \to A \to k \to 0$ can be written in this way: $0 \to A' \to \mathcal{S}_{|\{P\}} \xrightarrow{\beta} k(P) \to 0$. Thus β gives a point of $\mathbf{P}(\mathcal{S})$ lying on the fiber at P. For simplicity we denote by D the point of $\mathbf{P}(\mathcal{Q}^\vee)_P$ and $\mathbf{P}(\mathcal{S})_P$ defined by α and β.

In [**4**], the interpretation in terms of elementary transformations of a degeneration of curves in a projective space (see Proposition 1.3) is extended to curves in a Grassmannian ([**4**, Theorem 6.1]). Also in this case the proof works in any characteristic of k.

Lemma 2.4 ([**4**, Theorem 6.1]). *Fix integers g, r, ν with $g \geq 0$, $r > 0$ and $\nu > r$. Let Y be a smooth genus g curve and Q a rank r degree d vector bundle on Y giving a nondegenerate embedding $h : Y \to G(r, \nu)$. Define the kernel bundle S on X by the exact sequence $0 \to S \to \mathcal{O}_Y^{\oplus\nu} \to Q \to 0$. Assume $h^1(Y, S^\vee) = 0$. Let P be a point of Y and D a general line of $G(r, \nu)$ passing through $h(P)$. Let $Q' := \mathrm{elm}_D^+ Q$ be the positive elementary transformation of Q supported by P and defined by the line D.*

Then Q' is the flat limit of a flat family $\{Q_t\}$ of spanned vector bundles, each of them a quotient of $\mathcal{O}_Y^{\oplus\nu}$ and as kernel bundles a family $\{S_t\}$ having as flat limit the negative elementary transformation $\mathrm{elm}_D^- S$ of S supported by P and defined by the line D.

Lemma 2.5. *Let X be a smooth curve of genus $g \geq 2$, r and d integers with $r > 1$. Then for a general vector bundle E in $M(X; r, d)$ we have that a general negative elementary transformation of E and a general positive elementary transformation of E are both stable.*

Proof. Let E be a general stable bundle of $M(X; r, d)$ and F a rank i subbundle of E. Put F^+, the transformed bundle of F, in the positive elementary transformation $\mathrm{elm}_K^+ E$ of E. We recall that K is a point of $\mathbf{P}(E^\vee)$.

If $K \notin \mathbf{P}(F^\vee)$, then $\deg F^+ = \deg F$ and $(\deg F^+/i) < (d+1)/r$.

If $K \in \mathbf{P}(F^\vee)$, we have $\deg F^+ = \deg F + 1$. Since E is general, we have $r(\deg F) \leq id - i(r-i)(g-1)$ (see [**11**] and, in any characteristic of k, [**9**]). If the inequality is strict, we obtain $(\deg F + 1)/i < (d+1)/r$. The bundle E has only finitely many rank i subbundles F with $r(\deg F) = id - i(r-i)(g-1)$ ([**9**]) and then a general point K of $\mathbf{P}(E^\vee)$ is not contained in $\mathbf{P}(F^\vee)$ for such an F.

Thus, $\text{elm}_K^+ E$ is stable. Since $\text{elm}_K^- E \cong (\text{elm}_K^+ E^\vee)^\vee$, we also have the assertion on the negative elementary transformations.

Remark 2.6. Let $Z(X; r, d)$ be a ramified covering of $M(X; r, d)$ such that there is a semi-universal vector bundle $\mathcal{E}$ on $X \times Z(X, r, d)$.

The projective bundle $\mathbf{P}(\mathcal{E}^\vee) \to X \times Z(X; r, d)$ parameterizes the positive elementary transformations of the vector bundles corresponding to the points of $Z(X; r, d)$. Set $Z'(X; r, d) := \mathbf{P}(\mathcal{E}^\vee)$. By Lemma 2.5, we obtain a rational map from $Z'(X; r, d)$ to $X \times M(X; r, d+1)$. By using negative elementary transformations, we obtain that the above rational map is dominant.

We will pass several times to a finite ramified covering to get freely many sections of the various morphisms; to simplify the notations we will never change the name of the maps and spaces after such a procedure.

