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ORBIFOLD SPECTRAL THEORY

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ABSTRACT. In this paper we study Sobolev spaces for smooth closed orientable Riemannian orbifolds. In particular we prove the Sobolev embedding theorem, the Rellich-Kondrakov theorem and Poincare's inequalities. From these theorems we derive properties of the spectrum of the Laplacian. In particular, Weil's asymptotic formula and estimates from below of the eigenvalues of the Laplacian are proved in analogy with the manifold case.

0. Introduction. In this paper we study spectral theory for closed orientable orbifolds. (In the literature orbifolds are also called V-manifolds.) Orbifold Hilbert Sobolev spaces H_k^2 were first introduced by Chiang in [**5**]. Other orbifold Sobolev spaces are also considered in [**12**]. After defining general Sobolev spaces for closed orientable orbifolds, we establish Sobolev embedding theorems and the Rellich-Kondrakov theorem. By using these theorems we prove, in analogy with the manifold case, Weil's asymptotic formula for the eigenvalues of the orbifold Laplacian. We also prove Poincare's inequalities. Our presentation of this material follows closely [**9**], [**1**], which deal with the manifold case. By proving more refined Sobolev inequalities we also obtain estimates from below of the eigenvalues of the Laplacian, generalizing the results of [**4**] to the orbifold case.

This paper is the starting point of an ongoing project aiming at generalizing several well-known results of spectral theory for manifolds to orbifolds.

We will now recall a few basic definitions used throughout the paper [**10**], [**5**], [**7**]. Unless otherwise specified, all our orbifolds are assumed to be both smooth and Riemannian.

A closed orientable orbifold, M, can be covered by a finite number of charts $(\Omega_l, \phi_l)_{l=1,\dots,N}$, where $\Omega_l = \tilde{\Omega}_l/G_l$ with $\tilde{\Omega}_l$ homeomorphic to $\mathbf{R}^n$ and G_l a finite subgroup of $SO(n)$. The local lifts of the changes of charts are assumed to be smooth.

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A partition of unity $\{\eta_l\}_{l=1,\dots,N}$ subordinate to the cover $\{\Omega_l\}_{l=1,\dots,N}$ and to the shrinkage $\{V_l\}_{l=1,\dots,N}$ of $\{\Omega_l\}_{l=1,\dots,N}$ is constructed in analogy with the manifold case by using a family of G_l-invariant functions $\tilde{\mu}_l$ on $\tilde{\Omega}_l$ such that $\tilde{\mu}_l = 1$ on $\tilde{V}_l$ and $\tilde{\mu}_l = 0$ outside $\tilde{\Omega}_l$. This defines functions μ_l on M with $\mu_l = 1$ on V_l and $\mu_l = 0$ outside Ω_l. Moreover

$$\eta_l = \frac{\mu_l}{\sum_{j=1}^N \mu_j}, \quad l = 1, \dots, N, \quad \sum_{j=1}^N \eta_j = 1.$$

By choosing a G_l-invariant metric $\tilde{g}^l_{ij}$ on $\tilde{\Omega}_l$ and using a standard partition of unity construction, one gets a metric g on M in analogy with the manifold case.

Now, given any function $u \in C^\infty(M)$, one can define the integral of u over M by

$$\int_M u\, dv(x) \underset{\text{def}}{=} \sum_{l=1}^N \frac{1}{|G_l|} \int_{\tilde{\Omega}_l} \tilde{\eta}_l(\tilde{x})\tilde{u}(\tilde{x}) \det(\tilde{g}^l_{ij}(\tilde{x}))\, d\tilde{x},$$

where $\sim$ means lift to $\tilde{\Omega}_l$.

1. Sobolev spaces for orbifolds. The Sobolev spaces H^2_k were introduced by Chiang in [**5**]. More in general, one can define the Sobolev spaces H^p_k as follows.

Definition 1.1. Let M be an orbifold. Define the space $C^p_k(M)$ by

$$C^p_k(M) = \Big\{ u \in C^\infty(M) \mid \sup_{j=0,\dots,k} \int_M |\nabla^j u|^p\, dv(x) < \infty \Big\}.$$

The Sobolev space $H^p_k(M)$ is the completion of $C^p_k(M)$ with respect to the norm

$$\|u\|_{H^p_k} = \sum_{j=0}^k \left(\int_M |\nabla^j u|^p\, dv(x) \right)^{1/p}.$$

Remark 1.2. (a) If M is compact, then $C^p_k(M) = C^\infty(M)$ for all k and $p \geq 1$, and $H^p_k(M)$ does not depend on the metric.

(b) $H_k^p(M)$ is a closed subspace of $L^p(M)$.

