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COVARIANT REPRESENTABILITY FOR COVARIANT MULTILINEAR OPERATORS

JAESEONG HEO

ABSTRACT. In this paper the notion of a covariant multilinear map from a C^*-algebra to another is introduced. Covariant completely bounded symmetric multilinear maps are decomposed into covariant completely bounded and completely positive multilinear maps, and each covariant completely bounded map is covariantly representable in terms of covariant representations and bridging operators. We show that a covariant completely bounded multilinear map extends to a completely bounded multilinear map on the crossed product C^*-algebra.

1. Introduction and preliminaries. Christensen and Sinclair [**2**] were the first to formulate the notation of completely bounded (respectively, completely positive) multilinear operators from a C^*-algebra into $\mathcal{B}(\mathcal{H})$ and gave representations for completely bounded multilinear operators. In particular, they introduced the notion of a representable k-linear operator from A^k into $\mathcal{B}(\mathcal{H})$ and pioneered the representability of completely bounded k-linear operators. Paulsen and Smith [**5**] extended a representation of completely bounded multilinear maps to the case of subspaces of C^*-algebras using the correspondence between completely bounded multilinear maps and completely bounded linear maps on Haagerup tensor products.

In Section 2 the notion of a covariant multilinear map from a C^*-algebra to another is introduced. To prove the covariant representation theorem for covariant completely bounded and completely positive multilinear maps, we prove the technical lemmas which are covariant versions of Theorem 2.8 and Lemma 3.1 in [**2**]. In Section 3 we construct the covariant representations of covariant completely bounded symmetric multilinear maps and show that such maps are decomposed

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into covariant completely bounded and completely positive multilinear maps and that each covariant completely bounded map is covariantly representable in terms of covariant representations and bridging operators. In Theorem 3.5 we will show that, given a C^*-dynamical system (A, G, α) with G amenable, a covariant completely bounded multilinear map extends to a completely bounded multilinear map on the crossed product $A \times_\alpha G$.

We recall the definitions introduced in [**2**] about completely bounded (and completely positive) multilinear maps for our convenience. Let A and B be C^*-algebras, and let $\phi : A^k = A \times \cdots \times A \to B$ be a k-linear map. The k-linear map ϕ_n from $M_n(A)^k$ into $M_n(B)$ is defined by

$$\phi_n(A_1, A_2, \dots, A_k) = \left(\sum_{r,s,\dots,t} \phi(a_{1ir}, a_{2rs}, \dots, a_{ktj}) \right) \tag{1.1}$$

for all $A_p = (a_{pij}) \in M_n(A)$, $1 \le p \le k$. We define the norm of ϕ_n by

$$\|\phi_n\| = \sup\{\|\phi_n(A_1, A_2, \dots, A_n)\| : A_p \in M_n(A) \\ \text{with} \quad \|A_p\| \le 1 \quad \text{for} \quad 1 \le p \le k\}$$

and define the *completely bounded norm* of ϕ_n by

$$\|\phi\|_{\mathrm{cb}} = \sup\{\|\phi_n\| : n \in \mathbf{N}\}.$$

The k-linear map ϕ is called *completely bounded* if $\|\phi\|_{\mathrm{cb}} < \infty$. We denote $CB(A^k, B)$ by the set of all completely bounded k-linear maps.

The k-linear map ϕ^* from A^k into B is defined by

$$\phi^*(a_1, a_2, \dots a_k) = \phi(a_k^*, \dots, a_2^*, a_1^*)^* \tag{1.2}$$

for all $a_1, a_2, \dots a_k \in A$. The k-linear map ϕ is called *symmetric* (when $k = 1$, self-adjoint) if $\phi = \phi^*$. Note that if ϕ is symmetric, then ϕ is completely symmetric in that $\phi_n = (\phi_n)^*$ for all n and that if ϕ is completely bounded, then so is ϕ^* with $\|\phi^*\|_{\mathrm{cb}} = \|\phi\|_{\mathrm{cb}}$ [**2**]. $CB_s(A^k, B)$ will denote the set of all completely bounded symmetric k-linear maps.

A k-linear map $\phi : A^k \to B$ is said to be *completely positive* if

$$\phi_n(A_1, \dots, A_k) \ge 0$$

for all $(A_1, \dots, A_k) = (A_k^*, \dots, A_1^*) \in M_n(A)^k$ with $A_m \geq 0$ if k is odd where $m = [(k+1)/2]$, and all $n \in \mathbf{N}$. Though every completely positive linear map between C^*-algebras is completely bounded, this fails in the case of completely positive k-linear maps when $k \geq 2$. $CB^+(A^k, B)$ will denote the set of all completely bounded and completely positive k-linear maps from A^k into B. The set $CB^+(A^k, B)$ is a proper positive cone in $CB_s(A^k, B)$. $CB_s(A^k, B) = CB^+(A^k, B) - CB^+(A^k, B)$ if B is an injective C^*-algebra [**2**]. For details and other definitions, see [**2**].

2. Technical lemmas for covariant representations.

In this paper we will follow the notations in [**6**]. Let (A, G, α) be a C^*-dynamical system with a locally compact group G, and let $\mathcal{U}(\mathcal{H})$ be the unitary group of $\mathcal{B}(\mathcal{H})$. If (A, G, α) is a C^*-dynamical system, then the action $\alpha : G \to \mathrm{Aut}\,(A)$ induces the action $\tilde{\alpha} : G \to \mathrm{Aut}\,(A^k)$ by

$$\tilde{\alpha}_g(a_1, \dots, a_k) = (\alpha_g(a_1), \dots, \alpha_g(a_k)) \tag{2.1}$$

for all $a_1, \dots, a_k \in A$. Given a unitary representation $u : G \to \mathcal{U}(\mathcal{H})$, a k-linear map $\phi : A^k \to \mathcal{U}(\mathcal{H})$ is called *u-covariant* if

$$\phi(\tilde{\alpha}_g(a_1, \dots, a_m)) = \phi(\alpha_g(a_1), \dots, \alpha_g(a_m)) = u_g \phi(a_1, \dots, a_m) u_g^* \tag{2.2}$$

for each $a_1, \dots, a_m \in A$ and $g \in G$. A *covariant representation* of a C^*-dynamical system (A, G, α) is a triple $(\pi, \sigma, \mathcal{H})$ where $(\pi, \mathcal{H})$ is a representation of A on a Hilbert space $\mathcal{H}$ and $(\sigma, \mathcal{H})$ is a unitary representation of G into $\mathcal{U}(\mathcal{H})$ such that

$$\pi(\alpha_g(a)) = \sigma_g \pi(a) \sigma_g^* \quad \text{for each } a \in A, g \in G. \tag{2.3}$$

The following Lemmas 2.1 and 2.2 are covariant versions of Theorem 2.8 and Lemma 3.1 in [**2**], respectively.