Definition 2.7. Let E be a rank r vector bundle on X. Let $t(E)$ be the minimal integer t such that a general $G \in M(X; r, \deg E + t)$ is obtained from E by t positive elementary transformations, i.e., there is an exact sequence $0 \to E \to G$.

Remark 2.8. Here we will check that $t(E)$ is always defined.

We can prove that $t(E)$ is well-defined for every spanned bundle E, see Lemma 2.11 below.

Note that $t(E)$ is well-defined if and only if there is a line bundle L_0 on X such that $t(E \otimes L_0)$ is well-defined. In that case we have $t(E) = t(E \otimes L)$ for every line bundle L on X.

For every vector bundle E, a line bundle L_0 exists on X with a large degree such that $E \otimes L_0$ is spanned by its global sections. We obtain that $t(E) = t(E \otimes L_0)$ is well defined.

Remark 2.9. Let E be a rank r vector bundle on X. By Lemma 2.5, for every $t \geq t(E)$, a general $G \in M(X; r, \deg E + t)$ is obtained from E by t positive elementary transformations. Easy examples with direct sums of line bundles show that the integer $t(E)$ depends on the "instability degree" of the bundle E. The integer $t(E)$ is also the minimal integer t such that a general $G \in M(X; r, -\deg E - t)$ is obtained from $E^\vee$ making t general negative elementary transformations.

Lemma 2.10. *We have* $t(\mathcal{O}_X^{\oplus r}) = rg$, $r \geq 2$.

Proof. We have $t(\mathcal{O}_X^{\oplus r}) \geq rg$ because for every d a general $G \in M(X; r, d)$ has either $h^0(X, G) = 0$ or $h^1(X, G) = 0$ (Lemma 1.2) and hence it may contain $\mathcal{O}_X^{\oplus r}$ only if $d \geq rg$.

Conversely, if $d \geq rg$ and G is general in $M(X; r, d)$, we have $h^1(X, G) = 0$ (Lemma 1.2) and $h^0(X, G) \geq r$. If $d \geq rg + 1$ and $r \geq 3$, by Theorem 1.5 a general G is generated by its global sections and $H^0(X, G)$ spans a rank r subsheaf of G, i.e., we have an exact sequence $0 \to \mathcal{O}_X^{\oplus r} \to G$.

For $d = rg$ and $r \geq 3$, let E be a general bundle in $M(X; r, rg + 1)$; then $h^1(X, E) = 0$ and E is generated by its global sections. Thus $h^1(X, E(-P)) = 0$ for every $P \in X$. Fix $P_0 \in X$ for a general point P of X and a general point K of $\mathbf{P}(E)_P = \mathbf{P}(E(-P_0)_P$ take the negative elementary transformations $G := \mathrm{elm}_K^-(E)$ and $\mathrm{elm}_K^-(E(-P_0)) \cong G(-P_0)$. We have $h^1(X, G) = h^1(X, G(-P_0)) = 0$; then $h^0(X, G) = r$ and $h^0(G(-P_0)) = 0$. We have an injective morphism $\mathcal{O}_X^{\oplus r} \to G$.

For $r = 2$ and $d \geq 2g + 3$, by [**4**, Proposition 3.3], a general $E \in M(X; 2, d)$ is generated by its global sections. Taking general negative elementary transformations as above, we conclude

Lemma 2.11. *Let E be a rank r degree d vector bundle on X spanned by its global sections. Then* $t(E) \leq (r-1)d + rg$.

Proof. Since E is spanned, there is an exact sequence $0 \to \mathcal{O}_X^{\oplus r} \to E$, i.e., E is obtained from $\mathcal{O}_X^{\oplus r}$ by d positive elementary transformations, say supported by $P_1, \dots, P_d \in X$ (allowing repetitions). Hence, $(\mathcal{O}_X(P_1 + \cdots + P_d))^{\oplus r}$ is obtained from E making $(r-1)d$ positive elementary transformations. Since for every $m \in \mathbf{Z}$ twisting by a line bundle $M \in \mathrm{Pic}^d(X)$ induces a natural isomorphism between $M(X; r, m)$ and $M(X; r, m + rd)$, we conclude by Lemma 2.10.