Proposition 1.3. (a) *$H_k^2(M)$ is a Hilbert space when equipped with the norm $\|.\|_k$ (equivalent to $\|.\|_{H_k^p}$) defined by*

$$\|u\|_k = \sqrt{\sum_{j=0}^{k} \int_M |\nabla^j u|^2 \, dv(x)}.$$

The scalar product $\langle , \rangle$ associated to the given norm is defined by

$$\langle u, v \rangle = \sum_{j=1}^{k} \int_M \langle \nabla^j u, \nabla^j v \rangle \, dv(x).$$

(b) *If $p > 1$, $H_k^p(M)$ is reflexive.*

2. Sobolev embedding theorems. We will establish Sobolev embedding theorems for orbifolds, generalizing existing Sobolev embedding theorems for manifolds. Propositions 2.1 and 2.2 will deal with particular cases of the main theorem. This generalizes Chiang's results in [**5**] for H_k^2. For Sobolev inequalities of Gallot type, see [**12**].

Proposition 2.1. *Let M be a closed orientable orbifold of dimension n. Suppose that the embedding $H_1^1(M) \subset L^{n/(n-1)}(M)$ is continuous. Then, for any real numbers p, q with $1 \le q < n$ and $1/p = 1/q - 1/n$, the embedding $H_1^q(M) \subset L^p(M)$ is continuous.*

Proof. Let p and q be as above, and let $A \in \mathbf{R}$ be such that

$$\|u\|_{L^{n/(n-1)}} \le A\|u\|_{H_1^1} \quad \text{for all } u \in H_1^1(M).$$

For $u \in C^\infty(M)$, let $\phi = |u|^{p(n-1)/n}$. By continuity and Holder's inequality we have (where we set $p' = [p(n-1)/n]-1)$, $q' = (q/(q-1))$,

$$
\begin{aligned}
(\|u\|_{L^p})^{p(n-1)/n} &= \|\phi\|_{L^{(n-1)/n}} \leq A\|\phi\|_{H_1^1} \\
&= A\int_M (|\nabla\phi| + |\phi|)\,dv(x) \\
&= \frac{Ap(n-1)}{n}\int_M |u|^{p'}|\nabla u|\,dv(x) + A\int_M |u|^{p'+1}\,dv(x) \\
&\leq \frac{Ap(n-1)}{n}\bigg(\int_M |u|^{p'q'}\,dv(x)\bigg)^{1/q'}\bigg(\int_M |\nabla u|^q\,dv(x)\bigg)^{1/q} \\
&\quad + A\bigg(\int_M |u|^{p'q'}\,dv(x)\bigg)^{1/q'}\bigg(\int_M |u|^q\,dv(x)\bigg)^{1/q}.
\end{aligned}
$$

Since

$$
\begin{aligned}
p'q' &= \bigg(\frac{p(n-1)}{n} - 1\bigg)q' \\
&= \bigg(\frac{p(n-1)}{n} - 1\bigg)\bigg(1 - \frac{1}{q}\bigg)^{-1} \\
&= \bigg(\frac{p(n-1)}{n} - 1\bigg)\bigg(1 - \frac{1}{p} - \frac{1}{n}\bigg)^{-1} \\
&= \bigg(\frac{p(n-1)}{n} - 1\bigg)\bigg(\frac{pn-n-p}{pn}\bigg)^{-1} = p,
\end{aligned}
$$

we have that

$$
\begin{aligned}
\|u\|_{L^p}^{p(n-1)/n} \leq \|u\|_{L^p}^{p/q'}\bigg\{\frac{Ap(n-1)}{n}\bigg(\int_M |\nabla u|^q\,dv(x)\bigg)^{1/q} \\
+ A\bigg(\int_M |u|^q\,dv(x)\bigg)^{1/q}\bigg\}.
\end{aligned}
$$

Since $p(n-1)/n - p/(q') = 1$, it follows that $\|u\|_{L^p} \leq (Ap(n-1)/n)\|u\|_{H_1^q}$ for all $u \in C^\infty(M)$. As $C^\infty(M)$ is dense in $H_1^q(M)$, we are done. □

Proposition 2.2. *Let M be a closed orientable orbifold of dimension n. Suppose that the embedding $H_1^{q_0}(M) \subset L^{p_0}(M)$ is valid for any $p_0, q_0 \in \mathbf{R}$ such that $1 \leq q_0 < n$ and $1/p_0 = 1/q_0 - 1/n > 0$. Then* *for any real member p, q, such that $1 \leq q < n$ and integers m, k such that $0 \leq m < k$ satisfying $1/p = 1/q - (k-m)/n > 0$, the embedding $H^q_k(M) \subset H^p_m(M)$ is continuous.*

Proof. Let $r \in \mathbf{N}$. Then

$$|\nabla|\nabla^r\psi|| \leq |\nabla^{r+1}\psi|, \quad \forall\, \psi \in C^\infty(M).$$