Lemma 2.1. *Let (A, G, α) be a C^*-dynamical system with G amenable and $u : G \to \mathcal{U}(\mathcal{H})$ a unitary representation of G. If $\phi : A^k \to \mathcal{B}(\mathcal{H})$ is a u-covariant completely bounded symmetric k-linear map with $k \geq 2$, then there is a u-covariant completely positive linear map $\psi : A \to \mathcal{B}(\mathcal{H})$ such that*

$$-\psi_n(X^*X) \leq \phi_n(X^*, A_2, \dots, A_{k-1}, X) \leq \psi_n(X^*X) \tag{2.4}$$

for all $X \in M_n(A)$ *and* $\mathbf{A} = \mathbf{A}^* = (A_2, \dots , A_{k-1}) \in M_n(A)^{k-2}$ *(not occurring when* $k = 2$*) with* $\|\mathbf{A}\| \leq 1$*, and for all* n *and such that* $\|\psi\| = \|\psi\|_{\mathrm{cb}} \leq \|\phi\|_{\mathrm{scb}}$.

Proof. By [**2**, Theorem 2.8], there is a completely bounded and completely positive k-linear map $\varphi : A \to \mathcal{B}(\mathcal{H})$ such that

$$-\varphi_n(X^*X) \leq \phi_n(X^*, A_2, \dots , A_{k-1}, X) \leq \varphi_n(X^*X)$$

for all $X \in M_n(A)$ and $\mathbf{A} = \mathbf{A}^* = (A_2, \dots , A_{k-1}) \in M_n(A)^{k-2}$ with $\|\mathbf{A}\| \leq 1$ and all n and such that $\|\varphi\| = \|\varphi\|_{\mathrm{cb}} = \|\phi\|_{\mathrm{scb}}$.

Let m be a right invariant mean on G, and define $\psi : A \to \mathcal{B}(\mathcal{H})$ by

$$\langle \psi(a)\xi, \eta\rangle = m(t \mapsto \langle u_t^* \varphi(\alpha_t(a))u_t\xi, \eta\rangle) \tag{2.5}$$

for $a \in A$ and $\xi, \eta \in \mathcal{H}$. Since φ is completely positive and m is a positive linear functional, we see that ψ is completely positive. By the right invariance of m, we have that $m(t \mapsto f(ts)) = m(f)$ so that

$$\begin{aligned}\langle \psi(\alpha_s(a))\xi, \eta\rangle &= m(t \mapsto \langle u_t^* \varphi(\alpha_t(\alpha_s(a)))u_t\xi, \eta\rangle)\\ &= m(t \mapsto \langle u_s u_{ts}^* \varphi(\alpha_{ts}(a))u_{ts}u_s^*\xi, \eta\rangle)\\ &= m(t \mapsto \langle u_t^* \varphi(\alpha_t(a))u_t u_s^*\xi, u_s^*\eta\rangle)\\ &= \langle u_s\psi(a)u_s^*\xi, \eta\rangle,\end{aligned}$$

for every $s \in G$, where the third equality follows from the right invariance of m. Thus we have $\psi(\alpha_s(a)) = u_s\psi(a)u_s^*$ for each $s \in G$.

For any $x \in A$, $\mathbf{a} = \mathbf{a}^* = (a_2, \dots , a_{k-1}) \in A^{k-2}$ and $\xi \in \mathcal{H}$, we have

$$\begin{aligned}&\langle(\psi(x^*x) - \phi(x^*, a_2, \dots , a_{k-1}, x))\xi, \xi\rangle\\ &\qquad = m(t \mapsto \langle u_t^* \varphi(\alpha_t(x^*x))u_t\xi, \xi\rangle)\\ &\qquad\quad - m(t \mapsto \langle \phi(x^*, a_2, \dots , a_{k-1}, x)\xi, \xi\rangle)\\ &\qquad = m(t \mapsto \langle u_t^* \varphi(\alpha_t(x^*x))u_t - \phi(x^*, a_2, \dots , a_{k-1}, x)\xi, \xi\rangle)\\ &\qquad = m(t \mapsto \langle u_t^*\{\varphi(\alpha_t(x^*x)) - \phi(\alpha_t(x^*), \alpha_t(a_2), \dots ,\\ &\qquad\qquad\qquad \alpha_t(a_{k-1}), \alpha_t(x))\}u_t\xi, \xi\rangle)\\ &\qquad \geq 0,\end{aligned}$$

where the third equality follows from the u-covariance of ϕ. Similarly, we have $\langle(\psi(x^*x) + \phi(x^*, a_2, \dots, a_{k-1}, x))\xi, \xi\rangle \geq 0$. From the above two inequalities, we conclude that

$$-\psi_n(X^*X) \leq \phi_n(X^*, A_2, \dots, A_{k-1}, X) \leq \psi_n(X^*X)$$

for all $X \in M_n(A)$ and $\mathbf{A} = \mathbf{A}^* = (A_2, \dots, A_{k-1}) \in M_n(A)^{k-2}$ with $\|\mathbf{A}\| \leq 1$ and all n. Since $\|\varphi\| = \|\phi\|_{\mathrm{scb}}$, we can obtain $\|\psi\| \leq \|\phi\|_{\mathrm{scb}}$ by averaging the equation $\|\psi(\cdot)\| = \|u_s^*\varphi(\cdot)u_s\|$, and this completes the proof. □

Lemma 2.2. *Let (A, G, α) be a C^*-dynamical system and $u : G \to \mathcal{U}(\mathcal{H})$ a unitary representation of G. Let $\phi : A^k \to \mathcal{B}(\mathcal{H})$ be a u-covariant completely bounded symmetric k-linear map with $k \geq 2$. If $\varphi : A \to \mathcal{B}(\mathcal{H})$ is a u-covariant completely positive linear map such that*

$$-\varphi_n(X^*X) \leq \phi_n(X^*, A_2, \dots, A_{k-1}, X) \leq \varphi_n(X^*X)$$

for all $X \in M_n(A)$ and $\mathbf{A} = \mathbf{A}^* = (A_2, \dots, A_{k-1}) \in M_n(A)^{k-2}$ with $\|\mathbf{A}\| \leq 1$ and all n, then there exist