Lemma 2.12. *Fix integers d, r, ν with $1 < r < \nu$ such that* *there is a degree d_0 nondegenerate embedding $h : X \to G(r,\nu)$ with $h^1(X, h^*(\mathcal{S}^\vee)) = 0$. Then for every integer $d \geq d_0 + \max\{t(h^*(\mathcal{Q})), t(h^*(\mathcal{S}^\vee))\}$ (see Definition 2.7), and a general $L \in \mathrm{Pic}^d(X)$ there is an open dense subset $\mathcal{U}$ of $M(X; r, L)$ and an open dense subset $\mathcal{V}$ of $M(X; \nu - r, L^\vee)$ such that for every $Q \in \mathcal{U}$ and every $S \in \mathcal{V}$ there is an embedding $\varphi : X \to G(r,\nu)$ with $Q \cong \varphi^*(\mathcal{Q})$, $S \cong \varphi^*(\mathcal{S})$ and in particular $\varphi^*(T_{G(r,\nu)}) \cong Q \otimes S^\vee$.*

Proof. We have the exact sequence $0 \to h^*(\mathcal{S}) \to \mathcal{O}_X^{\oplus\nu} \to h^*(\mathcal{Q}) \to 0$. A general rank r degree d vector bundle Q on X is obtained from $h^*(\mathcal{Q})$ by $d - d_0$ positive elementary transformations. We have an exact sequence

$$0 \longrightarrow h^*(\mathcal{Q}) \longrightarrow Q \longrightarrow k(P_1, \dots, P_{d-d_0}) \longrightarrow 0. \tag{6}$$

We may assume that the points $P_1, \dots, P_{d-d_0}$ are general; in fact, the line bundle $L = \det Q$ is general and $\det Q \cong \det h^*(\mathcal{Q}) \otimes \mathcal{O}_X(P_1 + \cdots + P_{d-d_0})$. For a general vector bundle S on X of rank $\nu - r$ and degree $-d$ we have an exact sequence

$$0 \longrightarrow h^*(\mathcal{S}^\vee) \longrightarrow \mathcal{S}^\vee \longrightarrow k(Q_1, \dots, Q_{d-d_0}) \longrightarrow 0. \tag{7}$$

Since $\det h^*(\mathcal{Q}) \cong \det h^*(\mathcal{S}^\vee)$ and by our assumptions $\det S^\vee \cong \det Q$, the positive elementary transformations (6) and (7) are supported by the same points $P_1, \dots, P_{d-d_0}$ of X. The exact sequences (6) and (7) give the following exact sequences: $0 \to Q^\vee \to h^*(\mathcal{Q})^\vee \xrightarrow{\alpha} k(P_1, \dots, P_{d-d_0}) \to 0$ and $0 \to B \to h^*(\mathcal{S}) \xrightarrow{\beta} k(P_1, \dots, P_{d-d_0}) \to 0$.

For every point P_i, $1 \leq i \leq d - d_0$, the pair (α, β) gives a line D_i in $G(r,\nu)$ passing through $h(P_i)$ (Remark 2.3). By using Lemma 2.4 we can conclude

Theorem 2.13. *Fix a smooth projective curve X of genus $g \geq 2$ and integers r, ν with $2 \leq r < \nu$. If $(r,\nu) \neq (2,4)$, consider any integer d satisfying the condition $d \geq \max\{r((\nu - r + 1)g + r), (\nu - r)((\nu - r + 1)g + r)\}$. For $(r,\nu) = (2,4)$ consider any integer $d \geq 6g + 6$.*

Then for a general $L \in \mathrm{Pic}^d(X)$ there is an open dense subset $\mathcal{U}$ of $M(X; r, L)$ and an open dense subset $\mathcal{V}$ of $M(X; \nu - r, L^\vee)$ such *that for every $Q \in \mathcal{U}$ and for every $S \in \mathcal{V}$ there is a nondegenerate embedding $\varphi : X \to G(r,\nu)$ with $Q \cong \varphi^*(\mathcal{Q})$, $S \cong \varphi^*(\mathcal{S})$ and in particular $\varphi^*(T_{G(r,\nu)}) \cong Q \otimes S^\vee$.*

Proof. If $(r,\nu) \neq (2,4)$, let $d_0 = \nu + (\nu - r)(g-1)$. If $(r,\nu) = (2,4)$, take $d_0 = 2g+3$. Let F be a general bundle in $M(X;\nu - r, d_0)$ with $h^1(X,F) = 0$.