(Proof as in [**1**, p. 36].) Suppose first that $k - m = 1$, so that $1/p = 1/q - 1/n$. In this case the embedding $H^q_1(M) \subset L^p(M)$ is continuous by hypothesis so $A \in \mathbf{R}$ exists such that, for all $\phi \in H^q_1(M)$:

$$\|\phi\|_{L^p} \leq A(\|\nabla_\phi\|_{L^q} + \|\phi\|_{L^q}).$$

For $\phi = |\nabla^r\psi|$, $\psi \in C^\infty(M)$, we get

$$\begin{aligned} \|\nabla^r\psi\|_{L^p} &\leq A(\|\nabla|\nabla^r\psi|\|_{L^q} + \|\nabla^r\psi\|_{L^q}) \\ &\leq A(\|\nabla^{r+1}\psi\|_{L^q} + \|\nabla^r\psi\|_{L^q}). \end{aligned} \tag{1}$$

Now apply (1) for $r = k-1, k-2, \ldots, 0$, to find

$$\begin{aligned} \|\nabla^{k-1}\psi\|_{L^p} &\leq A(\|\nabla^k\psi\|_{L^q} + \|\nabla^{k-1}\psi\|_{L^q}) \\ \|\nabla^{k-2}\psi\|_{L^p} &\leq A(\|\nabla^{k-1}\psi\|_{L^q} + \|\nabla^{k-2}\psi\|_{L^q}) \\ \vdots \quad & \quad \vdots \\ \|\psi\|_{L^p} &\leq A(\|\nabla\psi\|_{L^q} + \|\psi\|_{L^q}). \end{aligned}$$

Hence,

$$\|\psi\|_{H^p_{k-1}} \leq 2A\|\psi\|_{H^q_k} \quad \text{for all } k. \tag{2}$$

Now suppose that $k - m = 2$ so that $1/p = 1/q - 2/n$. Define p_1 by $1/p_1 = 1/p + 1/n$. Notice that $1/p = 1/p_1 - 1/n = 1/q - 2/n$ so that $1/p_1 = 1/q - 1/n$ and hence, by (2), the embedding $H^q_k(M) \subset H^{p_1}_{k-1}(M)$ is continuous for all k. Now observe that $1/p = 1/p_1 - 1/n$ so that by (2) the embedding $H^{p_1}_{k-1}(M) \subset H^p_{k-2}(M)$ is continuous for all k. It follows that the embedding $H^q_k(M) \subset H^p_{k-2}(M)$ is continuous for all k. By repeating the above construction sufficiently many times, the theorem follows. $\square$

Theorem 2.3 (Sobolev inequality or Sobolev embedding theorem). *For any real numbers p, q such that $1 \leq q < n$ and integers m, k such that $0 \leq m < k$ satisfying $1/p = 1/q - (k-m)/n > 0$, the embedding $H_k^q(M) \subset H_m^p(M)$ is continuous.*

Proof. Because of Proposition 2.1 and Proposition 2.2, we only need to show that the embedding $H_1^1(M) \subset L^{n/(n-1)}(M)$ is continuous. Firstly, for $p = n/(n-1)$ and $u \in C^\infty(M)$ (we use the same notation as in the introduction in the formulas below)

$$\begin{aligned} \|u\|_{L^p(M)} &= \Big\| \sum_{l=1}^N \eta_l u \Big\|_{L^p(M)} \\ &\leq \sum_{l=1}^N \|\eta_l u\|_{L^p(M)} = \sum_{l=1}^N \bigg(\int_{\Omega_l} |\eta_l u|^p \, dv \bigg)^{1/p}, \end{aligned}$$

as $\mu_l = 0$ on Ω_l^c. Since $\eta_l u$ is supported on Ω_l and M is compact, we have

$$\int_{\Omega_l} |\eta_l u|^p \, dv(x) \leq \frac{C}{|G_l|} \int_{\tilde{\Omega}_l} |\tilde{\eta}_l \tilde{u}|^p \det \tilde{g}_{ij}^l(\tilde{x}) \, d\tilde{x} \quad \text{for some } C > 0.$$

Moreover, the inequality $a\delta_{ij} \leq \tilde{g}_{ij}^l \leq b\delta_{ij}$ for all l, i, j holds because M is compact. Hence,

$$\int_{\Omega_l} |\eta_l u|^p \, dv(x) \leq \frac{Cb^n}{|G_l|} \int_{\tilde{\Omega}_l} |\tilde{\eta}_l \tilde{u}|^p \, d\tilde{x}.$$