(i) *a covariant representation $(\pi, \sigma, \mathcal{K})$ of (A, G, α) into $\mathcal{B}(\mathcal{K})$,*

(ii) *a continuous linear operator $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ with $\|V\|^2 = \|\varphi\|$,*

(iii) *a σ-covariant completely bounded symmetric $(k-2)$-linear map ψ from A^{k-2} into $\mathcal{B}(\mathcal{K})$ with $\|\psi\|_{\mathrm{scb}} \leq 1$ (when $k = 2$, ψ is just a fixed self-adjoint element of $\mathcal{B}(\mathcal{K})$ commuting with σ_g)*

such that

(1) *$\phi(a_1, \dots, a_k) = V^*\pi(a_1)\psi(a_2, \dots, a_{k-1})\pi(a_k)V$ for all $a_1, \dots, a_k \in A$,*

(2) *$V(\mathcal{H})$ reduces σ and $Vu_g = \sigma_g V$ for each $g \in G$.*

Proof. By [**2**, Lemma 3.1], there exist a Hilbert space $\mathcal{K}$, a $*$-representation π of A on $\mathcal{K}$, a continuous linear operator $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ with $\|V\|^2 = \|\varphi\|$ and a completely bounded symmetric $(k-2)$-linear map $\psi : A^{k-2} \to \mathcal{B}(\mathcal{K})$ with $\|\psi\|_{\mathrm{scb}} \leq 1$ satisfying (1). Recalling the proof in [**2**, Lemma 3.1], we first form the algebraic tensor product $A \otimes \mathcal{H}$ and endow it with pre-inner product by setting

$$\langle x \otimes \xi, y \otimes \eta\rangle_{A\otimes\mathcal{H}} = (\varphi(y^*x)\xi|\eta)_{\mathcal{H}}$$

and extending linearly. To obtain $\mathcal{K}$ one divides by the kernel of $\langle\cdot,\cdot\rangle_{A\otimes\mathcal{H}}$ and completes. The representation π of A is defined by $\pi(a)(x\otimes\xi) = ax\otimes\xi$. If A is unital, the linear operator $V:\mathcal{H}\to\mathcal{K}$ is defined by $V\xi = 1_A\otimes\xi$. If A is nonunital, let $\{a_\lambda\}$ be a bounded approximate identity of positive elements of norm ≤ 1 in A, and define $V\in\mathcal{B}(\mathcal{H},\mathcal{K})$ by $V\xi = w^* - \lim(a_\lambda\otimes\xi)$. The completely bounded symmetric $(k-2)$-linear map $\psi : A^{k-2}\to\mathcal{B}(\mathcal{K})$ is given by

$$\langle\psi(a_2,\ldots,a_{k-1})(x\otimes\xi), y\otimes\eta\rangle_{A\otimes\mathcal{H}} = (\phi(y^*,a_2,\ldots,a_{k-1},x)\xi|\eta)_{\mathcal{H}}.$$

We first define a map $\sigma : G\to\mathcal{B}(\mathcal{K})$ by setting

$$\sigma_g(x\otimes\xi) = \alpha_g(x)\otimes u_g\xi, \quad g\in G$$

and extending linearly to $A\otimes\mathcal{H}$. Since

$$\begin{aligned}\langle\sigma_g(x\otimes\xi),\sigma_g(y\otimes\eta)\rangle &= \langle\alpha_g(x)\otimes u_g\xi,\alpha_g(y)\otimes u_g\eta\rangle\\ &= (\varphi(\alpha_g(y)^*\alpha_g(x))u_g\xi|u_g\eta)\\ &= (\varphi(y^*x)\xi|\eta)\\ &= \langle x\otimes\xi, y\otimes\eta\rangle,\end{aligned}$$

we have that σ_g extends to an isometry on $\mathcal{K}$. Further, σ_g is a unitary because

$$\begin{aligned}\langle\sigma_g(x\otimes\xi), y\otimes\eta\rangle &= \langle\alpha_g(x)\otimes u_g\xi, y\otimes\eta\rangle\\ &= \langle\varphi(y^*\alpha_g(x))u_g\xi|\eta)\\ &= (\varphi(\alpha_{g^{-1}}(y^*)x)\xi|u_g^*\eta)\\ &= \langle x\otimes\xi,\alpha_{g^{-1}}(y)\otimes u_g^*\eta\rangle\\ &= \langle x\otimes\xi,\sigma_{g^{-1}}(y\otimes\eta)\rangle.\end{aligned}$$

Since $\alpha_g(x)$ is norm continuous and u is strong continuous, σ is strong continuous on the finite sums of elementary tensors and the fact that $\|\sigma_g\|\leq 1$ allows one to pass to limits.

For each $g\in G$ and $a\in A$, we have

$$\begin{aligned}\sigma_g\pi(a)\sigma_g^*(x\otimes\xi) &= \sigma_g(a\alpha_{g^{-1}}(x)\otimes u_g^*\xi)\\ &= \alpha_g(a)x\otimes\xi\\ &= \pi(\alpha_g(a))(x\otimes\xi),\end{aligned}$$

which implies that $(\pi, \sigma, \mathcal{K})$ of (A, G, α) is a covariant representation of (A, G, α). Since $\sigma_g V\xi = (1_A \otimes u_g\xi) = Vu_g\xi$ for each $\xi \in \mathcal{H}$, we get $Vu_g = \sigma_g V$ for each $g \in G$. Similarly, this equality is also obtained in the nonunital case.