F gives a nondegenerate embedding $X \to G(\nu - r, \nu)$ (Theorem 2.1) and an exact sequence $0 \to G \to \mathcal{O}_X^{\oplus\nu} \to F \to 0$. Its dual sequence gives a nondegenerate embedding $h : X \to G(r,\nu)$. Applying Lemmas 2.11 and 2.12, we obtain the assertion.

Remark 2.14. If char $k = 0$, the tensor product of two stable bundles is polystable, i.e., it is a direct sum of stable bundles with the same slope. Moreover, if E and F are two general stable bundles, then the tensor product $E \otimes F$ is stable. This fact is well known. One may prove it by means of the irreducible unitary representations of suitable Fuchsian groups [**15**] and by following the proof of Theorem 10.5 contained in [**6**]. Hence we can take in Theorem 2.13 open dense subsets $\mathcal{U}$ and $\mathcal{V}$ giving stable restricted tangent bundles $\varphi^*(T_{G(r,\nu)})$.

Remark 2.15. Fix integers r, ν and d with $2 \leq r \leq \nu - 2$. Let L be a general degree d line bundle on X. In characteristic 0, the tensor product of two general stable bundles is stable, Remark 2.14. Hence, there are an open subset U of $M(X;r,L)$, an open subset V of $M(X;\nu - r, L^\vee)$ and a morphism of varieties $\phi : U \times V \to M(X; r(\nu - r), L^{\otimes\nu})$ defined by $\phi((E,F)) = E \otimes F^\vee$. ϕ is not dominant because we have $\dim M(X;r,L) + \dim M(X;\nu - r, L^\vee) < \dim M(X; r(\nu - r), L^{\otimes\nu})$. Hence, if $\nu \geq r+2 \geq 4$, for any integer d the restricted tangent bundles to degree d embeddings of X into $G(r,\nu)$ cannot cover a dense open subset of $M(X; r(\nu - r), \nu d)$.

3. Morphisms into flag varieties. X denotes a smooth projective curve of genus $g \geq 2$.

Theorem 3.1. *Fix integers ν, s and r_i, $1 \leq i \leq s$, with $\nu > r_1 > r_2 > \cdots > r_s > 0$. Let* Flag $(r_1, \ldots, r_s; \nu)$ *be the* (*partial*) *flag variety of* *quotient spaces of* k^ν *of dimension* $r_1, \ldots, r_s$. *Fix* s *integers* $d_1, \ldots, d_s$ *with* $d_1 \geq \nu + r_1(g-1)$ *and* $(\nu, r_1, d_1) \neq (4, 2, 2g+2)$.

a) *Suppose* $(d_i/r_i) + g \leq (d_{i+1}/r_{i+1})$ *for every* i *with* $1 \leq i < s$. *Then there is a nonempty open subset* U *of* $M(X; r_1, d_1) \times \cdots \times M(X; r_s, d_s)$ *such that for every* $(E_1, \ldots, E_s) \in U$ *there exists a nondegenerate embedding* $f : X \to \mathrm{Flag}\,(r_1, \ldots, r_s; \nu)$ *such that* $h^1(X, E_1) = 0$, $h^1(X, \mathrm{Hom}\,(E_i, E_{i+1})) = 0$ *for every* $1 \leq i < s$ *and* $E_1 \to E_2 \to \cdots \to E_s$ *are the pullbacks by* f *of the universal flag of quotients on* $\mathrm{Flag}\,(r_1, \ldots, r_s; \nu)$.

b) *Suppose* $(d_i/r_i) + 2g - 1 \leq (d_{i+1}/r_{i+1})$ *for every* i *with* $1 \leq i < s$. *Then there is a nonempty open subset* U' *of* $M(X; r_1, d_1)$ *such that for every* $E_1 \in U'$ *and for every* $E_i \in M(X; r_i, d_i)$ *with* $2 \leq i \leq s$ *there is a nondegenerate embedding* $f : X \to \mathrm{Flag}\,(r_1, \ldots, r_s; \nu)$ *such that* $h^1(X, E_1) = 0$ *and* $E_1 \to E_2 \to \cdots \to E_s$ *are the pullbacks by* f *of the universal flag of quotients on* $\mathrm{Flag}\,(r_1, \ldots, r_s; \nu)$.

