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ON SEMI-SYMMETRIC COMPLEX HYPERSURFACES OF A SEMI-DEFINITE COMPLEX SPACE FORM

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ABSTRACT. The purpose of this paper is to give a complete classification of semi-symmetric complex hypersurfaces M in an $(n+1)$-dimensional semi-definite complex space form $M_{s+t}^{n+1}(c)$. Moreover, we also give a classification of semi-symmetric complex hypersurfaces in a semi-definite complex Euclidean space C_t^{n+1}, $t = 0$ or 1 when M has no geodesic points.

1. Introduction. Theory of indefinite complex submanifolds of an indefinite complex space form is one of the most interesting topics in differential geometry, and it has been investigated by many geometers from various points of view ([**1**], [**2**], [**3**], [**6**], [**9**], [**12**], [**13**] and [**15**], etc.).

Let $M_t^m(c)$ be an m-dimensional semi-definite complex space form of constant holomorphic sectional curvature c and of index $2t$, $0 \leqq t \leqq m$. As is well known, it globally consists of the following three kinds of complex space forms: the semi-definite complex projective space CP_t^m, the semi-definite complex Euclidean space C_t^m or the semi-definite complex hyperbolic space CH_t^m, according to whether $c > 0$, $c = 0$ or $c < 0$.

Now let M be a semi-definite Kaehler manifold. We denote by R the Riemannnian curvature tensor defined on M. Then M is said to be *semi-symmetric* if it satisfies the condition $R(X,Y)R = 0$ for any vector field X and Y on M. Its notion is much wider than the notion of locally symmetric spaces, that is, $\nabla R = 0$. The notion of semi-symmetric Riemannian spaces was first introduced by Cartan and

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Szabó [**14**], who systematically studied this kind of manifold structure in detail.

Now in this paper we give a classification of semi-symmetric semi-definite complex hypersurfaces in a semi-definite complex space form $M_{s+t}^{n+1}(c)$ as follows:

Theorem 1. *Let M_s^n be an n-dimensional semi-symmetric and semi-definite complex hypersurface of index $2s$ in $M' = M_{s+t}^{n+1}(c)$, $0 \leqq s \leqq n$, $t = 0$ or 1, $c \neq 0$. Then M is totally geodesic with the scalar curvature $r = n(n+1)c$ or Einstein with the scalar curvature $r = n^2 c$.*

Moreover, for a semi-symmetric complex hypersurface in a semi-definite complex Euclidean space C_t^{n+1}, we assert the following.

Theorem 2. *Let M be an n-dimensional semi-symmetric complex hypersurface of C_t^{n+1}, $t = 0$ or 1. If it has no geodesic points, then for any point x in M there exists a totally geodesic hypersurface $M(x)$ of M through x.*

2. Semi-definite Kaehler manifolds. This section is concerned with recalling basic formulas on semi-definite Kaehler manifolds. Let M be a complex m ($\geqq 2$)-dimensional semi-definite Kaehler manifold equipped with semi-definite Kaehler metric tensor g and almost complex structure J. For the semi-definite Kaehler structure $\{g, J\}$, it follows that J is integrable and the index of g is even, say $2q$, $0 \leqq q \leqq m$. In such a case it is denoted by M_q^m.

When the index q is contained in the range $0 < q < m$, M is said to be an *indefinite Kaehler manifold* and the structure $\{g, J\}$ is called an *indefinite Kaehler structure* and in particular, in the case where $q = 0$ or m, M is only called a *Kaehler manifold*, and then the structure $\{g, J\}$ is called a *Kaehler structure*.

We can choose a local field $\{E_A, E_{A^*}\} = \{E_1, \dots, E_m, E_{1^*}, \dots, E_{m^*}\}$ of orthonormal frames on a neighborhood of M where $E_{A^*} = JE_A$ and $A^* = m + A$. Here the indices $A, B, \dots$ run from 1 to m. We set $U_A = (E_A - iE_{A^*})/\sqrt{2}$ and $\overline{U}_A = (E_A + iE_{A^*})/\sqrt{2}$, where i denotes the imaginary unit. Then $\{U_A\}$ constitutes a local field of unitary frames on the neighborhood of M. This is a complex linear frame which is orthonormal with respect to the semi-definite Kaehler metric, that is, $g(U_A, \overline{U}_B) = \varepsilon_A \delta_{AB}$, where

$$\varepsilon_A = -1 \text{ or } 1, \quad \text{according to whether } 1 \leqq A \leqq q \text{ or } q+1 \leqq A \leqq m.$$

Let $\{\omega_A\}$ be the dual coframe field with respect to the local field $\{U_A\}$ of unitary frames on the neighborhood of M. Then $\{\omega_A\} = \{\omega_1, \dots, \omega_m\}$ consists of complex-valued 1-forms of type $(1,0)$ on M such that $\omega_A(U_B) = \varepsilon_A \delta_{AB}$ and $\{\omega_A, \bar{\omega}_A\} = \{\omega_1, \dots, \omega_m, \bar{\omega}_1, \dots, \bar{\omega}_m\}$ are linearly independent.

The semi-definite Kaehler metric g of M can be expressed as $g = 2\sum_A \varepsilon_A \omega_A \otimes \bar{\omega}_A$. Associated with the frame field $\{U_A\}$, there exist complex-valued forms ω_{AB} which are usually called *connection forms* on M such that they satisfy the structure equations of M:

$$\begin{aligned} & d\omega_A + \sum_B \varepsilon_B \omega_{AB} \wedge \omega_B = 0, \qquad \omega_{AB} + \bar{\omega}_{BA} = 0, \\ & d\omega_{AB} + \sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} = \Omega_{AB}, \\ & \Omega_{AB} = \sum_{C,D} \varepsilon_C \varepsilon_D R_{\overline{A}BC\overline{D}} \omega_C \wedge \bar{\omega}_D, \end{aligned} \tag{2.1}$$

where $\Omega = (\Omega_{AB})$, respectively $R_{\overline{A}BC\overline{D}}$, denotes the curvature form, respectively the components of the semi-definite Riemannian curvature tensor R, of M.