To show the σ-covariance of ψ, let $a_2, \ldots, a_{k-1} \in A$. Then we have

$$\begin{aligned}
&\langle \psi(\alpha_g(a_2), \ldots, \alpha_g(a_{k-1}))(x \otimes \xi), y \otimes \eta \rangle \\
&\qquad = (\phi(y^*, \alpha_g(a_2), \ldots, \alpha_g(a_{k-1}, x)\xi | \eta) \\
&\qquad = (u_g \phi(\alpha_{g^{-1}}(y^*), a_2, \ldots, a_{k-1}, \alpha_{g^{-1}}(x)) u_g^* \xi | \eta) \\
&\qquad = \langle \psi(a_2, \ldots, a_{k-1})(\alpha_{g^{-1}}(x) \otimes u_g^* \xi), \alpha_{g^{-1}}(y) \otimes u_g^* \eta \rangle \\
&\qquad = \langle \sigma_g \psi(a_2, \ldots, a_{k-1}) \sigma_g^* (x \otimes \xi), y \otimes \eta \rangle.
\end{aligned}$$

When $k = 2$, we get a fixed self-adjoint operator S in $\mathcal{B}(\mathcal{K})$ such that

$$\langle S(x \otimes \xi), y \otimes \eta \rangle = (\phi(y^*, x)\xi | \eta).$$

For each $g \in G$, we have

$$\begin{aligned}
\langle \sigma_g^* S \sigma_g (x \otimes \xi), y \otimes \eta \rangle &= \langle S(\alpha_g(x) \otimes u_g \xi), \alpha_g(y) \otimes u_g \eta \rangle \\
&= (\phi(\alpha_g(y^*), \alpha_g(x)) u_g \xi | u_g \eta) \\
&= \langle S(x \otimes \xi), y \otimes \eta \rangle,
\end{aligned}$$

which completes the proof. □

3. Covariant representations for covariant completely bounded multilinear maps. In [**2**], the representability of k-linear operators and the representable norm are introduced as follows: Let $\pi_1, \ldots, \pi_k$ be representations of a C^*-algebra A on Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_k$, and let $V_j \in \mathcal{B}(\mathcal{H}_{j+1}, \mathcal{H}_j)$ for $j = 0, \ldots, k$, where $\mathcal{H}_0 = \mathcal{H} = \mathcal{H}_{k+1}$. Then

$$\phi(a_1, \ldots, a_k) = V_0 \pi_1(a_1) V_1 \cdots \pi_k(a_k) V_k$$

is clearly a k-linear map from A^k into $\mathcal{B}(\mathcal{H})$. Such a k-linear map ϕ is said to be *representable*. The *representable norm* $\| \cdot \|_{\mathrm{rep}}$ of a representable k-linear operator ϕ is defined by

$$\|\phi\|_{\mathrm{rep}} = \inf\{\|V_0\| \cdot \|V_1\| \cdots \|V_k\|\},$$

where the infimum is taken over all such representations of ϕ.

Similarly, we consider the covariant representability of covariant completely bounded k-linear operators. Let (A, G, α) be a C^*-dynamical system and $u : G \to \mathcal{U}(\mathcal{H})$ a unitary representation of G. Let $(\pi_i, \sigma_i, \mathcal{H}_i)$, $1 \leq i \leq k$, be covariant representations of (A, G, α) and $V_j \in \mathcal{B}(\mathcal{H}_{j+1}, \mathcal{H}_j)$ for $j = 0, \dots, k$, where $\mathcal{H}_0 = \mathcal{H} = \mathcal{H}_{k+1}$. Consider a k-linear map $\phi : A^k \to \mathcal{B}(\mathcal{H})$ defined by

$$\phi(a_1, \dots, a_k) = V_0\pi_1(a_1)V_1 \cdots \pi_k(a_k)V_k, \quad a_1, \dots, a_k \in A, \tag{3.1}$$

satisfying the relations

$$u_g V_0 = V_0\sigma_1(g), \quad V_k u_g = \sigma_k(g)V_k, \quad V_i\sigma_{i+1}(g) = \sigma_i(g)V_i \tag{3.2}$$

for each $g \in G$ and $i = 1, \dots, k-1$. Then we see that the k-linear map ϕ is u-covariant. Such a k-linear map ϕ is said to be *covariant representable.*

The following covariant representation theorem of a covariant completely bounded symmetric k-linear operator from A^k into $\mathcal{B}(\mathcal{H})$ is followed from [**2**, Theorem 4.1] except for the covariance, so that we only show the covariance.

Theorem 3.1. *Let (A, G, α) be a C^*-dynamical system with G amenable and $u : G \to \mathcal{U}(\mathcal{H})$ a unitary representation of G. Let $\phi : A^k \to \mathcal{B}(\mathcal{H})$ be a u-covariant completely bounded symmetric k-linear map and $m = [(k+1)/2]$.*

(A) *k odd. There exist covariant representations $(\pi_i, \sigma_i, \mathcal{H}_i)$, $1 \leq i \leq m-1$, and $(\theta_j, \tau_j, \mathcal{K}_j)$, $j = 1, 2$, of (A, G, α) and continuous linear operators $V_i \in \mathcal{B}(\mathcal{H}_i, \mathcal{H}_{i+1})$, $0 \leq i \leq m-2$, where $\mathcal{H}_0 = \mathcal{H}$ with*

$$\|V_0\| \cdot \|V_1\| \cdots \|V_{m-2}\| = \|\phi\|_{\mathrm{scb}}^{1/2}$$

and $W_j \in \mathcal{B}(\mathcal{H}_{m-1}, \mathcal{K}_j)$, $j = 1, 2$, with $\|W_1^*W_1 + W_2^*W_2\| = 1$ such that

$$\begin{aligned} \phi(a_1, \dots, a_k) = {} & V_0^*\pi_1(a_1)V_1^* \cdots V_{m-2}^*\pi_{m-1}(a_{m-1}) \\ & \times \{W_1^*\theta_1(a_m)W_1 - W_2^*\theta_2(a_m)W_2\} \\ & \times \pi_{m-1}(a_{m+1})V_{m-2} \cdots V_1\pi_1(a_k)V_0 \end{aligned} \tag{3.3}$$

for all $a_1, \dots, a_k \in A$ and

(3.4)

$$\sigma_1(g)V_0 = V_0u_g, \quad \sigma_{i+1}(g)V_i = V_i\sigma_i(g), \quad \sigma_{m-1}(g)W_j^* = W_j^*\tau_j(g)$$

for each $g \in G$, $i = 1, \ldots, m-1$ *and* $j = 1, 2$*. If, in addition,* ϕ *is completely positive, then the* W_2 *expression is zero.*

(B) k *even. There exist covariant representations* $(\pi_i, \sigma_i, \mathcal{H}_i)$, $1 \leq i \leq m$, *of* (A, G, α) *and continuous linear operators* $V_i \in \mathcal{B}(\mathcal{H}_i, \mathcal{H}_{i+1})$, $0 \leq i \leq m-1$,

$$\|V_0\| \cdot \|V_1\| \cdots \|V_{m-1}\| = \|\phi\|_{\mathrm{scb}}^{1/2}$$

where $\mathcal{H}_0 = \mathcal{H}$ *and* $W = W^* \in \mathcal{B}(\mathcal{H}_m)$ *with* $\|W\| = 1$ *such that*

$$\begin{aligned} \phi(a_1, \ldots, a_k) = V_0^* \pi_1(a_1) V_1^* \cdots V_{m-1}^* \pi_m(a_m) W \\ \times \pi_m(a_{m+1}) V_{m-1} \cdots V_1 \pi_1(a_k) V_0 \end{aligned} \tag{3.5}$$

for all $a_1, \ldots, a_k \in A$*, and*

$$V_0 u_g = \sigma_1(g) V_0, \quad \sigma_{i+1}(g) V_i = V_i \sigma_i(g), \quad \sigma_m(g) W = W \sigma_m(g) \tag{3.6}$$

for each $g \in G$ *and* $1 \leq i \leq m-2$*. If, in addition,* ϕ *is completely positive, then* W *is positive.*

Proof. The proof follows from Lemmas 2.1 and 2.2 as in [**2**].