Proof. By Theorem 2.1 there is a nonempty open subset U'' of $M(X; r_1, d_1)$ such that for every $E_1 \in U''$ we have $h^1(X, E_1) = 0$, hence $h^0(X, E_1) = d_1 + r_1(1-g) \geq \nu$, and such that there is a linear space $V \subset H^0(X, E_1)$ with $\dim V = \nu$, V spanning E_1 and such that the morphism $\varphi : X \to G(r_1, \nu)$ induced by the pair (E_1, V) is a nondegenerate embedding.

Applying $s-1$ times the first case of Lemma 3.2 below, we obtain the chain of surjections $E_1 \to E_2 \to \cdots \to E_s$ with (E_1, V) inducing a nondegenerate embedding of X into $G(r_1, \nu)$. Hence $(E_1 \to E_2 \to \cdots \to E_s, V)$ induces a nondegenerate embedding into the flag variety.

Part a) is proved. For part b) we proceed as above by using the second case of Lemma 3.2 below.

Lemma 3.2. *Fix integers* a, b, u, w *with* $u > w \geq 1$ *and* $(b/w) \geq (a/u) + g$, *respectively,* $(b/w) \geq (a/u) + 2g - 1$. *Then for a general, respectively for every, pair* (A, B) *in* $M(X; u, a) \times M(X; w, b)$ *there is a surjection* $f : A \to B$ *and we have* $h^1(X, \mathrm{Hom}\,(A, B)) = 0$.

Proof. Let us consider the first case. Since in arbitrary characteristic every bundle on X is the flat limit of a flat family of stable bundles

([**9**]), it is sufficient to find bundles A', B' with $\operatorname{rank} A' = u$, $\deg A' = a$, $\operatorname{rank} B' = w$, $\deg B' = b$, $h^1(X, \operatorname{Hom}(A', B')) = 0$ and such that there exists a surjection $A' \to B'$. Fix $P \in X$ and set $x := [a/u]$ (the integer part of (a/u)). Twisting by $\mathcal{O}_X(-xP)$ we may reduce to the case $0 \leq a < u$.

For $1 \leq i \leq u - a$, take general $M_i \in \operatorname{Pic}^0(X)$. For $u - a < i \leq u$, take general $M_i \in \operatorname{Pic}^1(X)$. Set $y := [b/w] \geq g$ and $m := b - yw$ (hence $0 \leq m < w$). For $1 \leq j \leq w - m$, take general $N_j \in \operatorname{Pic}^y(X)$. For $w - m < j \leq w$, take general $N_j \in \operatorname{Pic}^{y+1}(X)$. Set $A' := \oplus_{1 \leq i \leq u} M_i$ and $B' := \oplus_{1 \leq j \leq w} N_j$. Since $\deg(N_j \otimes M_i^\vee) \geq g - 1$ for all i, j and the line bundles M_i and N_j are general, we have $h^1(X, \operatorname{Hom}(A', B')) = 0$.

Set $A'' := \oplus_{1 \leq i \leq w+1} M_i$. It is sufficient to find a surjection $A'' \to B'$ and then send to 0 the remaining $u = w - 1$ factors (if any) of A'. For $1 \leq j \leq w$, we fix $s_j \in H^0(X, N_j \otimes M_{j+1}^\vee)$, $s_j \neq 0$ and such that their 0-loci $D(j)$'s, $1 \leq j \leq w$, are reduced and disjoint. This is possible by the generality of all N_j's and M_i's. We send the factor M_{j+1} to B' using s_j as map to N_j and 0 for maps to the other factors. Since $D(j) \cap D(l) = \varnothing$ for $j \neq l$, the corresponding map $\oplus_{2 \leq i \leq u} M_i \to B'$ has rank w outside $\cup_{1 \leq j \leq w} D(j)$ and rank $w - 1$ at each point of $\cup_{1 \leq j \leq w} D(j)$. Fix $a_j \in H^0(X, N_j \otimes M_1^\vee)$, $a_j \neq 0$ and such that their 0-loci are disjoint from $\cup_{1 \leq j \leq w} D(j)$ and map M_1 to B' using $(a_1, \ldots, a_w)$. The map $A' \to B'$ constructed in this way has rank w also at each point of $\cup_{1 \leq j \leq w} D(j)$.