By [**1**, p. 39], we know that for $1 \leq q < n$, $1/p = 1/q - 1/n$ and $v \in H_1^q(\mathbf{R}^n)$:

$$\|v\|_{L^p(\mathbf{R}^n)} \leq k(n,q) \|\nabla v\|_{H_1^q(\mathbf{R}^n)}.$$

Therefore, since $\tilde{\Omega}_l \approx \mathbf{R}^n$ and $q = 1$,

$$\begin{aligned} \int_{\Omega_l} |\eta_l u|^p \, dv(x) &\leq \frac{Cb^n k(n,q)}{|G_l|} \int_{\tilde{\Omega}_l} |\nabla(\tilde{\eta}_l \tilde{u})| \, d\tilde{x} \\ &= \frac{Cb^n k(n,q)}{|G_l|} \int_{\tilde{\Omega}_l} (\tilde{\eta}_l |\nabla \tilde{u}| + |\tilde{u}||\nabla \tilde{\eta}_l|) \, d\tilde{x} \\ &\leq D(\|\nabla u\|_{L^1(M)} + \|u \nabla \eta_l\|_{L^1(M)}) \\ &\leq D(\|\nabla u\|_{L^1(M)} + B\|u\|_{L^1(M)}), \end{aligned}$$

for some $B, D > 0$. It follows that the embedding $H_1^1(M) \subset L^p(M)$, $p = (n/(n-1))$ is continuous. □

Theorem 2.4 (Rellich-Kondrakov). *Let M be a closed orientable orbifold. Then for any integers $j \geq 0$, $m \geq 0$ and any real numbers $q \geq 1$ and p such that $1 \leq p < nq(n - mq)$, the embedding of $H_{j+m}^q(M)$ into $H_j^p(M)$ is compact, i.e., bounded subsets of $H_{j+m}^q(M)$ are precompact in $H_j^p(M)$. In particular, for any $1 \leq q < n$ and $p \geq 1$ such that $1/p > 1/q - 1/n$, the embedding of $H_1^q(M)$ into $L^p(M)$ is compact.*

Proof. We will do the proof for $m = 1$, $j = 0$. Let $\{f_m\}_{m\in\mathbf{N}}$ be a bounded sequence in $H_1^q(M)$. Consider the functions (notation as in the introduction and in the proof of Theorem 2.3), $h_m^l(\tilde{x}) = (\tilde{\eta}_l \tilde{f}_m)(\tilde{x})$, defined on $\tilde{\Omega}_l \approx \mathbf{D}^n$, where $\mathbf{D}^n$ is the unit disk. The set $\mathcal{A}_l$ of these functions is bounded in $H_1^q(\mathbf{D}^n)$. Since $\mathcal{A}_l$ is precompact [**1**, p. 55], a subsequence which is Cauchy in $L^p(\mathbf{D}^n)$ exists. By repeating this operation, we may select a subsequence $\{f_{m'}\}_{m'\in\mathbf{N}}$ such that $\{\tilde{\eta}_l \tilde{f}_{m'}\}_{m'\in\mathbf{N}}$ is Cauchy in $L^p(\tilde{\Omega}_l)$ for every l. But $\{\eta_l f_{m'}\}_{m'\in\mathbf{N}}$ is also Cauchy in $L^p(M)$ for every l. Therefore, $\{f_{m'}\}_{m'\in\mathbf{N}}$ is Cauchy in $L^p(M)$ as

$$\begin{aligned}\int_M |f_{m'} - f_{n'}|^p \, dv(x) &= \int_M \Big| \sum_{l=1}^N \eta_l (f_{m'} - f_{n'})|^p \, dv(x) \\ &\leq \sum_{l=1}^N \int_M |\eta_l (f_{m'} - f_{n'})|^p \, dv(x).\end{aligned}$$

Hence the embedding $H_1^q(M) \subset L^p(M)$ is compact. Since, by the Sobolev embedding theorem, the embedding $H_k^r(M) \subset H_1^q(M)$ is continuous for $1/q = 1/r - (k-1)/n$, and the composition of a continuous embedding with a continuous and compact one yields a compact embedding, we have that the embedding $H_1^r(M) \subset L^p(M)$ is compact for $1 \geq 1/p > 1/r - 1/n > 0$. To prove the general case, one applies a similar argument, using the fact that the embedding $H_{j+m}^q(\mathbf{D}^n) \subset H_j^p(\mathbf{D}^n)$ is compact for m, q, j, p as in the statement of the theorem (cf. [**9**, p. 25]). □

3. Properties of eigenvalues.

Theorem 3.1. *Let M be a closed orientable orbifold. The eigenvalues of the Laplacian are nonnegative and form a discrete set. The eigenfunctions, corresponding to the eigenvalue $\lambda_0 = 0$, are the constant functions and* $\ker(\Delta - \lambda I)$ *is finite-dimensional for all $\lambda \in \mathbf{R}$. The first nonzero eigenvalue λ_1 is equal to μ, defined by* $\mu \underset{\text{def}}{=} \inf\{\|\nabla\varphi\|_{L^2}^2 \mid \varphi \in \mathcal{A}\}$ *where* $\mathcal{A} = \{\varphi \in H_1^2(M)$ *such that* $\|\varphi\|_{L^2} = 1$ *and* $\int_M \varphi \, dv(x) = 0\}$.