The second relation of the equation (2.1) means that the skew-Hermitian symmetry of Ω_{AB}, which is equivalent to the symmetric condition $R_{\overline{A}BC\overline{D}} = \overline{R}_{\overline{B}AD\overline{C}}$. Moreover, the first Bianchi identity $\sum_B \varepsilon_B \Omega_{AB} \wedge \omega_B = 0$ is given by the exterior differential of the first and third equations of (2.1), which implies the further symmetric relations

$$R_{\overline{A}BC\overline{D}} = R_{\overline{A}CB\overline{D}} = R_{\overline{D}CB\overline{A}} = R_{\overline{D}BC\overline{A}}.$$

Now, relative to the frame field chosen above, the Ricci tensor S of M can be expressed as follows:

$$S = \sum_{A,B} \varepsilon_A \varepsilon_B (S_{A\overline{B}} \omega_A \otimes \bar{\omega}_B + S_{\overline{A}B} \bar{\omega}_A \otimes \omega_B),$$

where $S_{A\overline{B}} = \sum_C \varepsilon_C R_{\overline{C}CA\overline{B}} = S_{\overline{B}A} = \overline{S}_{\overline{A}B}$. The scalar curvature r is also given by $r = 2\sum_A \varepsilon_A S_{A\overline{A}}$. An n-dimensional semi-definite Kaehler manifold M is said to be *Einstein* if the Ricci tensor S is given by $S_{A\overline{B}} = (r/2n)\varepsilon_A\delta_{AB}$. The components $R_{\overline{A}BC\overline{D}:E}$ and $R_{\overline{A}BC\overline{D}:\overline{E}}$ of the covariant derivative of the Riemannian curvature tensor R are defined by

$$\sum_E \varepsilon_E (R_{\overline{A}BC\overline{D}:E}\omega_E + R_{\overline{A}BC\overline{D}:\overline{E}}\bar{\omega}_E) = dR_{\overline{A}BC\overline{D}}$$
$$-\sum_E \varepsilon_E (R_{\overline{E}BC\overline{D}}\bar{\omega}_{EA} + R_{\overline{A}EC\overline{D}}\omega_{EB} + R_{\overline{A}BE\overline{D}}\omega_{EC} + R_{\overline{A}BC\overline{E}}\bar{\omega}_{ED}).$$

The second Bianchi identity is given by $R_{\overline{A}BC\overline{D}:E} = R_{\overline{A}BE\overline{D}:C}$.

Let M be an m-dimensional semi-definite Kaehler manifold of index $2q$, $0 \leqq q \leqq m$. A plane section P of the tangent space T_xM of M at any point x is said to be *nondegenerate* provided that $g_x|_{T_xM}$ is nondegenerate. It is easily seen that P is nondegenerate if and only if it has a basis $\{X, Y\}$ such that

$$g(X,X)g(Y,Y) - g(X,Y)^2 \neq 0.$$

If the nondegenerate plane P is invariant by the complex structure J, it is said to be *holomorphic*. It is also trivial that the plane P is holomorphic if and only if it contains a vector X in P such that $g(X,X) \neq 0$. For the nondegenerate plane P spanned by X and Y in P, the sectional curvature $K(P)$ is usually defined by

$$K(P) = K(X,Y) = \frac{g(R(X,Y)Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2}.$$

The sectional curvature $K(P)$ of the holomorphic plane P is called the *holomorphic sectional curvature*, which is denoted by $H(P)$. The semi-definite Kaehler manifold M is said to be of *constant holomorphic sectional curvature* if its holomorphic sectional curvatures $H(P)$ are constant for all holomorphic planes at all points of M. Then M is called a *semi-definite complex space form*, which is denoted by $M_q^m(c)$ provided that it is of constant holomorphic sectional curvature c, of complex dimension m and of index $2q$ ($\geqq 0$).

It is seen in Wolf [**16**] that the standard models of semi-definite complex space forms are the following three kinds: the semi-definite complex projective space CP_q^m, the semi-definite complex Euclidean space C_q^m and the semi-definite complex hyperbolic space CH_q^m, according to whether $c > 0$, $c = 0$ or $c < 0$. For any integer q, $0 \leqq q \leqq m$, it is also seen by [**16**] that they are completely simply connected and connected semi-definite complex space forms of dimension m and of index $2q$. The Riemannian curvature tensor $R_{\overline{A}BC\overline{D}}$ of $M_q^m(c)$ is given by

$$R_{\overline{A}BC\overline{D}} = \frac{c}{2}\,\varepsilon_B\varepsilon_C(\delta_{AB}\delta_{CD} + \delta_{AC}\delta_{BD}).$$

3. Semi-definite complex submanifolds. This section is concerned with semi-definite complex submanifolds of an indefinite Kaehler manifold. First of all the basic formulas for the theory of semi-definite complex submanifolds are given.

Now let M' be an $(n+p)$-dimensional connected semi-definite Kaehler manifold of index $2(s+t)$, $0 \leqq s \leqq n$, $0 \leqq t \leqq p$, with semi-definite Kaehler structure (g', J'). Let M be an n-dimensional connected semi-definite complex submanifold of M', and let g be the induced semi-definite Kaehler metric tensor of index $2s$ on M from g'. We can choose a local field $\{U_A\} = \{U_j, U_x\} = \{U_1, \dots, U_{n+p}\}$ of unitary frames on a neighborhood of M' in such a way that, restricted to M, $U_1, \dots, U_n$ are tangent to M and the others are normal to M. Here and in the sequel, the following convention on the range of indices is used throughout this paper unless otherwise stated:

$$A, B, C, \dots = 1, \dots, n, n+1, \dots, n+p;$$
$$i, j, k, l, \dots = 1, \dots, n; \quad x, y, z, \dots = n+1, \dots, n+p.$$

