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UNIVALENCE AND CONVEXITY PROPERTIES FOR GAUSSIAN HYPERGEOMETRIC FUNCTIONS

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ABSTRACT. Let $\mathcal{A} = \{f : \Delta \to \mathbf{C} \,|\, f(z) = z + \sum_{n=2}^{\infty} A_n z^n\}$. We study sufficient/necessary conditions, in terms of the coefficients A_n, for a function $f \in \mathcal{A}$ to be member of well-known subclasses of the class $\mathcal{S}$ of univalent functions. Examples of these subclasses include starlike, convex, close-to-convex functions. In particular, functions of the form $z {}_2F_1(a, b; c; z)$ are considered, where ${}_2F_1(a, b; c; z)$ is the hypergeometric function.

1. Introduction. The class of normalized analytic functions

$$\mathcal{A} = \left\{ f : \Delta \to \mathbf{C} \,\middle|\, f(z) = z + \sum_{n=2}^{\infty} A_n z^n \right\} \tag{1.1}$$

has been studied extensively, together with its subclass of univalent (Schlicht) functions

$$\mathcal{S} = \{ f \in \mathcal{A} \,|\, f \text{ is one-to-one in } \Delta \}, \tag{1.2}$$

where Δ is the unit disc. Along with the classes $\mathcal{A}$ and $\mathcal{S}$ several subclasses of $\mathcal{S}$ have been widely studied. Two such subclasses are analytically characterized by

$$\mathcal{C}(\beta) = \left\{ f \in \mathcal{A} \,\middle|\, \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta,\ z \in \Delta \right\}, \quad \beta < 1, \tag{1.3}$$

and

$$\mathcal{S}^*(\beta) = \left\{ f \in \mathcal{A} \,\middle|\, \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta,\ z \in \Delta \right\}, \quad \beta < 1. \tag{1.4}$$

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The classes are called *convex of order* β and *starlike of order* β, respectively. If $\beta = 0$ these classes are called just convex and starlike and are denoted by $\mathcal{C}$ and $\mathcal{S}^*$, respectively. Given a convex function $g \in \mathcal{C}$ with $g'(z) \neq 0$, set

$$\mathcal{K}_g(\beta) = \left\{ f \in A \,\middle|\, \mathrm{Re}\left(\frac{f'(z)}{g'(z)}\right) > \beta,\ z \in \Delta \right\}, \quad \beta < 1. \tag{1.5}$$

The class $\mathcal{K}_g(0)$ is the class of functions *close-to-convex* with respect to g. Let $\mathcal{K} = \{\mathcal{K}_g(0) : g \in \mathcal{C}\}$ denote the class of all close-to-convex functions. The strict inclusions $\mathcal{C} \subsetneq \mathcal{S}^* \subsetneq \mathcal{K} \subsetneq \mathcal{S}$ hold. We prove in this paper univalence criteria for the Gaussian hypergeometric series

$$F(a,b;c;z) := {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)n!}\, z^n, \tag{1.6}$$

where

$$(a,0) = 1, \quad (a,n+1) = (a+n)(a,n), \quad n = 0,1,2,...,$$

and where we require that the denominators $\neq 0$ that is $c \neq 0,-1,-2,-3,\ldots$ and we give various univalence criteria for ${}_2F_1$ in terms of inequalities between the three parameters a, b and c. The hypergeometric function has found many applications and generalizations, [**19**]. The present work is in part motivated by our earlier work dealing with the case of the Gaussian hypergeometric function and its asymptotic behavior close to the singularity $r = 1$ for real values of r, see [**2, 16**].

Ozaki [**14**] (see also [**1, 10**]) proved the following theorems just by looking at certain monotonicity conditions on the coefficients A_n when these coefficients are real and nonnegative. Monotonicity of the sequence $\{A_n\}$, $A_n = (a,n)(b,n)/((c,n)n!)$ was used in [**2, 16**] to derive explicit bounds for the asymptotic behavior of $F(a,b;c;x)$ at $x = 1$.

Theorem 1.1 [**14**]. *Suppose that*

$$1 \geq 2A_2 \geq \cdots \geq nA_n \geq \cdots \geq 0 \tag{1.7}$$

or

$$1 \leq 2A_2 \leq \cdots \leq nA_n \leq \cdots \leq 2 \tag{1.8}$$

and f is defined by (1.1). *Then f is close-to-convex with respect to* $-\log(1-z)$.

Theorem 1.2 [**14**]. *Suppose that f is an odd function* (*i.e., A_{2n} in* (1.1) *is zero for each $n \geq 1$*) *such that*

$$1 \geq 3A_3 \geq \cdots \geq (2n+1)A_{2n+1} \geq \cdots \geq 0 \tag{1.9}$$

or

$$1 \leq 3A_3 \leq \cdots \leq (2n+1)A_{2n+1} \leq \cdots \leq 2. \tag{1.10}$$

Then $f \in \mathcal{S}$. Indeed, f is close-to-convex with respect to the convex function $(1/2)\log((1+z)/(1-z))$.

Theorem 1.3 [**6**]. *If $A_n \geq 0$, $\{nA_n\}$ and $\{nA_n - (n+1)A_{n+1}\}$ both are nonincreasing, then the function f defined by* (1.1) *is in $\mathcal{S}^*$.*

It is interesting to point out that a convex function in $\mathcal{A}$ need not be close-to-convex with respect to the identity function $g(z) = z$. The convex function $z/(1-z)$ clearly demonstrates this. More explicitly, even if $f \in \mathcal{A}$ is in $\mathcal{C}$ then it is not true [**8**] that $\operatorname{Re}(f'(z)) > 0$ in the whole of Δ. It is also known that if $f \in \mathcal{A}$ satisfies the condition $\operatorname{Re}(f'(z)) > 0$ for $z \in \Delta$ then the function f need not be starlike in the whole of the unit disc Δ. Indeed, not even the condition

$$|f'(z) - 1| < \lambda, \tag{1.11}$$

will imply the starlikeness if $\lambda > 2/\sqrt{5}$. Fournier [**7**] has shown that if $\lambda \leq 2/\sqrt{5}$, then (1.11) implies starlikeness, and the bound $\lambda = 2/\sqrt{5}$ is sharp.

Theorem 1.4 [**12**]. *Let $\Omega \subset \mathbf{C}$. Suppose that $\psi : \mathbf{C}^2 \times \Delta \longrightarrow \mathbf{C}$ satisfies the condition*

$$\psi(ir_2, s_1; z) \notin \Omega$$

when r_2 is real and $s_1 \leq -(1+r_2^2)/2$. If p is analytic in Δ, with $p(0) = 1$ and $\psi(p(z), zp'(z); z) \in \Omega$ for $z \in \Delta$, then $\operatorname{Re}(p(z)) > 0$ in Δ.

The above theorem is a special case of Theorem 1 due to Miller and Mocanu in [**12**].