Proof. For the first two statements, see [**5**, Theorem 2.5]. To prove that $\lambda_1 = \mu$, let $\{\varphi_i\}_{i\in\mathbf{N}}$ be a sequence in $\mathcal{A}$ such that $\|\nabla\varphi_i\|_{L^2}^2 \to \mu$ for $i \to +\infty$. By Rellich-Kondrakov's theorem (Theorem 2.4), a subsequence $\{\varphi_j\}_{j\in\mathbf{N}}$ of $\{\varphi_i\}_{i\in\mathbf{N}}$ exists such that $\varphi_j \to \varphi_0$ strongly in $L^2(M)$, i.e., $\lim_{j\to+\infty} \|\varphi_j - \varphi_0\|_{L^2} = 0$. Hence $\|\varphi_j - \varphi_0\|_{L^1} \to 0$ as $j \to +\infty$ since M is closed, and then φ_0 satisfies $\|\varphi_0\|_{L^2} = 1$ and $\int \varphi_0 \, dv(x) = 0$. Since $H_1^2(M)$ is reflexive, a subsequence, which we still call $\{\phi_k\}_{k\in\mathbf{N}}$, of $\{\phi_k\}_{k\in\mathbf{N}}$ exists, such that $\phi_k \to \tilde{\phi}_0$ weakly in $H_1^2(M)$. Hence $\phi_k \to \tilde{\phi}_0$ strongly. Also,

$$\|\varphi_0\|_{H_1^2} \leq \lim_{k\to+\infty} \inf \|\varphi_k\|_{H_1^2}$$

and

$$\|\varphi_0\|_{L^2}^2 \leq \lim_{k\to+\infty} \|\nabla\varphi_k\|_{L^2}^2 = \mu,$$

so the minimum is attained.

Now writing Euler's equation of our variational problem, $\alpha, \beta \in \mathbf{C}$ exist such that, for all $\psi \in H_1^2(M)$,

$$\int_M \nabla^\nu \varphi_0 \nabla_\nu \psi \, dv(x) = \alpha \int_M \varphi_0 \psi \, dv(x) + \beta \int_M \psi \, dv(x).$$

By choosing $\psi = 1$, we get $\beta = 0$, and by choosing $\psi = \varphi_0$, we get $\alpha = \mu$. So $\varphi_0 \in H_1^2(M)$ and satisfies weakly $\Delta\varphi_0 = \mu\varphi_0$.

By regularity [**5**, Theorem 2.5], $\varphi_0 \in C^\infty(M)$. Thus μ is an eigenvalue of Δ and φ_0 an eigenfunction.

Conversely, let γ be an eigenfunction satisfying $\Delta\gamma = \lambda_1\gamma$, $\int \gamma \, dv(x) = 0$, then $\lambda_1 = \|\nabla\gamma\|_{L^2}^2 \|\gamma\|_{L^2}^{-2} \geq \mu$. □

Theorem 3.2 (Weil's asymptotic formula) (cf. [**3**, p. 156]). *Let M be a closed orientable orbifold, with eigenvalues of the Laplacian $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$, each distinct eigenvalue repeated according to its multiplicity. Then, for*

$$N(\lambda) = \sum_{\lambda_j \leq \lambda} 1,$$

we have

$$N(\lambda) \sim \omega_n V \lambda^{n/2}/(2\pi)^n,$$

as λ tends to $+\infty$. In the latter formula, V denotes the volume of M and ω_n is the volume of the unit disk in dimension n.

Proof. We wish to estimate

$$\sum_{j=1}^{+\infty} e^{-\lambda_j t},$$

as the heat kernel H is given by

$$H(x, y, t) = \sum_{j=1}^{+\infty} e^{-\lambda_j t} \phi_j(x) \phi_j(y),$$

where ϕ_j is an eigenfunction of the Laplacian, with eigenvalue λ_j, of norm one [**5**]. Let $\mathcal{O} = \{\Omega_l\}_{l=1,\dots,N}$ be an orbifold cover of M and $\mathcal{U} = \{U_l\}_{l=1,\dots,N}$ a shrinkage of $\mathcal{O}$. By [**5**], we have that on lifted orbifold charts $\tilde{\Omega}_l$, $l = 1, \dots, N$, $H(\tilde{x}, \tilde{y}, t) = K(\tilde{x}, \tilde{y}, t) + k(\tilde{x}, \tilde{y}, t)$ where $k(\tilde{x}, \tilde{y}, t)$ is a bounded G_l-invariant function and $K(\tilde{x}, \tilde{y}, t)$ is the heat kernel on $\tilde{\Omega}_l$ satisfying the boundary condition $K(\tilde{x}, \tilde{y}, t) = 0$ for $\tilde{x}$ or $\tilde{y}$ belonging to the boundary of $\tilde{U}_l$ (cf. also [**10**], [**2**]). Hence we can substitute $K + k$ for H in the above formula, and we obtain