With respect to the frame field, let $\{\omega_A\} = \{\omega_j, \omega_y\}$ be its dual frame fields. Then the semi-definite Kaehler metric tensor g' of M' is given by $g' = 2\sum_A \varepsilon_A\omega_A \otimes \bar{\omega}_A$ where $\{\varepsilon_A\} = \{\varepsilon_j, \varepsilon_y\}$, $\varepsilon_A = \pm 1$. The connection forms on M' are denoted by $\{\omega_{AB}\}$. The canonical forms ω_A and the connection forms ω_{AB} of the ambient space M' satisfy the structure equations

$$
\begin{aligned}
d\omega_A + \sum_B \varepsilon_B \omega_{AB} \wedge \omega_B = 0, \qquad \omega_{AB} + \bar{\omega}_{BA} = 0, \\
d\omega_{AB} + \sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} = \Omega'_{AB}, \\
\Omega'_{AB} = \sum_{C,D} \varepsilon_C \varepsilon_D R'_{\overline{A}BC\overline{D}} \omega_C \wedge \bar{\omega}_D,
\end{aligned} \tag{3.1}
$$

where Ω'_{AB}, respectively $R'_{\overline{A}BC\overline{D}}$, denotes the curvature form, respectively the components of the Riemannian curvature tensor R', of M'. Restricting these forms to the submanifold M, we have

$$\omega_x = 0, \tag{3.2}$$

and the induced semi-definite Kaehler metric tensor g of index $2s$ of M is given by $g = 2\sum_j \varepsilon_j \omega_j \otimes \bar{\omega}_j$. Then $\{U_j\}$ is a local unitary frame field with respect to this metric and $\{\omega_j\}$ is a local dual frame field due to $\{U_j\}$ which consists of complex-valued 1-forms of type $(1,0)$ on M. Moreover, $\omega_1, \dots, \omega_n, \bar{\omega}_1, \dots, \bar{\omega}_n$ are linearly independent and $\{\omega_j\}$ is the canonical form on M. It follows from (3.2) and Cartan's lemma that the exterior derivative of (3.2) gives rise to

$$\omega_{xi} = \sum_j \varepsilon_j {h_{ij}}^x \omega_j, \quad {h_{ij}}^x = {h_{ji}}^x. \tag{3.3}$$

The quadratic form $\alpha = \sum_{i,j,x} \varepsilon_i \varepsilon_j \varepsilon_x {h_{ij}}^x \omega_i \otimes \omega_j \otimes U_x$ with values in the normal bundle NM on M in M' is called the *second fundamental form* of the submanifold M. The structure equations for M are similarly given by

$$
\begin{aligned}
d\omega_i + \sum_j \varepsilon_j \omega_{ij} \wedge \omega_j = 0, \qquad \omega_{ij} + \bar{\omega}_{ji} = 0, \\
d\omega_{ij} + \sum_k \varepsilon_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij}, \qquad \Omega_{ij} = \sum_{k,l} \varepsilon_k \varepsilon_l R_{\bar{i}jk\bar{l}} \omega_k \wedge \bar{\omega}_l.
\end{aligned} \tag{3.4}
$$

Moreover, the following relationships are obtained:

$$
\begin{aligned}
d\omega_{xy} + \sum_z \varepsilon_z \omega_{xz} \wedge \omega_{zy} = \Omega_{xy}, \\
\Omega_{xy} = \sum_{k,l} \varepsilon_k \varepsilon_l R_{\bar{x}yk\bar{l}} \omega_k \wedge \bar{\omega}_l,
\end{aligned} \tag{3.5}
$$

where Ω_{xy} is called the *normal curvature form* of M. For the Riemannian curvature tensors R and R' of M and M', respectively, it follows from (3.1), (3.3) and (3.4) that we have the Gauss equation

$$R_{\bar{i}jk\bar{l}} = R'_{\bar{i}jk\bar{l}} - \sum_x \varepsilon_x h_{jk}{}^x \bar{h}_{il}{}^x. \tag{3.6}$$

And by means of (3.3) and (3.5), we have

$$R_{\bar{x}yk\bar{l}} = R'_{\bar{x}yk\bar{l}} + \sum_r \varepsilon_r h_{kr}{}^x \bar{h}_{rl}{}^y. \tag{3.7}$$

The components $S_{i\bar{j}}$ of the Ricci tensor S and the scalar curvature r of M are given by

$$S_{i\bar{j}} = \sum_k \varepsilon_k R'_{\bar{j}ik\bar{k}} - h_{i\bar{j}}{}^2, \tag{3.8}$$

$$r = 2\Big(\sum_{k,j} \varepsilon_k \varepsilon_j R'_{\bar{k}kj\bar{j}} - h_2\Big), \tag{3.9}$$

where $h_{i\bar{j}}{}^2 = h_{\bar{j}i}{}^2 = \sum_{x,r} \varepsilon_x \varepsilon_r h_{ir}{}^x \bar{h}_{rj}{}^x$ and $h_2 = \sum_j \varepsilon_j h_{j\bar{j}}{}^2$.

Now the components $h_{ijk}{}^x$ and $h_{ij\bar{k}}{}^x$ of the covariant derivative of the second fundamental form on M are given by

$$\sum_k \varepsilon_k (h_{ijk}{}^x \omega_k + h_{ij\bar{k}}{}^x \bar{\omega}_k) = dh_{ij}{}^x - \sum_k \varepsilon_k (h_{kj}{}^x \omega_{ki} + h_{ik}{}^x \omega_{kj}) + \sum_y \varepsilon_y h_{ij}{}^y \omega_{xy}. \tag{3.10}$$

Then, substituting $dh_{ij}{}^x$ in this definition into the exterior derivative of (3.3) and using (3.1)–(3.4) and (3.8), we have

$$h_{ijk}{}^x = h_{ikj}{}^x, \qquad h_{ij\bar{k}}{}^x = -R'_{\bar{x}ij\bar{k}}. \tag{3.11}$$

Similarly, the components $h_{ijkl}{}^x$ and $h_{ijk\bar{l}}{}^x$, respectively $h_{ij\bar{k}l}{}^x$ and $h_{ij\bar{k}l}{}^x$, of the covariant derivative of $h_{ijk}{}^x$, respectively $h_{ij\bar{k}}{}^x$, can be defined by

$$\sum_l \varepsilon_l (h_{ijkl}{}^x \omega_l + h_{ijk\bar{l}}{}^x \bar{\omega}_l) = dh_{ijk}{}^x - \sum_l \varepsilon_l (h_{ljk}{}^x \omega_{li} + h_{ilk}{}^x \omega_{lj} + h_{ijl}{}^x \omega_{lk}) + \sum_y \varepsilon_y h_{ijk}{}^y \omega_{xy}. \tag{3.12}$$

$$(3.13)\quad \sum_l \varepsilon_l (h_{ij\bar{k}l}{}^x \omega_l + h_{ij\bar{k}\bar{l}}{}^x \bar{\omega}_l) = dh_{ij\bar{k}}{}^x$$
$$- \sum_l \varepsilon_l (h_{lj\bar{k}}{}^x \omega_{li} + h_{il\bar{k}}{}^x \omega_{lj} + h_{ij\bar{l}}{}^x \bar{\omega}_{lk}) + \sum_y \varepsilon_y h_{ij\bar{k}}{}^y \omega_{xy}.$$