Merkes and Scott [**11**] made use of continued fractions and showed that if $0 < a < c$, $0 \leq b \leq 2$ and $b \leq a$ then $z_2F_1(a,b;c;z)$ is starlike in Δ. Ruscheweyh and Singh (see [**19**] and [**21**, Theorems 1 and 2]) employed a refined version on continued fraction and convolution theory and obtained sufficient conditions for $z_2F_1(a,b;c;z)$ to be in $\mathcal{S}^*(\beta)$, $\beta < 1$, for various choices of the parameters a,b,c. In this paper we employ two different techniques: the first one makes use of the well-known criteria of Ozaki and Fejér dealing with the sufficiency for close-to-convexity and starlikeness, respectively. The second method is a slightly modified version of the idea of Miller and Mocanu [**13**] and it uses Theorem 1.4. Similar problems for confluent hypergeometric functions have been considered by the authors in detail in [**17**]. Lewis [**9**], using another approach, based on the so-called Julia-Jack-Clunie lemma, proved the following result:

Theorem 1.5 [**9**, Lemma 1]. *For $\alpha \geq \beta > -\infty$, the function $zF(1+\alpha+\beta, 1+\beta; 1+\alpha; z)$ is in $\mathcal{S}^*((1-\alpha-\beta)/2)$.*

We recall that the functions in $\mathcal{S}^*((1-\alpha-\beta)/2)$ need not be univalent if $\alpha+\beta > 1$. Because of normalization we require that $\alpha+\beta > -1$. These observations show that the starlikeness of $zF(1+\alpha+\beta, 1+\beta; 1+\alpha; z)$ in Theorem 1.5 holds only for the values of α, β with $\alpha \geq \beta > -\infty$ and $-1 < \alpha+\beta \leq 1$. In our notation this is equivalent to saying that the function $zF(a,b;c;z)$ is in $\mathcal{S}^*$ for $a,b,c > 0$ such that $a \in (0,2]$ and $1 < b+c \leq 3$, according to Theorem 1.5. On the other hand, in Section 3 of the present paper, for a given $a,b > 0$ we determine a condition on c so that $zF(a,b;c;z) \in \mathcal{S}^*$. (In fact, we conclude much more than this, see Theorems 3.1, and 3.2.) A similar situation arises when considering the results of Ruscheweyh and Singh [**21**].

At this place, we remark that the univalency question for $zF(a,b;c;z)$ and $F(a,b;c;z)$ are different, as the examples $1+z$ and $z+z^2$ show for instance. For a given $a,b > -1$, a condition on c has been established in [**15**], in particular, for the univalence of the function $F(a,b;c;z)$. In [**5**], Carlson and Shaffer introduced a convolution (Hadamard product) operator for the case $a = 1$ to study the properties of the hypergeometric function $zF(1,b;c;z)$ under this operator for a different purpose.

This paper is organized as follows. In Section 2, we investigate the geometric meaning of hypergeometric functions, i.e., we establish constraints on the coefficients a,b,c to obtain geometric properties such as close-to-convexity for $zF(a,b;c;z)$. In Section 3, we determine conditions on a,b,c so that the function $zF(a,b;c;z)$ is close-to-convex with respect to $z/(1-z)$ and is also starlike in Δ. In Section 4, we find conditions so that an odd hypergeometric function $zF(a,b;c;z^2)$ is close-to-convex with respect to the convex function $(1/2)\log((1+z)/(1-z))$. In Section 5, we use the method of differential subordination to obtain the order of convexity for $zF(a,b;c;z)$ and the order of starlikeness for the odd function $zF(a,b;c;z^2)$, respectively.

In the proof of Theorems 2.1 and 4.1, we use the following well-known fact that follows easily from the Stirling formula [**3**, p. 57, Equation 5]:

$$\lim_{n\to\infty}\frac{(a,n)(b,n)}{(c,n)(1,n)} = \begin{cases} \dfrac{\Gamma(c)}{\Gamma(a)\Gamma(b)}) & \text{if } c+1=a+b \\ 0 & \text{if } c+1>a+b \\ \infty & \text{if } c+1<a+b. \end{cases} \tag{1.12}$$

We shall give several explicit examples of the parameter combinations (a,b,c) that lead to simplified expressions for $zF(a,b;c;z)$ and $zF(a,b;c;z^2)$ illustrating our results, see Section 6. An extensive list of special cases of $F(a,b;c;z)$ is given in [**18**]. Note that the behavior of $zF(a,b;c;z)$ depends strongly on the mutual size of the coefficients. In fact, the hypergeometric series defined by (1.6) converges for $|z|<1$, converges absolutely for $|z|=1$ whenever $\operatorname{Re}(c-a-b)>0$. These simple observations show that for $a,b,c\in\mathbf{C}$ with $\operatorname{Re}c>\operatorname{Re}(a+b)$, the function $F(a,b;c;z)$ is bounded in $|z|<1$. In particular, $zF(a,b;c;z)$ is bounded in $|z|<1$ for $a,b,c>0$ and $c>a+b$, as the example $zF(1,1;3;z)=(2/z)[z+(1-z)\log(1-z)]$ shows. For $c\le a+b$, the function $zF(a,b;c;z)$ is unbounded in $|z|<1$, as the functions $zF(1,1;2;z)=-\log(1-z)$, $zF(1,1/2;3/2;z^2)=(1/2)\log((1-z)/(1+z))$, $zF(1/2,3;1;z)=(z/8)[8-8z+3z^2](1-z)^{-5/2}$ and $zF(1/2,3;2;z)=(z/4)[4-3z](1-z)^{-3/2}$ show. The behavior of the hypergeometric function $F(a,b;c;z)$ at $z=x=1$ in the three cases $c=a+b$, $c>a+b$ and $c<a+b$ has been studied recently in [**2, 16**].

Finally, we remark that Theorems 1.1 and 1.3 suggest a conjecture for starlike mappings, see Conjecture 6.1.

2. Close-to-convexity criterion for hypergeometric functions. In this section we determine conditions on $a, b, c > 0$ for the Gaussian hypergeometric function $zF(a, b; c; z)$ to be close-to-convex.

Theorem 2.1. *If $a, b > 0$, $T_1(a, b) = \max\{a+b, a+b+(a\,b-1)/2, 2a\,b\}$ and c satisfies either*

$$c \geq T_1(a, b) \tag{2.1}$$

or $c = a + b$ with

$$a\,b \geq 1, \quad a + b \leq 2a\,b \quad \text{and} \quad \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \leq 2 \tag{2.2}$$

then $zF(a, b; c; z)$ is close-to-convex with respect to $-\log(1 - z)$.

Proof. Let $f(z) = zF(a, b; c; z)$. Then $f \in \mathcal{A}$ and is of the form $f(z) = z + \sum_{n=2}^{\infty} A_n z^n$, where

$$A_n = \frac{(a, n-1)(b, n-1)}{(c, n-1)(1, n-1)}, \quad \text{for } n \geq 2$$

and $A_1 = 1$. From the definition of ascending factorial notation we observe that

$$A_{n+1} = \frac{(a+n-1)(b+n-1)}{(c+n-1)n}\, A_n. \tag{2.3}$$

To prove the theorem, we first use (2.1) and then apply Theorem 1.1. Therefore, in this case we need to show that $\{nA_n\}$ is a decreasing sequence of positive real numbers. By hypothesis and (2.3) we note that A_n is positive for each $n \geq 1$. Next, we use (2.3) and obtain

$$nA_n - (n+1)\, A_{n+1} = A_n \left[n - \frac{(n+1)(a+n-1)(b+n-1)}{(c+n-1)n} \right]$$