$$\sum_{j=1}^{+\infty} e^{-\lambda_j t} \sim \frac{V}{4\pi t^{n/2}},$$

as $t \to 0$. The Karamata theorem now implies Weil's theorem [**3**], [**6**]. $\square$

4. Poincare inequalities. We will first prove Poincare's inequality for $q > 1$.

Theorem 4.1 (Poincare inequality for $q > 1$). *Let M be a closed orientable orbifold, and let $1 < q < n$ be a real number. Then, for any $u \in H_1^q(M)$, we have*

$$\left(\int_M |u - \bar{u}|^q \, dv(x) \right)^{1/q} \leq A \left(\int_M |\nabla u|^q \, dv(x) \right)^{1/q},$$

for some $A > 0$ and $\bar{u} = (1/V) \int_M u \, dv(x)$.

Proof. We only need to prove that

$$\mathrm{Inf}_{u \in \mathcal{H}} \int_M |\nabla u|^q \, dv(x) > 0,$$

where

$$\mathcal{H} = \left\{ u \in H_1^q(M) \text{ such that } \int_M |u|^q \, dv(x) = 1 \text{ and } \int_M u \, dv(x) = 0 \right\}.$$

Let $\{u_k\}_{k \in \mathbf{N}}$ be a sequence of elements of $\mathcal{H}$ such that

$$\lim_{n \to +\infty} \int_M |\nabla u_k|^q \, dv(x) = \operatorname*{Inf}_{u \in \mathcal{H}} \int_M |\nabla u|^q \, dv(x).$$

By using Rellich-Kondrakov's theorem and the fact that $H_1^q(M)$ is reflexive for $q > 1$, a subsequence $\{u_{k'}\}_{k' \in \mathbf{N}}$ of $\{u_k\}_{k \in \mathbf{N}}$ exists which converges weakly in $H_1^q(M)$ and strongly in $L^q(M) \cap L^1(M)$. Let $v = \lim_{k' \to +\infty} u_{k'}$. The strong convergence in $L^q(M) \cap L^1(M)$ implies $v \in \mathcal{H}$, while the weak convergence yields

$$\int_M |\nabla v|^q \, dv(x) \leq \lim_{k' \to +\infty} \int_M |\nabla u_{k'}|^q \, dv(x).$$

Therefore, $\mathrm{Inf}_{u \in \mathcal{H}} \int_M |\nabla u|^q \, dv(x) > 0$ is attained by v. $\square$

Theorem 4.2 (Poincare inequality for $q = 1$). *Let M be a closed orientable orbifold. Then a positive constant $A > 0$ exists such that, for any $u \in H_1^1(M)$,*

$$\left(\int_M |u - \bar{u}| \, dv(x) \right) \leq A \left(\int_M |\nabla u| \, dv(x) \right),$$

where $\bar{u} = (1/V)\int_M u\,dv(x)$.

Proof. Let $G(x,y)$ be Green's function on M, $G : M \times M \to \mathbf{R}$ and $G(x,y) \in H_2^{m+2}(M)$. By definition, G inverts Δ on the orthogonal complement of its kernel. Let $u \in C^\infty(M)$ be such that $\int_M u\,dv(x) = 0$. We then have, for any $x \in M$,

$$u(x) = \int_M \Delta G(x,y)u(y)\,dv(y).$$

By Green's formulas

$$|u(x)| \leq \int_M |\nabla_y G(x,y)||\nabla u(y)|\,dv(y),$$

and so

$$\int_M |u(x)|\,dv(x) \leq \int_M \int_M |\nabla_y G(x,y)||\nabla u(y)|\,dv(x)\,dv(y).$$