Differentiating (3.10) exteriorly and using the properties $d^2 = 0$, (3.4), (3.5), (3.8), (3.10) and (3.11), we have the following Ricci formula for the second fundamental form:

$$(3.14)\quad h_{ijkl}{}^x = h_{ijlk}{}^x, \qquad h_{ij\bar{k}\bar{l}}{}^x = h_{ij\bar{l}\bar{k}}{}^x,$$

$$(3.15)\quad h_{ijk\bar{l}}{}^x - h_{ij\bar{l}k}{}^x = \sum_r \varepsilon_r (R_{\bar{l}ki\bar{r}} h_{rj}{}^x + R_{\bar{l}kj\bar{r}} h_{ir}{}^x) - \sum_y \varepsilon_y R_{\bar{x}yk\bar{l}} h_{ij}{}^y.$$

In particular, let the ambient space M' be an $(n+p)$-dimensional semi-definite complex space form $M_{s+t}^{n+p}(c)$ of constant holomorphic sectional curvature c and of index $2(s+t)$, $0 \leqq s \leqq n$, $0 \leqq t \leqq p$. Then we get

$$(3.16)\quad R_{\bar{i}jk\bar{l}} = \frac{c}{2}\, \varepsilon_j \varepsilon_k (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}) - \sum_x \varepsilon_x h_{jk}{}^x \bar{h}_{il}{}^x,$$

$$(3.17)\quad S_{i\bar{j}} = \frac{(n+1)c}{2}\, \varepsilon_i \delta_{ij} - h_{i\bar{j}}{}^2,$$

$$(3.18)\quad r = n(n+1)c - 2h_2,$$

$$(3.19)\quad h_{ij\bar{k}}{}^x = 0,$$

$$(3.20)\quad h_{ijk\bar{l}}{}^x = \frac{c}{2}\,(\varepsilon_k h_{ij}{}^x \delta_{kl} + \varepsilon_i h_{jk}{}^x \delta_{il} + \varepsilon_j h_{ki}{}^x \delta_{jl})$$
$$- \sum_{r,y} \varepsilon_r \varepsilon_y (h_{ri}{}^x h_{jk}{}^y + h_{rj}{}^x h_{ki}{}^y + h_{rk}{}^x h_{ij}{}^y) \bar{h}_{rl}{}^y.$$

For the sake of brevity, a tensor $h_{i\bar{j}}{}^{2m}$ and a function h_{2m} on M for any integer m ($\geqq 2$) are introduced as follows:

$$h_{i\bar{j}}{}^{2m} = \sum_{i_1,\dots,i_{m-1}} \varepsilon_{i_1} \cdots \varepsilon_{i_{m-1}} h_{i\bar{i}_1}{}^2 h_{i_1\bar{i}_2}{}^2 \cdots h_{i_{m-1}\bar{j}}{}^2,$$

$$h_{2m} = \sum_i \varepsilon_i h_{i\bar{i}}{}^{2m}.$$

In particular, if M is a hypersurface, then a tensor $h_{ij}{}^{2m+1}$ on M is introduced as follows:

$$h_{ij}{}^{2m+1} = \sum_k \varepsilon_k h_{i\bar{k}}{}^{2m} h_{kj}.$$

4. Semi-symmetric and semi-definite complex hypersurfaces. This section is concerned with semi-symmetric and semi-definite complex hypersurfaces in a semi-definite complex space form. Let M be an n-dimensional semi-definite complex hypersurface of index $2s$ of an $(n+1)$-dimensional semi-definite complex space form $M' = M^{n+1}_{s+t}(c)$, $0 \leqq s \leqq n$, $t = 0$ or 1, of index $2(s+t)$ and of constant holomorphic sectional curvature c.

Let us denote by R the Riemannian curvature tensor on M. Assume that the hypersurface M satisfies the Nomizu condition

$$R(X, Y)R = 0, \quad X, Y \in TM. \tag{4.1}$$

It is equivalent to

$$R_{\bar{h}ij\bar{k}l\bar{n}} - R_{\bar{h}ij\bar{k}\bar{n}l} = 0. \tag{4.2}$$

By the twice exterior differentiation of the Riemannian curvature tensor R, the Ricci formula for R is as follows:

$$R_{\bar{h}ij\bar{k}l\bar{n}} - R_{\bar{h}ij\bar{k}\bar{n}l} = \sum_r \varepsilon_r (-R_{\bar{n}lr\bar{h}} R_{\bar{r}ij\bar{k}} + R_{\bar{n}li\bar{r}} R_{\bar{h}rj\bar{k}} + R_{\bar{n}lj\bar{r}} R_{\bar{h}ir\bar{k}} - R_{\bar{n}lr\bar{k}} R_{\bar{h}ij\bar{r}}).$$

In fact (4.2) is derived from (4.1). For the unitary frame $\{U_j\}$ on M, the components $R_{\bar{h}ij\bar{k}}$ of the Riemannian curvature tensor R is given by

$$R(U_i,\overline{U}_j)U_k=\sum_r \varepsilon_r R_{\bar{r}ki\bar{j}}U_r,$$
$$R(U_i,\overline{U}_j)\overline{U}_k=\sum_r \varepsilon_r R_{r\bar{k}i\bar{j}}\overline{U}_r.$$