so that

$$nA_n - (n+1)\,A_{n+1} = \frac{A_n}{n(c+n-1)}\,X(n),$$

where

$$X(n) = n^2(c-a-b) + n(1-a\,b) - (a-1)(b-1). \tag{2.4}$$

Therefore, for the proof of our first part, it is sufficient to check that $X(n)$ is nonnegative. We first note that the condition (2.1) implies that $c \geq a+b$ and so the coefficient of n^2 in the above expression for $X(n)$ is nonnegative. Thus for all $n \geq 1$, we can write

$$\begin{aligned} X(n) &\geq (2n-1)(c-a-b) + n(1-a\,b) - (a-1)(b-1) \\ &= n[2(c-a-b)+1-a\,b] - c + 2(a+b) - 1 - a\,b = Y(n), \end{aligned}$$

say. By (2.1), we have $c \geq a+b+(a\,b-1)/2$ so that the coefficient of n in the expression for $Y(n)$ is nonnegative and so we obtain that

$$X(n) \geq Y(n) \geq Y(1) = c - 2a\,b.$$

Since $c \geq 2a\,b$, by (2.1), we get that $Y(1) \geq 0$ which yields the desired conclusion. This argument proves that if $c \geq T_1(a,b)$ then the function $zF(a,b;c;z)$ is close-to-convex with respect to $-\log(1-z)$.

For the proof of the second part, we need to show that $\{nA_n\}$ is a nondecreasing sequence and has a limit less than or equal to 2. From (2.2), we note that $c = a+b$ and $a\,b \geq 1$. So in this case

$$X(n) = Y(n) \leq Y(1) = a+b-2a\,b \leq 0,$$

by (2.2). Using this inequality, we obtain that $X(n)$ is nonpositive for each $n \geq 1$. In other words, the sequence $\{nA_n\}$ is increasing. Thus to complete the proof, according to Theorem 1.1, it suffices to show that the value of the limit is less than or equal to 2. Therefore, for the proof of the second part, we write $c = a+b$ and

$$nA_n = \frac{(n-1)(a,n-1)(b,n-1)}{(c,n-1)(1,n-1)} + \frac{(a,n-1)(b,n-1)}{(c,n-1)(1,n-1)}$$

which may be equivalently written in the form

$$nA_n = \frac{a\,b}{c}\,\frac{(a+1,n-2)(b+1,n-2)}{(c+1,n-2)(1,n-2)} + \frac{(a,n-1)(b,n-1)}{(c,n-1)(1,n-1)}$$

so that

$$\lim_{n\to\infty} nA_n = \frac{a\,b}{c} \lim_{n\to\infty} \frac{(a+1,n-2)(b+1,n-2)}{(c+1,n-2)(1,n-2)} + \lim_{n\to\infty} \frac{(a,n-1)(b,n-1)}{(c,n-1)(1,n-1)}.$$

From the property of gamma function, namely the equation (1.12), we have

$$\lim_{n\to\infty} nA_n = \frac{a\,b}{c} \frac{\Gamma(c+1)}{\Gamma(a+1)\Gamma(b+1)} + 0$$

and since $c = a + b$, the above gives

$$\lim_{n\to\infty} nA_n = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}.$$

The conclusion now follows from the above relation and (2.2). □

Corollary 2.1. *Let $a, b > -1$ and $a\,b \neq 0$. If c satisfies either $c \geq -1 + T_1(a+1, b+1)$, where $T_1(a, b)$ is defined in Theorem 2.1, or*

$$c = a + b + 1 \quad \textit{with} \quad a + b \geq \max\{-a\,b, 2a\,b\}$$
$$\textit{and} \quad \frac{\Gamma(a+b+2)}{\Gamma(a+1)\Gamma(b+1)} \leq 2$$

then $zF'(a, b; c; z)$ is univalent in Δ.

Proof. Differentiating (1.6) with respect to z and then using the definition of the ascending factorial notation $(a, n+1) = (a+n)(a, n)$ we obtain the well-known identity for the first derivative of the hypergeometric function, namely (see [**22**])

$$a\,b\,zF(a+1, b+1; c+1; z) = czF'(a, b; c; z).$$

From the above formula and Theorem 2.1 we deduce that the function $(c/a\,b)zF'(a, b; c; z)$ is close-to-convex with respect to $-\log(1 - z)$ and hence the function $zF'(a, b; c; z)$ is univalent in Δ. □

3. Starlikeness criterion for hypergeometric functions. In this section we determine conditions on $a, b, c > 0$ for the Gaussian hypergeometric function $zF(a, b; c; z)$ to be not only close-to-convex with respect to $-\log(1-z)$ but also starlike in Δ. As remarked in the introduction, there exist conditions in the literature to conclude that the function $zF(a, b; c; z)$ is starlike in Δ but not both. An interesting special case, the incomplete beta function will be considered at the end of this section.

For convenience, we denote by $\mathcal{KS}^*$ the family of functions in $\mathcal{A}$ which are close-to-convex with respect to $-\log(1-z)$ and also starlike in Δ.

Theorem 3.1. *Let $a, b > 0$,*

$$T(a, b) = \max\{a + b, a + b + (3a\,b - 1)/2, (2 + \sqrt{10}/2)a\,b\}$$

and $c \geq T(a, b)$. Then $zF(a, b; c; z)$ is in $\mathcal{KS}^*$.

Proof. We use the method of proof of Theorem 2.1. Define

$$f(z) = zF(a, b; c; z) = z + \sum_{n=2}^{\infty} A_n z^n,$$

where A_n is as in (2.3). Further, let $T_1(a, b) = \max\{a + b, a + b + (a\,b - 1)/2, 2a\,b\}$. Then the hypotheses imply that $T(a, b) \geq T_1(a, b)$. Therefore, from Theorem 2.1, we have that the sequence nA_n is nonincreasing. Now we use Theorem 1.3 to prove that f is starlike. For this, we need to show that the sequence $\{nA_n - (n+1)A_{n+1}\}$ is also nonincreasing. For convenience, we define

$$B_n = nA_n - (n+1)A_{n+1}.$$

Using the definition of A_n we find that

$$B_n - B_{n+1} = A_n\left[n + (n+2)\,\frac{A_{n+2}}{A_n} - 2(n+1)\,\frac{A_{n+1}}{A_n}\right].$$

After some elementary computation we obtain that

$$B_n - B_{n+1} = \frac{A_n}{n(n+1)(c+n-1)(c+n)}\,U(n),$$

where

$$U(n) = D_3 n^3 + D_2 n^2 + D_1 n + D_0. \tag{3.1}$$

Using simple calculation one can simplify the coefficients of $U(n)$ and obtain

$$\begin{aligned} D_3 &= (c-a-b)(c+1-a-b) \\ D_2 &= (c-a-b+2(1-a\,b))(c+1-a-b) \\ D_1 &= -2c(2a\,b-a-b)+2a\,b(a+b-2) \\ &\quad +(a-1)(b-1)(2(a+b)+a\,b-2) \\ D_0 &= -2(c-a\,b)(a-1)(b-1). \end{aligned}$$