To finish the proof, we will show that there is a $C > 0$ such that

$$\int_M |\nabla_y G(x,y)|\,dv(y) \leq C.$$

The Green function G can be defined in the following way. Let $H(x,y,t)$ be the fundamental solution of the heat equation. Then [**5**]

$$H(x,y,t) = \sum_{i=0}^{+\infty} e^{-\lambda_i t}\phi_i(x)\phi_i(y),$$

where $\{\phi_j\}$ is an orthonormal base for $L^2(M)$ and $\Delta\phi_j = \lambda_j\phi_j$. Now

$$G = \int_0^{+\infty} e^{-tH}\,dt.$$

First of all, we will prove that the latter integral converges. Let $\mathcal{O} = \{\Omega_l\}_{l=1,\dots,N}$ be an orbifold cover of M and $\mathcal{U} = \{U_l\}_{l=1,\dots,N}$ a shrinkage of $\mathcal{O}$. By [**5**], we have that on lifted orbifold charts $\tilde{\Omega}_l$, $l = 1,\dots,N$, $H(\tilde{x},\tilde{y},t) = K(\tilde{x},\tilde{y},t) + k(\tilde{x},\tilde{y},t)$ where $k(\tilde{x},\tilde{y},t)$ is a bounded G_l-invariant function with bounded derivatives (cf. also [**2**] and [**10**]), and $K(\tilde{x}, \tilde{y}, t)$ is the heat kernel on $\tilde{\Omega}_l$ satisfying the boundary condition $K(\tilde{x}, \tilde{y}, t) = 0$ for $\tilde{x}$ or $\tilde{y}$ belonging to the boundary of $\tilde{U}_l$. Hence the integral $\int_0^{+\infty} e^{-tH}\, dt$ converges, as $\int_0^{+\infty} e^{-tK}\, dt$ and $\int_0^{+\infty} e^{-tk}\, dt$ both converge. An easy computation then shows that $G(\phi_j) = \lambda_j^{-1}\phi_j$ which ends the proof of the integral formula for G.

Therefore, we can locally estimate ∇G on $\tilde{U}_l$ by using the standard manifold estimates which yield up to a constant arising from the k-term. Hence,

$$|\nabla_{\tilde{y}} G(\tilde{x}, \tilde{y})| \leq \frac{C}{\tilde{r}^{n-1}},$$

where $\tilde{r}$ is the distance of $\tilde{x}$ from $\tilde{y}$.

Hence, for $\tilde{x} \in \tilde{U}_l$,

$$\int_{\tilde{U}_l} |\nabla_{\tilde{y}} \tilde{G}(\tilde{x}, \tilde{y})|\, d\tilde{y},$$

is bounded, and so is $\int_M |\nabla_y G(x, y)|\, dv(x)$. Now the proof can be completed by applying compactness. □

A sharper Sobolev-Poincare's inequality can be obtained by combining Theorems 2.3 and 4.2.

Corollary 4.3 (Poincare-Sobolev inequality). *Let M be a closed orientable orbifold. Then, for any $u \in H_1^1$, we have*

$$\|u - \bar{u}\|_{L^{n/(n-1)}} \leq B\|\nabla u\|_{L^1}$$

for some $B > 0$ depending only on n.

Proof. By Theorem 2.3,

$$\begin{aligned} \|u - \bar{u}\|_{L^{n/(n-1)}} &\leq A\|u - \bar{u}\|_{H_1^1} \\ &= A\left(\int_M |u - \bar{u}|\, dv(x) + \int_M |\nabla u|\, dv(x)\right), \end{aligned}$$

for some $A > 0$.

Now, by applying Theorem 4.2, we get

$$\|u-\bar{u}\|_{L^{n/(n-1)}} \leq A(A'+1)\|\nabla u\|_{L^1}$$

for some $A'>0$. $\square$

5. More Sobolev inequalities. In this section we will present two additional Sobolev inequalities (Lemmas 5.1 and 5.2). We will then derive from Lemma 5.1 an estimate from below on the first eigenvalue of the Laplacian (Proposition 5.3). Our proof parallels the one given by Li in the manifold case [**11**].

Lemma 5.1. *Let M be a closed orientable orbifold of dimension $n\geq 3$. Then for some $D>0$ we have*

$$D\|f\|^2_{L^{2n/(n-2)}} \leq \|\nabla f\|^2_{L^2},$$

for all $f\in H^2_1(M)$ such that $\int_M \operatorname{sgn}(f)|f|^{2/(n-2)}\,dv(x)=0$.

Proof. Consider the function $g=\operatorname{sgn}(f)|f|^{[2(n-1)/(n-2)]}$. By hypothesis $\int_M \operatorname{sgn}(g)|g|^{1/(n-1)}\,dv(x)=0$. By a variational argument, the last equality implies

$$\|g\|_{L^{n/(n-1)}} = \inf_{a\in\mathbf{R}} \|g-a\|_{L^{n/(n-1)}}.$$

Hence, by Corollary 4.3,

$$\begin{aligned}\|f\|^{[2n(n-1)/(n-2)]}_{L^{2n/(n-2)}} &= \|g\|^n_{L^{n/(n-1)}}\\ &\leq B^n\|\nabla g\|^n_{L^1}\\ &= B^n\left\|\frac{2(n-1)}{n-2}f^{n/(n-2)}\nabla f\right\|^n_{L^1}.\end{aligned}$$