(A) k odd. Using the invariant mean to average as in Lemma 2.1, the case $k = 1$ is obtained from [**3**, Corollary 2.6]. Let $\phi : A^k \to \mathcal{B}(\mathcal{H})$, $k \geq 3$, be a u-covariant completely bounded symmetric k-linear map. By Lemma 2.1 there is a u-covariant completely positive linear map $\varphi : A \to \mathcal{B}(\mathcal{H})$ such that

$$-\varphi_n(X^* X) \leq \phi_n(X^*, A_2, \ldots, A_{k-1}, X) \leq \varphi_n(X^* X)$$

for all $X \in M_n(A)$ and $(A_2, \ldots, A_{k-1}) = (A_{k-1}^* \cdots A_2) \in M_n(A)^{k-2}$. By Lemma 2.2 there is a Hilbert space $\mathcal{H}_1$, a covariant representation $(\pi_1, \sigma_1, \mathcal{H}_1)$ of (A, G, α) and a σ_1-covariant completely bounded symmetric $(k-2)$-linear map $\psi : A^{k-2} \to \mathcal{B}(\mathcal{H}_1)$ such that

$$\phi(a_1, \ldots, a_k) = V_0^* \pi_1(a_1) \psi(a_2, \ldots, a_{k-1}) \pi_1(a_k) V_0$$

for all $a_1, \ldots, a_k \in A$ and that $V_0 u_g = \sigma_1(g) V_0$ for each $g \in G$, $\|V_0\|^2 \leq \|\phi\|_{\mathrm{scb}}$ and $\|\phi\|_{\mathrm{scb}} \leq 1$. In the proof of [**2**, Lemma 3.1], the equality

$$\phi_n(A_1, \ldots, A_k) = (V_0)_n^* (\pi_1)_n(a_1) \psi_n(A_2, \ldots, A_{k-1}) (\pi_1)_n(A_k) (V_0)_n$$

holds for all $A_1, \dots, A_k \in M_n(A)$. Hence $\|\phi\|_{\mathrm{scb}} \le \|\psi\|_{\mathrm{scb}} \cdot \|V_0\|^2$ so that we have $\|\phi\|_{\mathrm{scb}} = \|V_0\|^2$. The remainder is obtained by induction.

(B) k even. By Lemma 2.2 and the induction it suffices to consider only the case $k = 2$. Let ϕ be a u-covariant completely bounded symmetric two-linear map from A^2 into $\mathcal{B}(\mathcal{H})$. By Lemma 2.1 there is a u-covariant completely positive linear map $\psi : A^2 \to \mathcal{B}(\mathcal{H})$ dominating ϕ. From Lemma 2.2 we conclude that there exist a covariant representation $(\pi, \sigma, \mathcal{K})$ of (A, G, α), a continuous linear operator $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and a self-adjoint $W \in \mathcal{B}(\mathcal{K})$ with $\|W\| = 1$ and $\|V_0\|^2 \le \|\phi\|_{\mathrm{scb}}$ such that

$$\phi(a, b) = V^* \pi(a) W \pi(b) V \quad \text{and} \quad W = \sigma_g^* W \sigma_g$$

for all $a, b \in A$ and $g \in G$. The remainder is obtained by induction. □

Corollary 3.2. *Let (A, G, α) be a C^*-dynamical system with G amenable. Let B be an injective von Neumann algebra and u a unitary representation of G into the unitary group $\mathcal{U}(B)$ of B. If $\phi : A^k \to B$ is a u-covariant completely bounded symmetric k-linear map, then there are u-covariant completely bounded and completely positive k-linear maps ϕ_+ and ϕ_- from A^k into B such that*

$$\phi = \phi_+ - \phi_- \quad \textit{and} \quad \|\phi\|_{\mathrm{cb}} \ge \|\phi_+ + \phi_-\|_{\mathrm{cb}}.$$

Proof. By [**2**, Corollary 4.3] there are u-covariant completely bounded and completely positive k-linear maps ψ_+ and ψ_- from A^k into B such that

$$\phi = \psi_+ - \psi_- \quad \text{and} \quad \|\phi\|_{\mathrm{cb}} = \|\psi_+ + \psi_-\|_{\mathrm{cb}}.$$

First, represent B on a Hilbert space $\mathcal{H}$. Then there is a completely positive projection P from $\mathcal{B}(\mathcal{H})$ onto B so that we may regard ϕ, ψ_+ and ψ_- as being k-linear maps from A^k into $\mathcal{B}(\mathcal{H})$.

Let m be a right invariant mean on G, and define ϕ_+ and ϕ_- by

$$(\phi_\pm(a_1, \dots, a_k)\xi|\eta) = m(t \mapsto (u_t^* \psi_\pm(\alpha_t(a_1), \dots, \alpha_t(a_k)) u_t \xi|\eta)$$

for $a \in A$ and $\xi, \eta \in \mathcal{H}$. Then ϕ_+ and ϕ_- are completely bounded and completely positive k-linear maps. By the right invariance of m, we have

$$\begin{aligned}
&(\phi_\pm(\alpha_s(a_1),\dots,\alpha_s(a_k))\xi|\eta)\\
&\qquad\qquad = m(t\mapsto (u_t^*\psi_\pm(\alpha_{ts}(a_1),\dots,\alpha_{ts}(a_k))u_t\xi|\eta))\\
&\qquad\qquad = m(t\mapsto (u_s u_{ts}^*\psi_\pm(\alpha_{ts}(a_1),\dots,\alpha_{ts}(a_k))u_{ts}u_s^*\xi|\eta))\\
&\qquad\qquad = m(t\mapsto (u_t^*\psi_\pm(\alpha_t(a_1),\dots,\alpha_t(a_k))u_t u_s^*\xi|u_s^*\eta))\\
&\qquad\qquad = (u_s\phi_\pm(a_1,\dots,a_k)u_s^*\xi|\eta).
\end{aligned}$$