Our aim is to check that $U(n)$ is nonnegative for all $n \geq 1$. First we observe that

$$n^3 \geq 3n^2 - 3n + 1, \quad \text{for all } n \geq 1.$$

By hypothesis, $c \geq T_1(a,b) \geq a+b$ and so we have $D_3 \geq 0$. Thus for all $n \geq 1$, we can write

$$\begin{aligned} U(n) &\geq (3n^2-3n+1)D_3 + D_2 n^2 + D_1 n + D_0 \\ &= (3D_3+D_2)\,n^2 + (D_1-3D_3)\,n + D_0 + D_3 \\ &= V(n), \end{aligned}$$

say. Now we compute

$$3D_3 + D_2 = 2(c-a-b+1)[2(c-a-b)-(a\,b-1)] = \phi_1(c,a,b),$$

say, and since $c \geq T(a,b)$, we have $\phi_1(c,a,b) \geq 0$. Therefore, we obtain that

$$\begin{aligned} U(n) &\geq V(n) \\ &\geq (2n-1)(3D_3+D_2) + (D_1-3D_3)\,n + D_0 + D_3 \\ &= (3D_3+2D_2+D_1)\,n + (-2D_3-D_2+D_0) \\ &= W(n), \end{aligned}$$

say. We have

$$\begin{aligned} &3D_3 + 2D_2 + D_1 \\ &\quad = (c+1-a-b)(5c-3(a+b)+4-8a\,b)) + (a-1)(b-1)(a\,b-2). \end{aligned}$$

For convenience, we denote the right-hand side of the above expression by $\phi_2(c,a,b)$. Next we show that $\phi_2 \equiv \phi_2(c,a,b) \geq 0$. After some arithmetic calculation we find that

$$\phi_2 = [5c^2 + c(9 - 8(a+b) - 8a\,b)] + M_1(a,b)$$

where

$$M_1(a,b) = 2 + 3(a+b)^2 + a^2b^2 + 7a\,b(a+b) - 5(a+b) - 9a\,b.$$

To show $\phi_2 \geq 0$, we rewrite the square bracketed term in the above expression as

$$\left[(c+2a\,b)^2+5c+4\left\{c-\left(a+b+\frac{3a\,b-1}{2}\right)\right\}^2-4a^2b^2-4\left\{a+b+\frac{3a\,b-1}{2}\right\}^2\right]$$

and, for convenience, we denote the above expression by ϕ_3. We first give a lower bound for ϕ_3. Since $c \geq a+b+(3a\,b-1)/2$, we have

$$(c+2a\,b)^2 \geq (a+b+(7a\,b-1)/2)^2$$

and so

$$\phi_3 \geq (a+b+(7a\,b-1)/2)^2 + 5c - 4a^2b^2 - 4(a+b+(3a\,b-1)/2)^2.$$

From this inequality, we deduce, in particular, that

$$\phi_2 \geq (a+b+(7a\,b-1)/2)^2 + 5c - 4a^2b^2 - 4(a+b+(3a\,b-1)/2)^2 + M_1(a,b).$$

After some work we see that this inequality is equivalent to

$$\begin{aligned}\phi_2 \geq \frac{1}{4}\,[a^2b^2+1] &+ 3(c-2a\,b) + [2(c-a+b)+1-3a\,b]\\ &+ \frac{5a\,b}{2} + 2a\,b(a+b).\end{aligned}$$

In particular, by hypothesis, we get that $\phi_2 \geq 0$ and so $3D_3 + 2D_2 + D_1 \geq 0$. This observation shows that

$$U(n) \geq V(n) \geq W(n) \geq W(1) = D_0 + D_1 + D_2 + D_3. \tag{3.2}$$

Finally, to complete the proof we need to show that $D_0+D_1+D_2+D_3>0$. If we let $\phi_4(a,b,c)=D_0+D_1+D_2+D_3$, then we find that

$$\phi_4=[2c^2+2c(1-4a\,b)]-5a\,b+3a\,b(a+b)+3a^2b^2$$

which can be rewritten in the form

$$\phi_4=2\left[c-\left(2+\frac{\sqrt{10}}{2}\right)a\,b+\frac{\sqrt{10}-1}{2\sqrt{10}}\right]\left[c-\left(2-\frac{\sqrt{10}}{2}\right)a\,b+\frac{\sqrt{10}+1}{2\sqrt{10}}\right] \\ -\frac{9}{20}+3a\,b(a+b).$$

Since $c\geq(2+\sqrt{10}/2)a\,b$, we see that

$$\phi_4\geq 2\left\{\frac{\sqrt{10}-1}{2\sqrt{10}}\right\}\left\{\frac{\sqrt{10}+1}{2\sqrt{10}}\right\}-\frac{9}{20}+3a\,b(a+b)=3a\,b(a+b)>0$$

which shows that $D_0+D_1+D_2+D_3>0$. Thus, by (3.2), $U(n)$ is positive for all $n\geq 1$. Therefore the sequence $\{B_n\}$ and hence the sequence $\{nA_n-(n+1)A_{n+1}\}$ is nonincreasing, and by Theorem 1.3 we deduce that f is starlike. We also note that all the requirements of Theorem 1.1 have also been verified and so f is also close-to-convex with respect to $-\log(1-z)$. □

It is clear from the proof that the above theorem could be improved slightly in the following form:

Theorem 3.2. *Let $a,b>0$ and $\alpha=\alpha(a,b)$ be a least positive number such that*

$$[2\alpha^2+2\alpha(1-4a\,b)]-5a\,b+3a\,b(a+b)+3a^2b^2\geq 0. \tag{3.3}$$

If $T_2(a,b)=\max\{a+b,a+b+(3a\,b-1)/2,\alpha\}$ and $c\geq T_2(a,b)$, then $zF(a,b;c;z)$ is in $\mathcal{KS}^*$.

From Theorem 3.2, we can obtain several simple examples to point out the usefulness of the starlikeness criteria. We note that it is often difficult to check the analytic condition $\operatorname{Re}(zf'(z)/f(z))>0$ when f is the hypergeometric function. Because of its independent interest, as considered by Carlson and Shaffer [**5**], we state the result for the special case $a = 1$ of Theorem 3.2 separately in the following form:

Corollary 3.1. *If b and c are related by any one of the following:*

(i) $b \in (0, 1/3]$ *and* $c \geq b + 1$

(ii) $b \in [1/3, \infty)$ *and* $c \geq (5b + 1)/2$

then the incomplete beta function $\phi(b, c; z)$ is in $\mathcal{KS}^*$.

Proof. Choose $a = 1$ in Theorem 3.2. Then equation (3.3) reduces to

$$(\alpha - 2b)^2 + \alpha^2 + 2\alpha + 2b^2 - 2b \geq 0$$

which is clearly true whenever $\alpha \geq 1 + b$ or $\alpha \geq (5b + 1)/2$. Therefore, the condition $c \geq T_2(a, b)$ of Theorem 3.2 for $a = 1$ is equivalent to $c \geq \max\{1 + b, (5b + 1)/2\}$. This observation shows that the function $zF(1, b; c; z)$ is in $\mathcal{KS}^*$ and the proof is completed. $\square$

The above corollary is to indicate that Theorem 3.2 would be more suitable whenever we deal with special cases such as in Corollary 3.1.

The case $a = 1$ of [**21**, Theorem 2] shows that the incomplete beta function $\phi(b, c; z)$ is in $\mathcal{S}^*$ whenever $1 \leq b \leq c$. Thus, Theorem 2 in [**21**] applies to $\phi(b, c; z)$ only when $b \geq 1$ whereas the above example clearly demonstrates that our approach applies also to the case $b \in (0, 1)$ even though in this special case range for c becomes slightly smaller. On the other hand, the conclusion in our case is much stronger because, from Theorem 2 of Ruscheweyh and Singh [**21**], one cannot claim that the corresponding function is close-to-convex with respect to the convex function $-\log(1 - z)$.