Accordingly we have

$$\begin{aligned}
&(R(U_l,\overline{U}_n)R)(U_j,\overline{U}_k,U_i)\\
&\quad=R(U_l,\overline{U}_n)R(U_j,\overline{U}_k)U_i-R(R(U_l,\overline{U}_n)U_j,\overline{U}_k)U_i\\
&\qquad-R(U_j,R(U_l,\overline{U}_n)\overline{U}_k)U_i-R(U_j,\overline{U}_k)R(U_l,\overline{U}_n)U_i\\
&\quad=\sum_r \varepsilon_r\{R_{\bar{r}ij\bar{k}}R(U_l,\overline{U}_n)U_r-R_{\bar{r}jl\bar{n}}R(U_r,\overline{U}_k)U_i\\
&\qquad\qquad+R_{\bar{k}rl\bar{n}}R(U_J,\overline{U}_r)U_i-R_{\bar{r}il\bar{n}}R(U_j,\overline{U}_k)U_r\}\\
&\quad=\sum_{r,h}\varepsilon_r\varepsilon_h(R_{\bar{r}ij\bar{k}}R_{\bar{h}rl\bar{n}}-R_{\bar{r}jl\bar{n}}R_{\bar{h}ir\bar{k}}+R_{\bar{k}rl\bar{n}}R_{\bar{h}ij\bar{r}}-R_{\bar{r}il\bar{n}}R_{\bar{h}rj\bar{k}})U_h.
\end{aligned}$$

So the condition (4.1) is equivalent to

$$\sum_r \varepsilon_r(R_{\bar{r}ij\bar{k}}R_{\bar{h}rl\bar{n}}-R_{\bar{r}jl\bar{n}}R_{\bar{h}ir\bar{k}}+R_{\bar{k}rl\bar{n}}R_{\bar{h}ij\bar{r}}-R_{\bar{r}il\bar{n}}R_{\bar{h}rj\bar{k}})=0. \tag{4.3}$$

It means that (4.1) is equivalent to (4.3). By (3.16) and (4.3) we have

$$\begin{aligned}
&2({h_{l\bar{k}}}^2\bar{h}_{nh}+{h_{l\bar{h}}}^2\bar{h}_{nk})h_{ij}-2({h_{j\bar{n}}}^2h_{li}+{h_{i\bar{n}}}^2h_{lj})\bar{h}_{hk}\\
&\quad-c(\varepsilon_l\delta_{lh}h_{ij}\bar{h}_{nk}-\varepsilon_i\delta_{in}h_{jl}\bar{h}_{hk}-\varepsilon_j\delta_{jn}h_{li}\bar{h}_{hk}+\varepsilon_l\delta_{lk}h_{ij}h_n)=0.
\end{aligned} \tag{4.4}$$

Transvecting $\varepsilon_k\varepsilon_r h_{kr}$ to (4.4) and summing up with respect to indices k and r, we have

$$\begin{aligned}
&2({h_{l\bar{h}}}^4+h_2{h_{l\bar{h}}}^2)h_{ij}-2({h_{j\bar{h}}}^4h_{li}+{h_{i\bar{h}}}^4h_{lj})\\
&\qquad-c(\varepsilon_l\delta_{lh}h_2h_{ij}-h_{lj}{h_{i\bar{h}}}^2-h_{il}{h_{j\bar{h}}}^2+h_{ij}{h_{l\bar{h}}}^2)=0.
\end{aligned} \tag{4.5}$$

Then, putting $h=l$ in (4.5), multiplying ε_l and summing up with respect to the index l, we have

$$4{h_{ij}}^5-2c{h_{ij}}^3-\{2(h_4+{h_2}^2)-(n+1)ch_2\}h_{ij}=0. \tag{4.6}$$

On the other hand, putting $l = n$ in (4.4), multiplying ε_l and summing up with respect to the index l, and then putting $j = h$ in (4.5), multiplying ε_j and summing up with respect to the index j, we have

$$h_{ij}{}^3\bar{h}_{hk} = \bar{h}_{hk}{}^3 h_{ij}, \qquad h_2 h_{ik}{}^3 = h_4 h_{ik}, \tag{4.7}$$

respectively. Multiplying $h_2{}^2$ to (4.6) and making use of (4.7), we have

$$[4h_4{}^2 - 2ch_2h_4 - h_2{}^2\{2(h_4 + h_2{}^2) - (n+1)ch_2\}]h_{ij} = 0, \tag{4.8}$$

and hence we get

$$h_2[4h^24 - 2ch_2h_4 - h_2{}^2\{2(h_4 + h_2{}^2) - (n+1)ch_2\}] = 0. \tag{4.9}$$

Theorem 4.1. *Let $M = M_s^n$ be an n-dimensional semi-symmetric and semi-definite complex hypersurface of index $2s$ in $M' = M_{s+t}^{n+1}(c)$, $0 \leqq s \leqq n$, $t = 0$ or 1, $c \neq 0$. Then M is totally geodesic with $r = n(n+1)c$ or Einstein with $r = n^2c$, where r denotes the scalar curvature.*

Proof. Since it satisfies the condition $RR = 0$, equation (4.4) holds. Putting $l = k$ in (4.4), transvecting ε_l to the equation and summing up with respect to the index l, we get

$$\begin{aligned}2(h_2\bar{h}_{nh} + \bar{h}_{nh}{}^3)h_{ij} &- 2(h_{j\bar{n}}{}^2 h_{i\bar{h}}{}^2 + h_{i\bar{n}}{}^2 h_{j\bar{h}}{}^2)\\ &- c\{(n+1)h_{ij}\bar{h}_{hn} - \varepsilon_i\delta_{in}h_{j\bar{h}}{}^2 - \varepsilon_j\delta_{jn}h_{i\bar{h}}{}^2\} = 0.\end{aligned} \tag{4.10}$$

Furthermore, putting $h = i$ in (4.10), transvecting ε_h to the equation and summing up with respect to the index h, we get

$$c(nh_{j\bar{i}}{}^2 - h_2\varepsilon_j\delta_{ji}) = 0,$$

which implies that M is Einstein because of $c \neq 0$ and (3.17).

Next we investigate the scalar curvature r on M. Since $h_{j\bar{h}}{}^2 = (h_2/n)\varepsilon_j\delta_{jh}$, the equation (4.10) is reduced to

$$(2h_2 - nc)\{n(n+1)h_{ij}\bar{h}_{lh} - h_2\varepsilon_i\varepsilon_j(\delta_{il}\delta_{jh} + \delta_{jl}\delta_{ih})\} = 0. \tag{4.11}$$

Since M is Einstein, h_2 is a constant. So first of all let us consider the case where $2h_2 - nc \neq 0$ on M; then (4.11) gives

$$n(n+1)h_{ij}\bar{h}_{lh} - h_2\varepsilon_i\varepsilon_j(\delta_{il}\delta_{jh} + \delta_{jl}\delta_{ih}) = 0. \tag{4.12}$$

Transvecting $\varepsilon_h h_{hk}$ to (4.12) and summing up with respect to the index h and using $h_{k\bar{l}}{}^2 = (h_2/n)\varepsilon_k\delta_{kl}$, we have

$$h_2\{(n+1)\varepsilon_k\delta_{kl}h_{ij} - (\varepsilon_i\delta_{il}h_{jk} + \varepsilon_j\delta_{jl}h_{ik})\} = 0,$$

from which it follows that we have $(n+2)(n-1)h_2h_{ij} = 0$. Thus we get $h_2 = 0$ on M, from which together with (4.12) it follows that we have $h_{ij} = 0$ on M. It means that M is totally geodesic.