Hence we have $\phi_\pm(\alpha_s(a_1),\dots,\alpha_s(a_k)) = u_s\phi_\pm(a_1,\dots,a_k)u_s^*$ for each $s\in G$ so that ϕ_+ and ϕ_- are u-covariant. For all $a_1,\dots,a_k\in A$ and $\xi,\eta\in\mathcal{H}$, we have

$$\begin{aligned}
&((\phi_+(a_1,\dots,a_k)-\phi_-(a_1,\dots,a_k))\xi|\eta)\\
&\qquad = m(t\mapsto (u_t^*\{\psi_+(\alpha_t(a_1),\dots,\alpha_t(a_k))\\
&\qquad\qquad\qquad -\psi_-(\alpha_t(a_1),\dots,\alpha_t(a_k))\}u_t\xi|\eta))\\
&\qquad = m(t\mapsto (u_t^*\phi(\alpha_t(a_1),\dots,\alpha_t(a_k))u_t\xi|\eta))\\
&\qquad = m(t\mapsto (\phi(a_1,\dots,a_k)\xi|\eta))\\
&\qquad = (\phi(a_1,\dots,a_k)\xi|\eta),
\end{aligned}$$

so that $\phi=\phi_+-\phi_-$. Furthermore, we get the desired result by averaging the equation

$$\|\phi\|_{\mathrm{cb}} = \|\psi_+ + \psi_-\|_{\mathrm{cb}} = \|u_g^*\psi_+u_g + u_g^*\psi_-u_g\|_{\mathrm{cb}},$$

which completes the proof. □

Let B be a C^*-algebra and u a unitary representation of G into $\mathcal{U}(B)$. Let $[\phi_{ij}]$ be a k-linear map from A^k into $M_n(B)$, and let $\tilde{u}_g\in\mathcal{U}(M_n(B))$ be a diagonal matrix with all the diagonal entries u_g. If the map $[\phi_{ij}]: A^k\to M_n(B)$ is $\tilde{u}$-covariant with respect to the dynamical system $(A^k,G,\tilde{\alpha})$, we say that $[\phi_{ij}]$ is a *u-covariant* k-linear map. Note that a k-linear map $[\phi_{ij}]$ is u-covariant if and only if

$$\begin{gathered}\phi_{ij}(\alpha_g(a_1),\dots,\alpha_g(a_k)) = u_g\phi_{ij}(a_1,\dots,a_k)u_g^*\\ i,j=1,\dots,n\end{gathered}\tag{3.7}$$

for each $(a_1,\dots,a_k)\in A^k$ and $g\in G$, cf. [**3**].

Corollary 3.3. *Let (A,G,α) be a C^*-dynamical system with G amenable and u a unitary representation of G into $\mathcal{U}(\mathcal{H})$. If $\phi : A^k \to \mathcal{B}(\mathcal{H})$ is a u-covariant completely bounded k-linear map, then ϕ is covariant representable.*

Proof. We get the proof via a slight modification. Let $S\phi$ be a k-linear map from A^k into $M_2(\mathcal{B}(\mathcal{H})) = \mathcal{B}(\mathcal{H}\oplus\mathcal{H})$ defined by

$$S\phi = \begin{bmatrix} 0 & \phi^* \\ \phi & 0 \end{bmatrix}. \tag{3.8}$$

Then $S\phi$ is a completely symmetric k-linear map.

For each $(a_1,\dots,a_k) \in A^k$ and $g \in G$, we have

$$\begin{aligned} &S\phi(\alpha_g(a_1),\dots,\alpha_g(a_k)) \\ &\qquad = \begin{bmatrix} 0 & \phi^*(\alpha_g(a_1),\dots,\alpha_g(a_k)) \\ \phi(\alpha_g(a_1),\dots,\alpha_g(a_k)) & 0 \end{bmatrix} \\ &\qquad = \begin{bmatrix} 0 & u_g\phi(a_k^*,\dots,a_1^*)u_g^* \\ u_g\phi(a_1,\dots,a_k)u_g^* & 0 \end{bmatrix} \\ &\qquad = \begin{bmatrix} u_g & 0 \\ 0 & u_g \end{bmatrix} \begin{bmatrix} 0 & \phi^*(a_1,\dots,a_k) \\ \phi(a_1,\dots,a_k) & 0 \end{bmatrix} \begin{bmatrix} u_g^* & 0 \\ 0 & u_g^* \end{bmatrix} \\ &\qquad = \tilde{u}_g S\phi(a_1,\dots,a_k)\tilde{u}_g^*, \end{aligned}$$

so that $S\phi$ is u-covariant. Since the norm of the symmetrization operator S is 1, $S\phi$ is completely bounded. Using Theorem 3.1 and restricting to the lower left corner of a 2×2-matrix defining $S\phi$ gives the desired result. □

Corollary 3.4. *Let (A,G,α) be a C^*-dynamical system with G amenable and u a unitary representation of G into $\mathcal{U}(\mathcal{H})$. If $\phi : A^k \to \mathcal{B}(\mathcal{H})$ is a u-covariant completely bounded k-linear map, then there are a covariant representation $(\pi,\sigma,\mathcal{K})$ of (A,G,α), a u-covariant completely bounded symmetric $(k-2)$-linear map $\phi : A^{k-2} \to \mathcal{B}(\mathcal{K})$ and continuous linear operators $V : \mathcal{K} \to \mathcal{H}$ and $W : \mathcal{H} \to \mathcal{K}$ such that*

$$\phi(a_1,\dots,a_k) = V\pi(a_1)\psi(a_2,\dots,a_{k-1})\pi(a_k)W, \quad a_1,\dots,a_k \in A, \tag{3.9}$$

and

$$V\sigma_g = u_g V, \quad W u_g = \sigma_g W, \quad g \in G \tag{3.10}$$

with $\|\phi\|_{\mathrm{cb}} = \|V\| \cdot \|W\| \cdot \|\psi\|_{\mathrm{cb}}$.