If we restrict only to close-to-convexity of the incomplete beta function, then we have the following example which follows from Theorem 2.1:

Example 3.1. The incomplete beta function defined by $\phi(b, c; z) = zF(1, b; c; z)$ is close-to-convex with respect to $-\log(1 - z)$ if b and c are related by any one of the following:

(i) $b \in (0, 1]$ and $c \geq b + 1$

(ii) $b \in [1,2]$ and $c = b+1$

(iii) $b \in [1,\infty)$ and $c \geq 2b$.

Remark 3.1. There is yet another simple criteria for functions in $\mathcal{S}^*$ due to Ruscheweyh [**20**]: If $A_n \geq 0$, $\{nA_n\}$ is nonincreasing and $(2n+1)A_{2n+1} \leq (2n-1)A_{2n}$ for $n \geq 1$, then the function f defined by (1.1) is starlike univalent in Δ. However, our calculation shows that the result of Ruscheweyh [**20**] does not yield better information, at least not for the case where $f(z) = zF(a,b;c;z)$.

4. Close-to-convexity criterion for odd hypergeometric functions. In this section we determine conditions on $a, b, c > 0$ for the Gaussian hypergeometric function $zF(a,b;c;z^2)$ to be close-to-convex with respect to the convex function $(1/2)\log((1+z)/(1-z))$. We observe that the results of this type seem completely new in the case of the odd hypergeometric function $zF(a,b;c;z^2)$.

Theorem 4.1. *If $a,b > 0$, $N(a,b) = \max\{a+b, a+b+(2a\,b-1)/3, 3a\,b\}$ and c satisfies either $c \geq N(a,b)$ or $c = a+b$ with*

$$a\,b \geq 1/2, \quad a+b \leq 3a\,b \quad \text{and} \quad \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \leq 1$$

then $zF(a,b;c;z^2)$ is close-to-convex with respect to $(1/2)\log((1+z)/(1-z))$.

Proof. As in the proof of Theorem 2.1, we let $A_1 = 1$ and

$$f(z) = zF(a,b;c;z^2) = z + \sum_{n=2}^{\infty} A_{2n-1} z^{2n-1},$$

where

$$A_{2n-1} = \frac{(a,n-1)(b,n-1)}{(c,n-1)(1,n-1)} \quad \text{for } n \geq 2.$$

Using this, we estimate

$$(2n-1)A_{2n-1} - (2n+1)A_{2n+1} = \frac{A_{2n-1}}{n(c+n-1)}\,Y(n)$$

where

$$Y(n) = 2n^2(c-a-b) + n(1+a+b-c-2a\,b) - (a-1)(b-1).$$

By hypothesis, $A_{2n-1} > 0$ for all $n \geq 1$ and the coefficient of n^2 in $Y(n)$ is nonnegative.

First we assume that $c \geq N(a,b)$. Thus, we note that the sequence $\{A_{2n-1}\}$ is nonincreasing, if the inequality

$$2(2n-1)(c-a-b) + n(1+a+b-c-2a\,b) - (a-1)(b-1) \geq 0,$$

or equivalently,

$$n[3(c-a-b) - (2a\,b-1)] - 2c + 3(a+b) - 1 - a\,b \geq 0$$

holds for all $n \geq 1$. From the condition on c, we, in particular, have $c \geq a+b+(2a\,b-1)/3$. Since $3(c-a-b)-(2a\,b-1) \geq 0$, the last inequality continues to hold for all $n \geq 1$ if this holds for $n = 1$. Therefore putting $n = 1$, we infer that $c \geq 3a\,b$ which is true by hypothesis and therefore if $c \geq N(a,b)$ then the sequence $\{A_{2n-1}\}$ is nonincreasing for all $n \geq 1$, and by Theorem 1.2 we deduce that the function $zF(a,b;c,z^2)$ is close-to-convex with respect to the convex function $(1/2)\log((1+z)/(1-z))$. Hence, the proof of the first part follows.

For the proof of the second part, we write $c = a+b$ and consider the case $a\,b \geq 1/2$, $a+b \leq 3a\,b$, and $\Gamma(a+b) \leq \Gamma(a)\Gamma(b)$. Therefore, in this case the coefficient of n^2 in the expression for $Y(n)$ is zero and, since $a\,b \geq 1/2$, the coefficient of n in $Y(n)$ is nonpositive. This observation gives that $Y(n) \leq Y(1) = a+b-3a\,b$ and since $a+b \leq 3a\,b$, we obtain that $Y(n) \leq 0$ for all $n \geq 1$. These arguments show that

$$(2n-1)A_{2n-1} - (2n+1)A_{2n+1} \leq 0$$

for all $n \geq 1$ and hence the sequence $\{A_{2n-1}\}$ is increasing for all $n \geq 1$. Using the definition of A_{2n-1}, we have

$$\begin{aligned}(2n-1)A_{2n-1} &= \frac{(2(n-1)+1)(a,n-1)(b,n-1)}{(c,n-1)(1,n-1)} \\ &= \frac{2a\,b}{c}\,\frac{(a+1,n-2)(b+1,n-2)}{(c+1,n-2)(1,n-2)} + \frac{(a,n-1)(b,n-1)}{(c,n-1)(1,n-1)}\end{aligned}$$

so that

$$\lim_{n\to\infty}(2n-1)A_{2n-1} = \frac{2a\,b}{c}\lim_{n\to\infty}\frac{(a+1,n-2)(b+1,n-2)}{(c+1,n-2)(1,n-2)} + \lim_{n\to\infty}\frac{(a,n-1)(b,n-1)}{(c,n-1)(1,n-1)}.$$

Using (1.12), we deduce that

$$\lim_{n\to\infty}(2n-1)A_{2n-1} = \frac{2a\,b}{c}\frac{\Gamma(c+1)}{\Gamma(a+1)\Gamma(b+1)} = 2\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}.$$

Since $\Gamma(a+b)\le\Gamma(a)\Gamma(b)$, the value of the limit, namely $\lim_{n\to\infty}(2n-1)A_{2n-1}$, is less than or equal to 2. The result now follows from the above facts and part 2 of Theorem 1.2. $\square$

Using the method of proof of Corollary 2.1 we have

Corollary 4.1. *Let $a, b > -1$ and $a\,b \neq 0$. If c satisfies either $c \ge -1 + N(a+1, b+1)$, where $N(a,b)$ is defined in Theorem* 4.1, *or $c = a+b+1$ with*

$$2(a+b+a\,b)+1 \ge \max\{0, -a\,b\} \quad \textit{and} \quad \frac{\Gamma(a+b+2)}{\Gamma(a+1)\Gamma(b+1)} \le 1$$

then $zF'(a,b;c;z^2)$ is univalent in Δ.

5. Order of convexity for hypergeometric functions. In our next theorem, we obtain the order of convexity of the hypergeometric function $F(a,b;c;z)$ for certain values of the parameters a, b and c. The results of this section extend the results of Miller and Mocanu [**13**].

Theorem 5.1. *Let a, b, c be real, $(a+1)(b+1)\beta \le 0$ and such that $F'(a,b;c;z)\neq 0$ in Δ. Let $c \ge M_\beta(a,b)$, where*

$$M_\beta(a,b) = \max\left\{2(1-\beta)+|a+b+2\beta|,\ 1-a\,b-\frac{(a+1)(b+1)\beta}{1-\beta}\right\}. \tag{5.1}$$

Then $F(a,b;c;z)$ is convex of order β.