Now let us consider the second case. Since A and B are stable and $\mu(B) \geq \mu(A) + 2g - 1$ (for any bundle A, $\mu(A)$ denotes the slope of A), we have $h^1(X, \operatorname{Hom}(A, B)) = h^1(X, \operatorname{Hom}(A, B(-P))) = 0$ for every $P \in X$. Hence $h^0(X, \operatorname{Hom}(A, B(-P))) = h^0(X, \operatorname{Hom}(A, B)) - uw$ by the Riemann-Roch Theorem. Set $H := H^0(X, \operatorname{Hom}(A, B))$. Fix $P \in X$. The vector space $H(-P) := \{f \in H / f(A_P) = 0\}$ has codimension uw in H. Since the vector space B_P given by the fiber of B at P has dimension w, for every codimension 1 linear subspace M of B_P the algebraic set $H_M = \{f \in H / f(A_P) \subseteq M\}$ has codimension $\geq u$ in H. Thus the set $H'(P) := \{f \in H / f(A_P) \neq B_P\}$ has codimension $\geq u - w + 1 \geq 2$ in H. Since $\dim X = 1$, $\cup_{P \in X} H'(P) \neq H$, i.e., a general $f \in H$ is surjective.

4. On the elliptic curves in a Grassmannian. In this section we will consider briefly the stability of the restricted tangent bundle of maps from an elliptic curve Y to a Grassmannian $G(r,\nu)$ with $1 \le r \le \nu - 1$ and $\nu \ge 4$.

We show that in characteristic 0 there is a degree d embedding of Y in $G(r,\nu)$ with stable restricted tangent bundle if and only if there is not a numerical restriction to its existence.

For $r = 1$ and $r = \nu - 1$ we have maps from Y to a projective space $\mathbf{P}^n$; these cases are easy and well known.

We assume $\operatorname{char} k = 0$ and use the multiplicative structure of vector bundles on elliptic curves proved by Atiyah in [**2**, Part III].

We need the following well-known lemma.

Lemma 4.1. *Every vector bundle E on Y is the flat limit of a family of semi-stable vector bundles with fixed determinant.*

Proof. By [**2**] there is a semi-stable vector bundle F on Y with $\operatorname{rank} F = \operatorname{rank} E$ and $\det F \cong \det E$. Since semi-stability is an open condition, it is sufficient to show that E and F have a common specialization by flat families of vector bundles with fixed determinant. We may use induction on $b := \operatorname{rank} E$, the case $b = 1$ being trivial. Assume $b > 1$. Fix an ample line bundle L and a large integer n such that both $E \otimes L^{\otimes n}$ and $F \otimes L^{\otimes n}$ are spanned. Hence we obtain exact sequences $0 \to L^{\otimes(-n)} \to E \to E' \to 0$ and $0 \to L^{\otimes(-n)} \to F \to F' \to 0$, with $\det E' \cong \det F'$ and $\operatorname{rank} E' = \operatorname{rank} F' = b - 1$. E has $L^{\otimes(-n)} \oplus E'$ as a flat specialization. Hence we conclude by the inductive assumption applied to E' and F'.

Theorem 4.2. *Assume* $\operatorname{char} k = 0$. *Let Y be an elliptic curve. Fix integers d, r and ν with $1 \le r \le \nu - 1$, $\nu \ge 4$. Then a nondegenerate degree d exists embedding $i : Y \to G(r,\nu)$ with $i^*(T_{G(r,\nu)})$ stable if and only if $d \ge \nu$ and $(r,\nu) = (r,d) = (\nu - r, d) = (1)$.*

Proof. Fix a nondegenerate degree d embedding i from Y to $G(r,\nu)$.