Secondly, let us consider the case where $2h_2 = nc$. That is, the square norm h_2 of the second fundamental form is given by $(n/2)c$. Then in this case by (3.18) we know that M is Einstein with the constant scalar curvature $r = n^2c$. This completes the proof. □

Now let us introduce the following theorem due to Nakagawa and Takagi [**8**].

Theorem A. *Let M be a complete Kaehler submanifold imbedded into CP^N with parallel second fundamental form. If M is irreducible, then M is congruent to one of the following Kaehler submanifolds imbedded into CP^N, $N = n+p$, with parallel second fundamental form:*

$$CP^n = SU(n+1)/S(U(n)\times U(1)),$$
$$Q^n = SO(n+2)/S(O(n)\times SO(2)),$$
$$SU(r+2)/S(U(r)\times U(2)), \quad r \geqq 3,$$
$$SO(10)/U(5), \quad E_6/\mathrm{Spin}\,(10)\times T, \quad E_7/E_6\times T,$$

where $U(n)$, $SU(n)$ and $SO(n)$ denote the unitary group, the special unitary group and the special orthogonal group of order n, respectively, and E_6, Spin (10) *and T denote the exceptional group, the spin group and the torus group, respectively. If M is reducible then M is congruent to $(CP^{n_1}\times CP^{n_2}, f)$ for some n_1 and n_2 with $\dim M = n_1+n_2$, where $f : CP^{n_1}\times CP^{n_2} \to CP^{n_1+n_2+n_1n_2}$ is the Kaehler imbedding. The corresponding local version is also true.*

In general, the Kaehler submanifold with parallel second fundamental form is known to be locally symmetric. So it naturally is semi-symmetric. Now let us consider only a complex hypersurface in CP^{n+1} and give a complete classification of semi-symmetric hypersurfaces. Then, as an application of Theorem 4.1 and Theorem A, we assert the following:

Theorem 4.2. *Let M be an n-dimensional complex hypersurface of an $(n+1)$-dimensional complex projective space CP^{n+1}. If it is semi-symmetric, then M is locally congruent to a complex quadric Q^n or a complex projective space CP^n.*

Proof. Let M be an n-dimensional complex hypersurface of an $(n+1)$-dimensional complex space form $M^{n+1}(c)$, $c \neq 0$. We assume that it is semi-symmetric. Then we have ${h_{i\bar{j}}}^2 = h_2\delta_{i\bar{j}}/n$ where h_2 is constant, and so M is Einstein by Theorem 4.1. Since M is a hypersurface, we see that ${h_{i\bar{j}}}^2 = \sum_r h_{ir}\bar{h}_{rj}$. Differentiating this relation covariantly, by (3.11), (3.19) and the fact that h_2 is constant, we obtain $\sum_r h_{ikr}\bar{h}_{rj} = 0$. Since $h_2 = nc/2$ if M is not totally geodesic, we see that $h_{ijk} = 0$, which means that the second fundamental form of M is parallel. Combining this result with Theorem A, we have the results of the main theorem. This completes the proof. □

Remark. By using a method quite different from ours, Ryan [**12**] also has verified that complex hypersurfaces in $P^{n+1}C$ satisfying $R \cdot R = 0$ is Einstein. Moreover, Smith [**13**] has classified the class of Kaehler Einstein hypersurfaces in CP^{n+1} and showed that they are congruent to CP^n or Q^n.

Also, in the proof of Theorem 4.1, if we compare both cases concerned with the length of the second fundamental form h_2, we can easily verify the following:

Corollary 4.3. *Let $M = M^n$ be an n-dimensional complex hypersurface of $M' = M^{n+1}(c)$, $c < 0$. If it is semi-symmetric, then M is totally geodesic.*

Corollary 4.4. *Let $M = M^n$ be an n-dimensional complex hypersurface of $M' = M^{n+1}_{0+1}(c)$, $c > 0$. If it is semi-symmetric, then M is totally geodesic.*

5. Semi-symmetric complex hypersurface in $M^{n+1}_{0+t}(0)$. In the paper [**2**], Aiyama, et al., studied an n-dimensional semi-definite complex hypersurface M of index $2s$ in $M^{n+1}_{s+t}(0)$ of index $2(s+t)$, $0 \leqq s \leqq n$, $t = 0$ or 1, and proved that (4.1) implies ${h_{i\bar{j}}}^2 = 0$ or

$$h_{ij}h_{kl} = h_{il}h_{jk} \quad \text{on } M. \tag{5.1}$$

As a direct consequence of (3.17), ${h_{i\bar{j}}}^2 = 0$ is equivalent to the fact that the Ricci tensor is flat, but in their paper the geometric meaning of the second one was not stated. In this section we are concerned with the geometric meaning of semi-symmetric complex hypersurfaces in a semi-definite complex space form $M^{n+1}_{0+t}(0)$, $t = 0$ or 1; namely, we will discuss the different case from the topics treated in the previous sections.

We prove the following theorem.

Theorem 5.1. *Let $M = M^n$ be an n-dimensional semi-symmetric complex hypersurface of $M' = M^{n+1}_{0+t}(0)$, $t = 0$ or 1. If it has no geodesic points, then, for any point x in M, there exists a totally geodesic hypersurface $M(x)$ of M through x.*

In order to prove this theorem, we establish some steps. We remark here that the equations (4.4)–(4.9) hold under the assumption of Theorem 5.1. First of all, we require the relation between the functions h_2 and h_4.