Proof. Likewise, as in the proof of Corollary 3.3, the symmetrization

$$S\phi = \begin{bmatrix} 0 & \phi^* \\ \phi & 0 \end{bmatrix}$$

gives a u-covariant completely bounded symmetric k-linear map from A^k into $\mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ with $\|S\phi\|_{\mathrm{cb}} = \|\phi\|_{\mathrm{cb}}$. By Lemma 2.1 there is a u-covariant completely positive linear map $\varphi : A \to \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ dominating $S\phi$. By Lemma 2.2 there exist a covariant representation $(\pi, \sigma, \mathcal{K})$ of (A, G, α), a u-covariant completely bounded symmetric $(k-2)$-linear map $\psi : A^{k-2} \to \mathcal{B}(\mathcal{K})$ and a continuous linear operator $U \in B(\mathcal{H} \oplus \mathcal{H}, \mathcal{K})$ such that

$$S\phi(a_1, \dots, a_k) = U^* \pi(a_1) \psi(a_2, \dots, a_{k-1}) \pi(a_k) U, \quad a_1, \dots, a_k \in A,$$

and $U\tilde{u}_g = \sigma_g U$ for each $g \in G$. Let P be an orthogonal projection from $\mathcal{H} \oplus \mathcal{H}$ onto $\mathcal{H} \oplus 0$. Letting $V = (I - P)U^*$ and $W = U|_{\mathcal{H} \oplus 0}$, we get

$$\phi(a_1, \dots, a_k) = V \pi(a_1) \psi(a_2, \dots, a_{k-1}) \pi(a_k) W.$$

Furthermore, we have $V\sigma_g = u_g V$ and $W u_g = \sigma_g W$ for each $g \in G$. □

Theorem 3.5. *Let (A, G, α) be a unital C^*-dynamical system with G amenable and u a unitary representation of G into $\mathcal{U}(\mathcal{H})$. If $\phi : A^k \to \mathcal{B}(\mathcal{H})$ is a u-covariant completely bounded k-linear map, then a completely bounded k-linear map $\psi : (A \times_\alpha G)^k \to \mathcal{B}(\mathcal{H})$ exists given by*

(3.11)
$$\psi(f_1, \dots, f_k) = \int_{G^k} \phi(f_1(s_1), \alpha_{s_1}(f_2(s_2)), \dots, \alpha_{s_1 \cdots s_{k-1}}(f_k(s_k))) \times u_{s_1} u_{s_2} \cdots u_{sk}\, d\mu(s_1)\, d\mu(s_2) \cdots d\mu(s_k)$$

for all $f_1, \dots, f_k \in K(G,A)$ *where* $K(G,A)$ *is the set of continuous functions from* G *to* A *with compact supports.*

Proof. The proof is divided into both the k odd and k even cases.

(A) k odd. By Theorem 3.1, there exist covariant representations $(\pi_i, \sigma_i, \mathcal{H}_i)$, $(\theta_j, \tau_j, \mathcal{K}_j)$ of (A, G, α) and continuous linear operators $V_i \in \mathcal{B}(\mathcal{H}_i, \mathcal{H}_{i+1})$ where $\mathcal{H}_0 = \mathcal{H}$ and $W_j \in \mathcal{B}(\mathcal{H}_{m-1}, \mathcal{K}_j)$ satisfy (3.3) and (3.4). We define $\pi_i \times \sigma_i$ and $\theta_j \times \tau_j$ by

$$\begin{aligned}(\pi_i \times \sigma_i)(f) &= \int_G \pi_i(f(s))\sigma_i(s)\, d\mu(s), \quad 1 \le i \le m-1,\\ (\theta_j \times \tau_j)(f) &= \int_G \theta_j(f(s)\tau_j(s)\, d\mu(s), \quad j = 1, 2,\end{aligned}$$

for every $f \in K(G,A)$. From [**6**, Proposition 7.6.4], we see that $\pi_i \times \sigma_i$ (respectively, $\theta_j \times \tau_j$) extends to a representation, again denoted by $\pi_i \times \sigma_i$ (respectively, $\theta_j \times \tau_j$) from $L^1(G,A)$ to $\mathcal{B}(\mathcal{H}_i)$ (respectively, $\mathcal{B}(\mathcal{K}_j)$). By the universal property of the crossed product $A \times_\alpha G$, the representation $\pi_i \times \sigma_i$ (respectively, $\theta_j \times \tau_j$) extends to a representation of $A \times_\alpha G$ into $\mathcal{B}(\mathcal{H}_i)$ (respectively, $\mathcal{B}(\mathcal{K}_j)$) still denoted by $\pi_i \times \sigma_i$ (respectively, $\theta_j \times \tau_j$).

We define a k-linear map $\psi : (A \times_\alpha G)^k \to \mathcal{B}(\mathcal{H})$ by

$$\begin{aligned}\psi(x_1, \dots, x_k) = V_0^*&(\pi_1 \times \sigma_1)(x_1)\\ &\times V_1^* \cdots V_{m-2}^*(\pi_{m-1} \times \sigma_{m-1})(x_{m-1})\\ &\times \{W_1^*(\theta_1 \times \tau_1)(x_m)W_1 - W_2^*(\theta_2 \times \tau_2)(x_m)W_2\}\\ &\times (\pi_{m-1} \times \sigma_{m-1}(x_{m+1})V_{m-2} \cdots\\ &\times V_1(\pi_1 \times \sigma_1)(x_k)V_0\end{aligned}$$

for each $x_1, \dots, x_k \in A \times_\alpha G$. The case $k = 1$ is obtained from [**3**, Proposition 3.2]. We only consider the case $k = 3$ because the general case is similar. For $f_1, f_2, f_3 \in K(G,A)$, we have

$$\begin{aligned}\psi(f_1, f_2, f_3) = V^*&(\pi \times \sigma)(f_1)\{W_1^*(\theta_1 \times \tau_1)(f_2)W_1\\ &- W_2^*(\theta_2 \times \tau_2)(f_2)W_2\}(\pi \times \sigma)(f_3)V\end{aligned}$$