Put $Q := i^*(\mathcal{Q})$ and $S := i^*(\mathcal{S})$. If either Q or S are not stable, then $i^*(T_{G(r,\nu)}) \cong Q \otimes S^\vee$ is not stable. By Atiyah's classification of vector bundles on Y ([**2**, Part II]) there is a stable vector bundle A

with rank x and degree y if and only if $(x, y) = 1$. Hence a necessary condition for the stability of $i^*(T_{G(r,\nu)})$ is that $(r, d) = (\nu - r, d) = 1$. Note that $\deg(Q \otimes S^\vee) = \nu d$ and $\operatorname{rank}(Q \otimes S^\vee) = r(\nu - r)$. Hence $(r, \nu) = 1$ is another necessary condition for the stability of $i^*(T_{G(r,\nu)})$ and we may assume $(r, \nu - r) = 1$.

Let A be a stable vector bundle with rank r and B a stable vector bundle of rank $\nu - r$. By [**2**, Lemma 28], $A \otimes B$ is indecomposable and hence it is stable if and only if $\deg(A \otimes B)$ and $\operatorname{rank}(A \otimes B)$ are coprime. Hence if $\deg A = \deg B = d$, $A \otimes B$ is stable if and only if $(\nu d, r(\nu - r)) = 1$. This is the case because $(d, r) = (d, \nu - r) = (\nu, r) = 1$.

Now fix a stable bundle A on Y with $\operatorname{rank} A = r \geq 2$ and $\deg A = d > 0$. Note that $h^0(Y, A) = d$. We claim that A is spanned by its global sections if and only if $d > r$. Indeed, this condition is obviously necessary and, to see that it is sufficient, just use the fact that A and $A(-P)$ for every $P \in Y$ are nonspecial and the Riemann-Roch Theorem to show that for every $P \in Y$ we have $h^0(Y, A(-P)) = h^0(Y, A) - r$. Furthermore, for $d > r \geq 2$ and $P, P' \in Y$ (we may have $P = P'$), we obtain $h^0(Y, A(-P - P')) < h^0(Y, A(-P))$.

In fact we have $\deg A(-P - P') = d - 2r$.

If $d - 2r > 0$, then $A(-P - P')$ is nonspecial and $h^0(Y, A(-P - P')) = d - 2r$. If $d - 2r \leq 0$, we have $h^0(Y, A(-P - P')) = 0$. If $d - 2r = 0$, we have $h^0(Y, A(-P - P')) \leq 1 < h^0(Y, A(-P)) = r$ ([**2**, Lemma 15]).

Thus the map induced by $(A, H^0(Y, A))$ into the Grassmannian $G(r, d)$ is an embedding if $d > r$. Hence, for all integers ν with $r < \nu \leq d$, for a general subspace V of $H^0(Y, A)$ with $\dim V = \nu$ the pair (A, V) induces an embedding into $G(r, \nu)$. By construction this embedding has stable universal quotient bundle.

For $r = \nu - 1$, we have obtained an embedding $i : Y \to \mathbf{P}^{\nu-1}$ with $i^*(T_{\mathbf{P}^{\nu-1}}(-1)) \cong A$ stable. The case $r = 1$ is dual to the above case and the theorem is proved for $r = 1$ and $r = \nu - 1$.

Consider now the case $2 \leq r \leq \nu - 2$. We want to check that for a general V the corresponding universal subbundle S_V is stable. We fix any S_V. We have $h^0(Y, S_V) = 0$ because the map $V \to H^0(Y, A)$ is injective. Hence, by Serre duality, we have $h^1(Y, S_V^\vee) = 0$. Fix a flat family $\{B_t\}_{t \in T}$ of vector bundles on Y parameterized by an integral affine curve T with $o \in T$ such that $B_o \cong S_V^\vee$, B_t is semi-stable for general t and $\det(B_t) \cong \det(A)$ for every t (Lemma 4.1). By semi-continuity we have $h^1(Y, B_t) = 0$ for general t. Since, to be spanned by global sections is an open condition in a flat family of bundles with constant cohomology, for general t we obtain surjections $\sigma_t : V^\vee \otimes \mathcal{O}_Y \to B_t$. Hence we have a flat family of bundles $\ker \sigma_t$ with fixed determinant which are deformations of $A^\vee$. By Atiyah's classification ([**2**]), every stable bundle on Y is rigid in flat families with fixed determinant. We conclude that we have an exact sequence

$$0 \to S_V \to V \otimes \mathcal{O}_Y \to A \to 0$$

with A and S_V stable. Hence we have proved the theorem.

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