Lemma 1. *Let M be as in Theorem 5.1. Then we have*

$$h_4 = {h_2}^2 \quad \text{or} \quad -\frac{{h_2}^2}{2}. \tag{5.2}$$

Proof. In (4.7) we have

$$\bar{h}_{ij}{h_{kl}}^3 = {\bar{h}_{ij}}{}^3 h_{kl}. \tag{5.3}$$

On the other hand, by (4.6) we get

$$2{h_{ij}}^5 - (h_4 + {h_2}^2)h_{ij} = 0, \tag{5.4}$$

because of $c = 0$. From (5.3) and (5.4), we have

$$2\bar{h}_{ij}{h_{kl}}^5 = 2{\bar{h}_{ij}}{}^3{h_{kl}}^3 = (h_4 + {h_2}^2)\bar{h}_{ij}h_{kl}.$$

From the second equality it follows that we have $2h_4{h_{ij}}^3 = (h_4 + {h_2}^2)h_2h_{ij}$. Thus we obtain $2{h_4}^2 = (h_4 + {h_2}^2){h_2}^2$, which yields that the conclusion of this lemma is derived. □

Lemma 2. *Let M be as in Theorem* 5.1*. If $h_4 = -{h_2}^2/2$, then we have $h_2 = h_4 = 0$.*

Proof. Putting $i = h$ in (4.5) and summing up with respect to the index h, we have

$$h_2(2{h_{ij}}^3 + h_2h_{ij}) = 0. \tag{5.5}$$

On the other hand, from (4.4) we get

$$h_2({h_{l\bar{k}}}^2\bar{h}_{nh} + {h_{l\bar{h}}}^2\bar{h}_{nk})h_{ij} = h_2(h_{li}{h_{j\bar{n}}}^2 + h_{lj}{h_{i\bar{n}}}^2)\bar{h}_{hk}.$$

Transvecting h_{kp} to the above equation, summing up with respect to the index k, and then replacing the index p with k, we can obtain

$$h_2({h_{lk}}^3\bar{h}_{nh} + {h_{l\bar{h}}}^2{h_{k\bar{n}}}^2)h_{ij} = h_2(h_{li}{h_{j\bar{n}}}^2 + h_{lj}{h_{i\bar{n}}}^2){h_{k\bar{h}}}^2.$$

Repeating the similar discussion, we get

$$h_2({h_{lk}}^3{h_{nh}}^3 + {h_{lh}}^3{h_{nk}}^3)h_{ij} = h_2(h_{li}{h_{nj}}^3 + h_{lj}{h_{ni}}^3){h_{kh}}^3,$$

from which together with (5.5) it follows that we have

$$h_2(h_{lk}h_{nh} + h_{lh}h_{kn})h_{ij} = h_2(h_{li}h_{nj} + h_{lj}h_{ni})h_{kh}.$$

Transvecting $\bar{h}_{hk}$ to the above equation, summing up with respect to the indices h and k, we have

$$2h_2{h_{nl}}^3h_{ij} = {h_2}^2(h_{li}h_{nj} + h_{lj}h_{ni}),$$

i.e.,

$$h_2{}^2(h_{ln}h_{ij} + h_{li}h_{nj}h_{lj}h_{nj}) = 0,$$

with the help of (5.5). Putting $i = j = l = n$, we get $h_2{}^2 h_{jj}h_{jj} = 0$ and then putting $i = l \neq j = n$, we have $h_2{}^2 h_{ij}h_{ij} = 0$. It means that we have $h_2{}^2 h_{ij} = 0$. This completes the proof. □

Lemma 3. *Let M be as in Theorem* 5.1. *If $h_4 = h_2{}^2 \neq 0$, then M satisfies condition* (5.1).

Proof. By the second equation of (4.7), we have

$$h_{ij}{}^3 - h_2 h_{ij} = 0 \quad \text{and} \quad h_{i\bar{j}}{}^4 - h_2 h_{i\bar{j}}{}^2 = 0.$$

By (4.5) and the above equation, we have $2h_{l\bar{h}}{}^2 h_{ij} = h_{li}h_{j\bar{h}}{}^2 + h_{lj}h_{i\bar{h}}{}^2$, and hence $2h_{lh}h_{ij} = h_{li}h_{jh} + h_{lj}h_{ih}$. Interchanging cyclically the indices i, j and h and then summing up the two equations, we have condition (5.1). This completes the proof. □

With these preparations now complete, we are able to give a geometric meaning of condition (5.1).

Proposition 5.2. *Let M be as in Theorem* 5.1. *If $h_2 \neq 0$, then condition* (5.1) *is equivalent to $S^2 = rS/2$, where r denotes the scalar curvature of M and S the matrix of the Ricci tensor.*

Proof. Under the condition (5.1) we have $h_{ij}{}^3 = h_2 h_{ij}$ and so $h_{i\bar{j}}{}^4 = h_2 h_{i\bar{j}}{}^2$. Since the Ricci tensor $S_{i\bar{j}}$ is given by $S_{i\bar{j}} = -h_{i\bar{j}}{}^2$, the equation $h_{i\bar{j}}{}^4 = h_2 h_{i\bar{j}}{}^2$ is equivalent to $S^2 = rS/2$. On the other hand, the condition implies $h_4 = h_2{}^2$. By Lemma 3, we get (5.1). This completes the proof. □

Proof of Theorem 5.1. Since M has no geodesic points, we have $h_2 \neq 0$ on M. Then it turns out by Lemmas 1 and 2 that we have $h_4 = h_2{}^2$. Accordingly, we get by Lemma 3 and Proposition 5.2 that it satisfies condition (5.1) and so $h_{i\bar{j}}{}^4 = h_2 h_{i\bar{j}}{}^2$. We can regard $(h_{i\bar{j}}{}^2)$ as a semi-definite Hermitian matrix of order n. If $t = 0$, that is, if the ambient space is complex Euclidean, then it is positive semi-definite;