Proof. If we put $1+(zF''(z)/F'(z)) = \beta+(1-\beta)p(z)$ then p is regular in Δ, $p(0) = 1$. Therefore to prove the theorem we need to show that $\mathrm{Re}\,(p(z)) > 0$ in Δ. Since the hypergeometric function $w = F(a, b; c; z)$ satisfies the differential equation

$$z(1-z)w''(z) + [c-(a+b+1)z]w'(z) - a\,b\,w(z) = 0,$$

it follows that p satisfies the first order differential equation

$$\begin{aligned}(1-z)\,zp'(z) + (1-\beta)(1-z)\,p^2(z) + \{c-2(1-\beta)-(a+b+2\beta)\,z\}\,p(z)\\ +1-c-\beta-\left\{a\,b-\beta+(a+1)(b+1)\,\frac{\beta}{1-\beta}\right\}z = 0.\end{aligned}$$

From the condition on c we note that $c-2(1-\beta)-(a+b+2\beta)z \neq 0$ in Δ. We adopt the method used in [**13**]. We can rewrite the above differential equation in the form $\psi(p(z), zp'(z); z) = 0$, where

$$\psi(r, s; z) = J(z)\,[s+(1-\beta)r^2] + r + \frac{J(z)-H(z)}{2}$$

with

$$J(z) = \frac{1-z}{c-2(1-\beta)-(a+b+2\beta)z}$$

and

$$H(z) = \frac{2c-1+2\beta-[1-2a\,b+2\beta-2(a+1)(b+1)\beta/(1-\beta)]z}{c-2(1-\beta)-(a+b+2\beta)z}.$$

According to Theorem 1.4, it suffices to show that $\psi(ir_2, s_1; z) \neq 0$ for $z \in \Delta$ and real r_2 and $s_1 \leq -(1+r_2^2)/2$. We actually show a stronger inequality, namely that $\mathrm{Re}\,\psi(ir_2, s_1; z) < 0$. Recall that the function

$$W(z) = \frac{1+Az}{1+Bz}, \quad -1 \leq A, B \leq 1,$$

maps the unit disc Δ conformally onto the disc

$$\left|\omega - \frac{1-AB}{1-B^2}\right| < \frac{|B-A|}{1-B^2} \quad \text{if } B \neq \pm 1, \tag{5.2}$$

and onto the half-plane

$$\mathrm{Re}\,(\omega) > \begin{cases} \dfrac{1+A}{2} & \text{if } B = 1 \\ \dfrac{1-A}{2} & \text{if } B = -1. \end{cases}$$

Further from (5.2) we also note that for $B \neq \pm 1$,

$$\mathrm{Re}\,(\omega) > \begin{cases} \dfrac{1+A}{1+B} & \text{if } B > A \\ \dfrac{1-A}{1-B} & \text{if } B < A. \end{cases}$$

The condition on c, in particular, gives $c > 2 + a + b$. Choose

$$A = -1 \quad \text{and} \quad B = -\frac{a+b+2\beta}{c-2(1-\beta)}.$$

Then $A < B$ and so the function $J(z)$ satisfies the inequality $\mathrm{Re}\, J(z) > 0$ for $z \in \Delta$. To get a similar condition for $H(z)$, we choose

$$\begin{aligned} A &= -\left(1 - 2a\,b + 2\beta - \frac{2(a+1)(b+1)\beta}{1-\beta}\right)\frac{1}{2c-1+2\beta}, \\ B &= -\frac{a+b+2\beta}{c-2(1-\beta)} \end{aligned}$$

so that $H(z)$ takes the form

$$H(z) = \frac{2c-1+2\beta}{c-2(1-\beta)}\left(\frac{1+Az}{1+Bz}\right).$$

Since $c > 2(1-\beta) + |a+b+2\beta|$, it follows that

$$1 \pm B = \frac{c-2(1-\beta) \mp (a+b+2\beta)}{c-2(1-\beta)} > 0.$$

This shows that $B \neq \pm 1$. Since $c > 1 - a\,b - [(a+1)(b+1)\beta]/[1-\beta]$, we have

$$1 > \left\{1 - 2a\,b + 2\beta - \frac{2(a+1)(b+1)\beta}{1-\beta}\right\}\frac{1}{2c-1+2\beta}$$

and so $1 + A > 0$. Further it is easy to see that

$$c > 1 - a\,b - \frac{(a+1)(b+1)\beta}{1-\beta} \Longleftrightarrow \beta < \frac{c-1+a\,b}{c-2-(a+b)},$$

and to note that

$$\frac{c-1+a\,b}{c-2-(a+b)} \le 1 \Longleftrightarrow (a+1)(b+1) \le 0.$$

Similarly, under the hypothesis of Theorem 5.1, we also get that $1 - A$ is positive. These observations show that $\operatorname{Re} H(z) > 0$ for $z \in \Delta$. Now for all real r_2 and $s_1 \le -(1+r_2^2)/2$ and $z \in \Delta$, we have

$$\begin{aligned}\operatorname{Re}\psi(ir_2, s_1; z) &= \operatorname{Re}\left\{J(z)[s_1 - (1-\beta)r_2^2] + \frac{J(z) - H(z)}{2}\right\}\\ &\quad - \left[\frac{1+r_2^2}{2} + (1-\beta)r_2^2\right]\operatorname{Re} J(z) + \operatorname{Re}\left(\frac{J(z)-H(z)}{2}\right)\\ &\le -\left[\frac{r_2^2}{2} + (1-\beta)r_2^2\right]\operatorname{Re} J(z) - \operatorname{Re}\left(\frac{H(z)}{2}\right).\end{aligned}$$

Now using the fact that $\operatorname{Re} J(z) > 0$ and $\operatorname{Re} H(z) > 0$ in Δ, we deduce that $\operatorname{Re}\psi(ir_2, s_1; z) < 0$. Hence, by Theorem 1.4, we obtain $\operatorname{Re}(p(z)) > 0$ in Δ, which shows that $F(a, b; c; z)$ is convex of order β. $\square$

To get more applications of the above theorem we need to obtain conditions on the coefficients a, b, c for which $F'(a, b; c; z) \neq 0$ in Δ. In [**13**] Miller and Mocanu have shown that if

$$-1 \le b \le c \quad \text{and} \quad a \in [-2, 0) \cup [c-1, c+1], \tag{5.3}$$

then $F'(a, b; c; z) \neq 0$ in Δ. Thus, the fact that $F'(a, b; c; z) \neq 0$ for $|z| < 1$ and Theorem 5.1 give the following result which generalizes Theorem 4 in [**13**].

Theorem 5.2. *If $-2 \le a < 0$, $-1 \le b$, $b \neq 0$, $(a+1)(b+1)\beta \le 0$ and $c \ge M_\beta(a, b)$, where $M_\beta(a, b)$ is given by* (5.1)*, then $F(a, b; c; z)$ is convex of order β.*

The case $\beta = 1/2$ of Theorem 5.2 gives

Corollary 5.1. *If* $-2 \leq a < -1$, $-1 \leq b$, $b \neq 0$ *and* $c \geq M_{1/2}(a,b)$, *where*

$$M_{1/2}(a,b) = \max\{1 + |a+b+1|,\ 1 - a\,b - (a+1)(b+1)\}$$

then $F(a,b;c;z)$ *is convex of order* $1/2$.