$$
\begin{aligned}
&= \int_{G^3} V^*\pi(f_1(s_1))\sigma(s_1)\{W_1^*\theta_1(f_2(s_2))\tau_1(s_2)W_1 \\
&\qquad\qquad\qquad\qquad - W_2^*\theta_2(f_2(s_2))\tau_2(s_2)W_2\} \\
&\qquad \times \pi(f_3(s_3))\sigma(s_3)V\,d\mu(s_1)\,d\mu(s_2)\,d\mu(s_3)W_1 \\
&= \int_{G^3} V^*\pi(f_1(s_1))\{W_1^*\theta_1(\alpha_{s_1}(f_2(s_2))) \\
&\qquad\qquad\qquad \times -W_2^*\theta_2(\alpha_{s_1}(f_2(s_2)))W_2\} \\
&\qquad \times \sigma(s_1s_2)\pi(f_3(s_3))\sigma(s_3)V\,d\mu(s)\,d\mu(s_2)\,d\mu(s_3) \\
&= \int_{G^3} V^*\pi(f_1(s_1))\{W_1^*\theta_1(\alpha_{s_1}(f_2(s_2)))W_1 \\
&\qquad\qquad\qquad \times -W_2^*\theta_2(\alpha_{s_1}(f_2(s_2)))W_2\} \\
&\qquad \times \pi(\alpha_{s_1s_2}(f_3(s_3)))Vu_{s_1}u_{s_2}u_{s_3}\,d\mu(s_1)\,d\mu(s_2)\,d\mu(s_3) \\
&= \int_{G^3} \phi(f_1(s_1), \alpha_{s_1}(f_2(s_2)), \alpha_{s_1s_2}(f_3(s_3))) \\
&\qquad\qquad\qquad \times u_{s_1}u_{s_2}u_{s_3}\,d\mu(s_1)\,d\mu(s_2)\,d\mu(s_3).
\end{aligned}
$$

(B) k even. By Theorem 3.1, there exist covariant representations $(\pi_i, \sigma_i, \mathcal{H}_i)$ of (A, G, α), continuous linear operators $V_i \in \mathcal{B}(\mathcal{H}_i, \mathcal{H}_{i+1})$,where $\mathcal{H}_0 = \mathcal{H}$, and $W = W^* \in \mathcal{B}(\mathcal{H}_m)$ satisfying (3.5) and (3.6). Likewise, as in (A), we define $\pi_i \times \sigma_i$

$$
(\pi_i \times \sigma_i)(f) = \int_G \pi_i(f(s))\sigma_i(s)\,d\mu(s), \quad 1 \le i \le m-1,
$$

for every $f \in K(G, A)$. From [**6**, Proposition 7.6.4], we see that $\pi_i \times \sigma_i$ extends to a representation, again denoted by $\pi_i \times \sigma_i$, from $L^1(G, A)$ to $\mathcal{B}(\mathcal{H}_i)$. By the universal property of the crossed product $A \times_\alpha G$, the representation $\pi_i \times \sigma_i$ extends to a representation of $A \times_\alpha G$ into $\mathcal{B}(\mathcal{H}_i)$, still denoted by $\pi_i \times \sigma_i$.

We define a k-linear map $\psi : (A \times_\alpha G)^k \to \mathcal{B}(\mathcal{H})$ by

$$
\begin{aligned}
\psi(x_1, \dots, x_k) &= V_0^*(\pi_1 \times \sigma_1)(x_1)V_1^* \cdots V_{m-1}^* \\
&\quad \times (\pi_m \times \sigma_m)(x_m)W(\pi_m \times \sigma_m)(x_{m+1}) \\
&\quad \times V_{m-1} \cdots V_1(\pi_1 \times \sigma_1)(x_k)V_0
\end{aligned}
$$

for each $x_1, \dots, x_k \in A \times_\alpha G$. We only consider the case $k = 4$ because the general case is similar. Let $f_i \in K(G, A)$, $1 \le i \le 4$. Then we have

$$\begin{aligned}
&\psi(f_1, f_2, f_3, f_4)\\
&\quad = V_0^*(\pi_1 \times \sigma_1)(f_1)V_1^*(\pi_2 \times \sigma_2)(f_2)W(\pi_2 \times \sigma_2)(f_3)V_1(\pi_1 \times \sigma_1)(f_4)V_0\\
&\quad = \int_{G^4} V_0^*\pi_1(f_1(s_1))\sigma_1(s_1)V_1^*\pi_2(f_2(s_2))\sigma_2(s_2)W\pi_2(f_3(s_3))\sigma_2(s_3)V_1\\
&\qquad\quad \times \pi_1(f_4(s_4))\sigma_1(s_4)V_0 \, d\mu(s_1)\, d\mu(s_2)\, d\mu(s_3)\, d\mu(s_4)\\
&\quad = \int_{G^4} V_0^*\pi_1(f_1(s_1))V_1^*\pi_2(\alpha_{s_1}(f_2(s_2)))W\sigma_2(s_1s_2)\pi_2(f_3(s_3))\sigma_2(s_3)V_1\\
&\qquad\quad \times \pi_1(f_4(s_4))\sigma_1(s_4)V_0 d\mu(s_1)\, d\mu(s_2)\, d\mu(s_3)\, d\mu(s_4)\\
&\quad = \int_{G^4} V_0^*\pi_1(f_1(s_1))V_1^*\pi_2(\alpha_{s_1}(f_2(s_2)))\\
&\qquad\quad \times W\pi_2(\alpha_{s_1s_2}(f_3(s_3)))V_1\sigma_1(s_1s_2s_3)\pi_1(f_4(s_4))\sigma_1(s_4)\\
&\qquad\quad \times V_0 \, d\mu(s_1)\, d\mu(s_2)\, d\mu(s_3)\, d\mu(s_4)\\
&\quad = \int_{G^4} V_0^*\pi_1(f_1(s_1))V_1^*\pi_2(\alpha_{s_1}(f_2(s_2)))\\
&\qquad\quad \times W\pi_2(\alpha_{s_1s_2}(f_3(s_3)))V_1\pi_1(\alpha_{s_1s_2s_3}(f_4(s_4)))\\
&\qquad\quad \times V_0u_{s_1}u_{s_2}u_{s_3}u_{s_4} \, d\mu(s_1)\, d\mu(s_2)\, d\mu(s_3)\, d\mu(s_4)\\
&\quad = \int_{G^4} \phi(f_1(s_1), \alpha_{s_1}(f_2(s_2)), \alpha_{s_1s_2}(f_3(s_3)), \alpha_{s_1s_2s_3}(f_4(s_4)))\\
&\qquad\quad \times u_{s_1}u_{s_2}u_{s_3}u_{s_4} \, d\mu(s_1)\, d\mu(s_2)\, d\mu(s_3)\, d\mu(s_4),
\end{aligned}$$

which completes the proof. □

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Department of Mathematics, Seoul National University, Seoul 151-742, Korea
E-mail address: `hjs@math.snu.ac.kr`