and if $t = 1$, that is, if the ambient space is complex Minkowski, then it is negative semi-definite. We denote by λ_i an eigenvalue of the Hermitian matrix $({h_{i\bar{j}}}^2)$. It is a nonnegative or nonpositive real-valued function on M. For the unitary frame $\{U_j, U_0\}$ the matrix $({h_{i\bar{j}}}^2)$ can be diagonalized as follows: ${h_{i\bar{j}}}^2 = \lambda_i \delta_{ij}$. Because of $h_4 = \sum_i {\lambda_i}^2$ and $h_2 = \sum_i \lambda_i$, we see that $0 = h_4 - {h_2}^2 = -\sum_{i \neq j} \lambda_i \lambda_j \leqq 0$. This yields that there exist at most two distinct eigenvalues, one of which is equal to 0 and of multiplicity $n-1$. Without loss of generality, we can interchange the number and we may suppose that $\lambda_1 = \lambda \neq 0$ and $\lambda_r = 0$, $r \geqq 2$. Since it satisfies that $\lambda_r = \sum_j h_{rj} \bar{h}_{rj} = 0$, we have that $h_{rj} = 0$ for any index j. Accordingly, $\lambda = h_{11} \bar{h}_{11} \neq 0$ because M has no geodesic points on M. Let D be the distribution defined by $\omega_0 = 0$ and $\omega_1 = 0$. For the unitary frame $\{U_j, U_0\}$, the connection form $\{\omega_{0j}\}$ satisfies

$$\omega_{01} = \sum_j h_{1j} \omega_j = h_{11} \omega_1,$$
$$\omega_{0r} = \sum_j h_{rj} \omega_j = 0, \quad r \geqq 2.$$

From the structure equation (3.1) of M' we have

$$\begin{aligned} d\omega_0 &= -\sum_A \omega_{0A} \wedge \omega_A = -\omega_{00} \wedge \omega_0 - \omega_{01} \wedge \omega_1 \\ &\equiv 0 \pmod{\omega_0 \text{ and } \omega_1}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} d_{\omega_{01}} &= -\sum_A \omega_{0A} \wedge \omega_{A1} + \Omega'_{01} \\ &= -\omega_{00} \wedge \omega_{01} - \omega_{01} \wedge \omega_{11} \\ &\equiv 0 \pmod{\omega_0 \text{ and } \omega_1}, \end{aligned}$$

because the ambient space is complex Euclidean. On the other hand, it satisfies

$$d\omega_{01} = dh_{11} \wedge \omega_1 + h_{11}\, d\omega_1,$$

from which together with the property $d\omega_{01} \equiv 0 \pmod{\omega_0 \text{ and } \omega_1}$ it follows that we have

$$d\omega_1 \equiv 0 \pmod{\omega_0 \text{ and } \omega_1}, \tag{5.6}$$

where the fact that h_{11} has no zero points is used. Thus the distribution D is completely integrable and, for any point x in M, the maximal integral submanifold $M(x)$ through x is of $(n-1)$-dimension. So it is the hypersurface of M. By the structure equation on M, we have

$$\begin{aligned} d\omega_1 &= -\sum_A \omega_{1,A} \wedge \omega_A \\ &= -\omega_{10} \wedge \omega_0 - \omega_{11} \wedge \omega_1 - \sum_{r \geqq 2} \omega_{1r} \wedge \omega_r \\ &\equiv -\sum_{r \geqq 2} \omega_{1r} \wedge \omega_r \ \ (\text{mod } \omega_0 \text{ and } \omega_1). \end{aligned}$$

By (5.6) we have

$$\sum_{r \geqq 2} \omega_{1r} \wedge \omega_r \equiv 0 \quad (\text{mod } \omega_0 \text{ and } \omega_1).$$

Similarly, we have

$$\sum_{r \geqq 2} \omega_{0r} \wedge \omega_r \equiv 0 \quad (\text{mod } \omega_0 \text{ and } \omega_1).$$

These yield that the connection forms ω_{1r} and ω_{0r} can be expressed as the linear combination of the 1-forms ω_0 and ω_1. Thus, restricted to the maximal integral submanifold $M(x)$, the connection forms ω_{1r} and ω_{0r} satisfy $\omega_{1r} = 0$ and $\omega_{0r} = 0$. This means that $M(x)$ is totally geodesic on M'. It turns out that so is $M(x)$ in M. This completes the proof. □

REFERENCES

1. R. Aiyama, T. Ikawa, J.-H. Kwon and H. Nakagawa, *Complex hypersurfaces in an indefinite complex space form*, Tokyo J. Math. **10** (1987), 349–361.

2. R. Aiyama, H. Nakagawa and Y.J. Suh, *Semi-Kaehlerian submanifolds of an indefinite complex space form*, Kodai Math. J. **11** (1988), 325–343.

3. M. Barros and A. Romero, *Indefinite Kaehler manifolds*, Math. Ann. **261** (1982), 55–62.

4. C.S. Houh, *On totally real bisectional curvatures*, Proc. Amer. Math. Soc. **56** (1976), 261–263.

5. T. Ikawa, H. Nakagawa and A. Romero, *Product complex submanifolds of indefinite complex space forms*, Rocky Mountain J. Math. **18** (1988), 601–615.

6. S. Kobayashi, *Hypersurfaces of complex projective space with constant scalar curvature*, J. Differential Geom. **1** (1967), 369–370.

7. S. Montiel and A. Romero, *Complex Einstein hypersurfaces of an indefinite complex space form*, Math. Proc. Camb. Philos. Soc. **94** (1983), 495–508.

8. H. Nakagawa and R. Takagi, *On locally symmetric Kaehler submanifolds in a complex space form*, J. Math. Soc. Japan **28** (1976), 638–667.

9. A Romero, *Differential geometry of complex hypersurfaces in an indefinite complex space form*, Univ. de Granada, 1986.

10. ———, *On a certain class of complex Einstein hypersurfaces in indefinite complex space forms*, Math. Z. **192** (1986), 627–635.

11. ———, *Some examples of indefinite complex Einstein hypersurfaces not locally symmetric*, Proc. Amer. Math. J. **98** (1986), 283–286.

12. P.J. Ryan, *A class of complex hypersurfaces*, Colloq. Math. **26** (1972), 175–182.

13. B. Smith, *Differential geometry of complex hypersurfaces*, Ann. of Math. **85** (1967), 246–256.

14. Z.I. Szabó, *Structure theorems on Riemannian spaces satisfying $R(X,Y)R = 0$, I, The local version*, J. Differential Geom. **17** (1982), 531–582.

15. T. Takahashi, *Complex hypersurfaces with $RS = 0$ in C^{n+1}*, Tôhoku Math. J. **25** (1973), 527–533.

16. J.A. Wolf, *Spaces of constant curvature*, McGraw-Hill, New York, 1967.

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