If we use the identity

$$(a-1)(b-1)zF(a,b;c;z) = (c-1)zF'(a-1,b-1;c-1;z),$$

then Theorem 5.2 takes the following equivalent form.

Theorem 5.3. *If* $-1 \leq a < 1$, $0 \leq b$, $b \neq 1$, $a\,b\,\beta \leq 0$ *and* $c \geq 1 + M_\beta(a-1,b-1)$, *where* $M_\beta(a,b)$ *is given by* (5.1), *then* $zF(a,b;c;z) \in \mathcal{S}^*(\beta)$.

If we let $f(z) = zF(a,b;c;z)$ and $h(z) = f(z^2)/z$ then we have

$$\frac{zh'(z)}{h(z)} = 2\,\frac{z^2 f'(z^2)}{f(z^2)} - 1.$$

This observation and Theorem 5.3 immediately yield

Theorem 5.4. *If* $-1 \leq a < 0$, $0 \leq b$, $b \neq 1$ *and* $c \geq 1 + M_\beta(a-1,b-1)$, *where* $M_\beta(a,b)$ *is given by* (5.1) *then, for* $1/2 \leq \beta < 1$, *the odd hypergeometric function* $zF(a,b;c;z^2)$ *is in* $\mathcal{S}^*(2\beta-1)$.

Example 5.1. If we take $a = -2$, $b = 1$ and $\beta = \delta/(\delta+2)$ in Theorem 5.2, then for $c \geq 3 + \delta$, we have

$$F(-2,1;c;z) = 1 - \frac{2}{c}\,z + \frac{2}{c(c+1)}\,z^2 \in \mathcal{C}(\delta/(\delta+2)).$$

We note that as δ increases from 0 to ∞, the order of convexity increases from 0 to 1. We also remark that the order of convexity of the function

$F(-2,1;c;z)$ cannot be obtained from the results of Merkes and Scott [**11**] or Ruscheweyh and Singh [**21**].

Supposing that $a \in (-1,\infty)$, $b \in (-1,-a/(a+1)]$ and $c \geq a+b+1$, then we get by Theorem 2.1 that the function $zF(a+1,b+1;c+1;z)$ (and hence $zF'(a,b;c;z)$) is univalent in Δ. This fact together with Theorem 5.1 gives the following corollary.

Corollary 5.2. *If $a \in (-1,\infty)$, $b \in (-1,-a/(a+1)]$, $b \neq 0$, and $c \geq M_0(a,b)$, where $M_0(a,b)$ is given by* (5.1), *then $F(a,b;c;z)$ is convex for $|z|<1$.*

Applying Corollaries 2.1 and 4.1 in the same way as above, one can easily get several new results. In particular, the order of convexity of the hypergeometric function $F(a,b;c;z)$ and the order of starlikeness of the odd hypergeometric function $zF(a,b;c;z^2)$ can be obtained for various values of the parameters a, b and c. These results are easy to derive and so we do not give the details. However, the following simple case is useful:

Taking $a=1$ and $\beta=0$ in Theorem 5.1 we obtain that if $b \in (-1,\infty)$, $b \neq 0$ and $c \geq b+3$ such that $F'(1,b;c;z) \neq 0$ in Δ, then $F(1,b;c;z)$ is convex in Δ. Further, if we choose $a=1$ in Corollary 2.1 then we have

Corollary 5.3. *Let $b>-1$, $b \neq 0$ and $c>0$. If b and c are related by any one of the following*:

(i) $b \in (-1,\infty)$ *and* $c \geq \max\{b+1,(3b+2)/2,2b+1\}$

(ii) $b \in [-1/2,0)$ *and* $c=b+1$.

then the function $(c/b)zF'(1,b;c;z)$ is close-to-convex with respect to $-\log(1-z)$.

In particular, the above observation together with Corollary 5.3 yield the following example dealing with the incomplete beta functions:

Example 5.2. If $b \in (-1,\infty)$, $b \neq 0$ and $c \geq \max\{b+3,(3b+2)/2,2b+1\}$, then $(c/b)zF'(1,b;c;z)$ is close-to-convex with respect to $-\log(1-z)$ and $F(1,b;c;z)$ is convex in Δ. On letting $g(z) = (c/b)[F(1,b;c;z)-1]$, we find that the latter condition is equivalent to saying that the function $zg'(z) = zF(2,b+1;c+1)$ is starlike in Δ whenever $b \in (-1,\infty)$, $b \neq 0$ and $c \geq \max\{b+3, (3b+2)/2, 2b+1\}$.

This section ends with the following observation: We remark that the criteria of Ozaki and Alexander are useful for generalized hypergeometric functions. For example, consider an odd analytic function defined by

$$f(z) = z + \frac{1}{3.5}z^3 + \frac{1.3}{5.7.9}z^5 + \frac{1.3.5}{7.9.11.13}z^7 + \cdots = z + \sum_{n=2}^{\infty} A_{2n-1}z^{2n-1}$$

where

$$A_{2n-1} = \frac{1.3.5\cdots(2n-3)}{(2n-1)(2n+1)\cdots(4n-3)}.$$

Now

$$\begin{aligned}
(2n-1)A_{2n-1} &> (2n+1)A_{2n+1} \\
&\Longleftrightarrow \frac{1}{2n+1} > \frac{2n-1}{(4n-3)(4n+1)} \\
&\Longleftrightarrow (4n-3)(4n+1) > (2n+1)(2n-1) \\
&\Longleftrightarrow 12n^2 - 8n - 2 = 4(n-1)[3n+1] + 2 > 0.
\end{aligned}$$

This shows that $\{(2n-1)A_{2n-1}\}$ is a strictly increasing sequence of positive real numbers. Thus f defined above is close-to-convex for $|z| < 1$. We also observe that f can be written in the form

$$f(z) = z\,{}_3F_2(1/2, 1/2, 1;\ 3/4, 5/4;\ z^2/4).$$

Thus we have

Problem 5.1. The results of this paper have a counterpart for the generalized hypergeometric function ${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z)$ for the case $a_j > 0$, $b_j > 0$ and $p = q+1$.

6. Concluding remarks. Consider $zF(a,b;c;z) = z + \sum_{n=2}^{\infty} A_n z^n$. It is known that [**3**, p. 57, Equation 5]

$$A_{n+1} = \frac{(a,n)(b,n)}{(c,n)(1,n)} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\, n^{a+b-c-1}\left[1 + O\left(\frac{1}{n}\right)\right].$$

From this relation, we note that $|A_{n+1}| > n+1$ as $n \to \infty$ whenever $\mathrm{Re}\, a > 0$, $\mathrm{Re}\, b > 0$ and $0 < \mathrm{Re}\, c < \mathrm{Re}\,(a+b-2)$. Using this observation and de Branges' theorem [**4**], $|a_n| \leq n$, we deduce that the function $zF(a,b;c;z)$ is not univalent for $a,b,c \in \mathbf{C}\backslash\{0\}$, $\mathrm{Re}\, a > 0$, $\mathrm{Re}\, b > 0$, and $0 < \mathrm{Re}\, c < \mathrm{Re}\,(a+b-2)$. In particular, when a,b,c are positive real numbers then the function $zF(a,b;c;z)$ is not univalent in Δ for $0 < c < a+b-2$. Further, according to Bieberbach's theorem for the second coefficient, for $zF(a,b;c;z)$ to be in $\mathcal{S}$ we must have $|A_2| = |a\,b/c| \leq 2$ and therefore, if $a,b,c \in \mathbf{C}$, $c \neq 0,-1,-2,\cdots$, then the function $zF(a,b;c;z)$ is not univalent whenever $|c| < |a\,b|/2$. We also note that if $a,b > 0$ are related by one of the following

