ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 31, Number 3, Fall 2001

ON THE NUMBER OF PARTITIONS WITH A FIXED LEAST PART

NEVILLE ROBBINS

ABSTRACT. Let $P(n)$, $Q(n)$ denote, respectively, the set of all unrestricted partitions of n and the set of all partitions of n into distinct parts. For $1 \leq j \leq n$, we derive formulas that permit the computation of the number of partitions in $P(n), Q(n)$ respectively whose least part is j.

1. Introduction. If $1 \leq j \leq n$, let $F_j(n)$, $f_j(n)$ denote, respectively, the number of partitions of the natural number n whose least part is j, the number of partitions of n into distinct parts whose least part is j. In this note, we derive formulas for the $F_j(n)$. We also derive recurrences that permit the evaluation of the $f_j(n)$ and present asymptotic formulas for the $f_j(n)$. In addition, we determine the parity of $f_1(n)$.

2. Preliminaries.

Definition 1. Let $p(n)$, $q(n)$ denote, respectively, the numbers of unrestricted partitions of n, partitions of n into distinct parts.

Definition 2. If $1 \leq j \leq n$, let $F_j(n)$ denote the number of partitions of n whose least part is j.

Definition 3. If $1 \leq j \leq n$, let $f_j(n)$ denote the number of partitions of n into distinct parts whose least part is j; let $f_j(0) = 0$.

Definition 4. Let $\omega(k) = k(3k-1)/2$.

(1) $q(n) \equiv 1 \pmod 2$ if and only if $n = \omega(\pm k)$ for some $k \geq 1$.

(2) $q(n) \sim 18^{-1/4}(24n+1)^{-3/4}e^{(\pi\sqrt{48n+2}/12)}$.

Remarks. (1) is well known; (2) was proven by Hagis [**1**].

Received by the editors on April 24, 2000.

3. Main results.

Theorem 1. *If* $1 \leq j \leq n$, *then*

$$f_j(n) = q(n-j) - \sum_{k=1}^{j} f_k(n-j).$$

Proof.

$$f_n(n) = 1 = q(0) = q(0) - \sum_{k=1}^{j} f_k(0).$$

If $n \geq j+1$, let a partition of $n-j$ into distinct parts exceeding j be given by

$$n - j = n_1 + n_2 + \cdots + n_r$$

where

$$n_1 > n_2 > \cdots > n_r > j.$$

Then a corresponding partition of n into distinct parts with j as least part is given by

$$n = n_1 + n_2 + \cdots + n_r + j,$$

and conversely. The conclusion now follows.

Theorem 2. $f_j(n) \leq f_j(n+1)$ *for all* $n \geq 1$.

Proof. If a partition of n into distinct parts with j as least part is given by

$$n = n_1 + n_2 + \cdots + n_r + j,$$

then a similar partition of $n+1$ is given by

$$n + 1 = (n_1 + 1) + n_2 + \cdots + n_r + j.$$

The conclusion now follows.

Theorem 3. $f_1(n) \leq q(n)/2$.

Proof. By Theorems 2 and 1, we have

$$2f_1(n) \leq f_1(n) + f_1(n+1) = q(n),$$

from which the conclusion follows.

The Table following lists $q(n)$ and $f_j(n)$ for $1 \leq j \leq 10$ and $0 \leq n \leq 22$.

Theorem 4. $f_1(n) \equiv 0 \pmod 2$ *if and only if there exists* $k \geq 1$ *such that* $\omega(k) < n \leq \omega(-k)$.

Proof. This follows from (1) and from Theorem 1, with $j = 1$.

Theorem 5. *For fixed* j *we have*

$$\frac{q(n-j)}{q(n)} \sim 1.$$

Proof. This follows from (2).

Theorem 6. *If* $0 < j < n$, *then* $f_j(n) \sim 2^{-j} q(n)$.

Proof (Induction on n). By Theorems 1 and 2, we have

$$\frac{1}{2}\, q(n-1) \leq f_1(n) \leq \frac{1}{2}\, q(n)$$

so that

$$\frac{q(n-1)}{q(n)} \leq \frac{f_1(n)}{q(n)/2} \leq 1.$$

Now Theorem 5 implies $f_1(n) \sim q(n)/2$, so Theorem 6 holds for $j = 1$. Similarly, one can show that

$$q(n-j) - \sum_{k=1}^{j-1} f_k(n-j) \leq 2f_j(n) \leq q(n) - \sum_{k=1}^{j-1} f_k(n).$$

By induction hypothesis, we have

$$\sum_{k=1}^{j-1} f_k(n) \sim \sum_{k=1} 2^{-k} q(n),$$

that is,

$$\sum_{k=1}^{j-1} f_k(n) \sim (1 - 2^{-(j-1)}) q(n)$$

and hence also

$$\sum_{k=1}^{j-1} f_k(n-j) \sim (1 - 2^{-(j-1)}) q(n)$$

Thus we have

$$2 f_j(n) \sim q(n)/2^{j-1}$$

from which the conclusion follows.

Remarks. It is easily seen that $f_j(j) = 1$ and that $f_j(n) = 0$ if $[(n+1)/2] \le j \le n-1$.

Theorem 7. *If* $1 \le j \le n$, *then*

$$F_j(n) = p(n-j) - \sum_{k=1}^{j-1} F_k(n-j).$$

Proof. Clearly, $F_j(j) = 1$. If $n \ge j+1$, let a partition of $n-j$ whose least part is at least j be given by

$$n - j = n_1 + n_2 + \cdots + n_r$$

where $n_1 \ge n_2 \ge \cdots \ge n_r \ge j$. Then a corresponding partition of n whose least part is j is given by

$$n = n_1 + n_2 + \cdots + n_r + j$$

and vice versa. The conclusion now follows.

Remarks. As particular cases of Theorem 7, we have

$$F_1(n) = p(n-1)$$

$$F_2(n) = p(n-2) - p(n-3)$$

$$F_3(n) = p(n-3) - p(n-4) - p(n-5) + p(n-6).$$

REFERENCES

1. P. Hagis, *On a class of partitions with distinct summands*, Trans. Amer. Math. Soc. **112** (1964), 401–415.

Mathematics Department, San Francisco State University, San Francisco, CA 94132
E-mail address: robbins@math.sfsu.edu