(i) $a \in (0,2]$ and $b \in (0,2]$

(ii) $a \in (2,\infty)$ and $b \in (2,\infty)$

then it is easy to check that the inequality $a\,b/2 \geq a+b-2$ holds, and if $a,b > 0$ are related by

(iii) $a \in (0,2)$ and $b \in (2,\infty)$

then we have $a\,b/2 < a+b-2$. More precisely, these observations give the following:

Theorem 6.1. *If $a,b,c > 0$ satisfy* (i) *or* (ii)*, then the function $zF(a,b;c;z)$ is not univalent in Δ for $0 < c < a\,b/2$. A similar conclusion holds if a and b are related by* (iii) *and $0 < c < a+b-2$. In particular, $zF(a,b;a+b;z)$ is not univalent when $a > 2$ and $b > 2a/(a-2)$.*

It is also known that a necessary condition for $f \in \mathcal{A}$ to be in $\mathcal{C}$ is that the modulus of the n-th coefficient of the Maclaurin series for f is bounded by 1. This fact and the idea used above immediately imply that the function $zF(a,b;c;z)$ is not convex for $a,b,c \in \mathbf{C}\backslash\{0\}$, $\mathrm{Re}\, a > 0$, $\mathrm{Re}\, b > 0$, and $0 < \mathrm{Re}\, c < \mathrm{Re}\,(a+b-1)$. In particular, the function $zF(a,b;c;z)$ is not convex whenever $a,b,c > 0$ with $0 < c < a+b-1$. Further, for $zF(a,b;c;z)$ to belong to $\mathcal{C}$ we must have $|c| > |a\,b|$. Using the fact that

(iv) $a \in (0,1]$ and $b \in (0,1] \Longrightarrow a\,b \geq a+b-1$,

(v) $a \in (1,\infty)$ and $b \in (1,\infty) \Longrightarrow a\,b \geq a+b-1$,

(vi) $a \in (0,1)$ and $b \in (1,\infty) \Longrightarrow a\,b < a+b-1$,

we summarize the above observations, for the case $a,b,c>0$, as follows:

Theorem 6.2. *If $a,b \in (0,1]$ or $a,b \in (1,\infty)$, then the function $zF(a,b;c;z)$ is not convex in Δ for $0<c<a\,b$. If a and b are related by $a \in (0,1)$ and $b \in (1,\infty)$ then for $0<c<a+b-1$, the function $zF(a,b;c;z)$ is not convex in Δ. In particular, $zF(a,b;a+b;z)$ is not convex when $a>1$ and $b>a/(a-1)$.*

For convenience, we denote by $\mathcal{K}_1$ the family of functions in $\mathcal{A}$ which are close-to-convex with respect to $-\log(1-z)$ and we denote by $\mathcal{K}_2$ the family of odd functions in $\mathcal{A}$ which are close-to-convex with respect to $(1/2)\log((1+z)/(1-z))$. As pointed out in the introduction these two classes $\mathcal{K}_1$ and $\mathcal{K}_2$ are in $\mathcal{S}$. Further, the results of this paper provide us with numerous examples of close-to-convex, starlike and convex functions. In the following example we list some simple cases for the close-to-convex case and we remark that in the same way one can obtain several results for starlikeness and convexity properties of the hypergeometric function $zF(a,b;c;z)$ and the odd hypergeometric function $zF(a,b;c;z^2)$, respectively.

Examples 6.1.

(i) On taking $b=a+1/2$ and $c=2a+1$ from Theorem 2.1, we see that if $0<a\le(\sqrt{17}-1)/4$ then [**18**, p. 461, Formula 105]

$$zF(a,a+1/2;\,2a+1;\,z) = z\left(\frac{2}{1+\sqrt{1-z}}\right)^{2a} \in \mathcal{K}_1.$$

(ii) On substituting $a=b=1$ and $c=3$ in Theorem 2.1, we see that [**18**, p. 477, Formula 150]

$$zF(1,1;\,3;\,z) = 2+2\,\frac{(1-z)}{z}\,\log(1-z) \in \mathcal{K}_1.$$

(iii) If we put $a=1$ and $c=b+1$ in Theorem 2.1, then we see that the function [**18**, p. 462, Formula 122] $zF(1,b;b+1;z)$ (and, in particular, $zF(1,2;\,3;\,z) = -(2/z)\log(1-z)-2\,)$ is in $\mathcal{K}_1$.

(iv) From Theorem 2.1, we have that the function [**18**, p. 455, Formula 28] $zF(1,b;c;z) \in \mathcal{K}_1$ whenever $b \in (1,\infty)$ and $c \geq 2b$. In particular, the function [**18**, p. 477, Formula 158]

$$zF(1,3/2;\,3;\,z) = 4z(1+\sqrt{1-z})^{-2}$$

and the function [**18**, p. 477]

$$zF(1,2;\,4;\,z) = (3/z^2)[z(2-z)+2(1-z)\log(1-z)]$$

are in $\mathcal{K}_1$.

Based on our experiments we have the following conjecture to study the geometry of the image domains under the normalized functions $z_2F_1(a,b;a+b;z)$ and $z_2F_1(a,b;a+b;z^2)$.

Problem 6.1. There exist positive numbers δ_1,δ_2 such that for $a \in (0,\delta_1)$ and $b \in (0,\delta_2)$ the normalized function $z_2F_1(a,b;a+b;z)$, $z_2F_1(a,b;a+b;z^2)$, respectively, satisfies the property that maps the unit disc Δ into a strip domain. For example, the functions

$$-\log(1-z) = z_2F_1(1,1;2;z)$$

and

$$\frac{1}{2}\log\left(\frac{1+z}{1-z}\right) = z_2F_1(1,1/2;3/2;z^2)$$

map the Δ into a strip. Therefore, the problem here is to find the exact range of the constants δ_1,δ_2 and conditions on a and b satisfying the stated property.

We recall that the Koebe function $z/(1-z)^2 = zF(1,2;1;z)$ maps Δ into the compliment of the ray $\{w = u+iv \in \mathbf{C} : u = 0,\ v \leq -1/4\}$. This function raises the following question: Suppose $a,b,c > 0$ with $c < a+b$. Do there exist $\delta_3,\delta_4 > 0$ such that for $a \in (0,\delta_3)$ and $b \in (0,\delta_4)$ the function $z_2F_1(a,b;c;z)$, $z_2F_1(a,b;c;z^2)$ respectively, has the property that the image domain is completely contained in a sector type domain where the "angle" depends on $a+b-c$?

The paper ends with the following conjecture:

Conjecture 6.1. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ satisfies conditions*

$$\sum_{n\geq 1} |na_n - (n+1)a_{n+1}| \leq 1,$$

and

$$\sum_{n\geq 1} |(n-1)a_{n-1} - 2na_n + (n+1)a_{n+1}| \leq 1,$$

then $f \in \mathcal{S}^*$.

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