By Holder's inequality,

$$\|f\|^{[2n(n-1)/(n-2)]}_{L^{2n/(n-2)}} \leq B^n\frac{2(n-1)}{n-2}\|f\|^{n^2/(n-2)}_{L^{2n/(n-2)}}\|\nabla f\|^n_{L^2}.$$

Now, by dividing both sides by $\|f\|_{L^{2n/(n-2)}}^{n^2/(n-2)}$, we obtain

$$\|f\|_{L^{2n/(n-2)}}^{n} \le B^n \frac{2(n-1)}{n-2} \|\nabla f\|_{L^2}^{n}. \qquad \square$$

Lemma 5.2. *Let M be a closed orientable orbifold of dimension $n \ge 3$. Then for some $E > 0$ depending only on n, we have*

$$E\|\nabla f\|_{L^2}^2 \ge [\|f\|_{L^{2n/(n-2)}}^2 - V^{-(2/n)}\|f\|_{L^2}^2],$$

for all $f \in H_1^2(M)$. (V is the volume of M.)

Proof. First note that, for some $k \in \mathbf{R}$,

$$\int_M \operatorname{sgn}(f-k)|f-k|^{2/(n-2)}\, dv(x) = 0.$$

(We can assume $k > 0$, otherwise we are done by Lemma 5.1.) Now, by Lemma 5.1,

$$(*) \qquad \begin{aligned} \|\nabla f\|_{L^2}^2 &\ge D\|f-k\|_{L^{2n/(n-2)}}^2 \\ &\ge D\Big[2^{-(n+2)/(n-2)}\|f\|_{L^{2n/(n-2)}}^{2n/(n-2)} - Vk^{2n/(n-2)}\Big]^{(n-2)/n} \\ &\ge D\Big[2^{-(n+2)/n}\|f\|_{L^{2n/(n-2)}}^2 - V^{(n-2)/n}k^2\Big]. \end{aligned}$$

Set $M_+ = \{x \in M | (f-k)(x) > 0\}$ and $M_- = \{x \in M | (f-k)(x) < 0\}$. Then

$$\int_{M_+} |f-k|^{2/(n-2)}\, dv(x) = \int_{M_-} |f-k|^{2/(n-2)}\, dv(x).$$

But

$$\int_{M_+} |f-k|^{2/(n-2)}\, dv(x) \le 2^\alpha \int_{M_+} f^{2/(n-2)}\, dv(x) - V_+ k^{2/(n-2)},$$

where $\alpha = 3$ if $n = 3, 4$, and $\alpha = (n-2)/(n-4)$ if $n \ge 5$. Also

$$\int_{M_-} |f-k|^{2/(n-2)}\, dv(x) \ge 2^\beta V_- k^{2/(n-2)} - \int_{M_-} |f|^{2/(n-2)}\, dv(x),$$

where $\beta = (n-4)/(n-2)$ if $n = 3, 4$ and $\beta = 0$ if $n \geq 4$.

Therefore, since $\int_{M_+} |f-k|^{2/(n-2)}\, dv(x) = \int_{M_-} |f-k|^{2/(n-2)}\, dv(x)$,

$$2^\alpha \int_{M_+} f^{2/(n-2)}\, dv(x) - V_+ k^{2/(n-2)} \geq 2^\beta V_- k^{2/(n-2)} - \int_{M_-} |f|^{2/(n-2)}\, dv(x),$$

which implies

$$\begin{aligned}\int_{M_-} |f|^{2/(n-2)}\, dv(x) &+ 2^\alpha \int_{M_+} |f|^{2/(n-2)}\, dv(x)\\ &\geq 2^\beta V_- k^{2/(n-2)} + V_+ k^{2/(n-2)} = (2^\beta V_- + V_+) k^{2/(n-2)}.\end{aligned}$$

Hence,

$$\|f\|_{L^{2/(n-2)}}^{2/(n-2)} \geq 2^{-\alpha+\beta} V k^{2/(n-2)}.$$

By applying Holder's inequality, we obtain

$$\begin{aligned}\|f\|_{L^2}^{2/(n-2)} V^{(n-3)/(n-2)} &\geq 2^{-\alpha+\beta} V k^{2/(n-2)},\\ \|f\|_{L^2}^2 V^{(n-3)} &\geq 2^{(-\alpha+\beta)(n-2)} V^{(n-2)} k^2.\end{aligned}$$

Combining this last inequality with $(*)$ ends the proof. □

Proposition 5.3. *Let M be a closed orientable orbifold of dimension $n \geq 3$. Then*

$$\lambda_1 \geq \frac{V^{-2/n} A}{(n-1)^2},$$

where λ_1 is the first nonzero eigenvalue of the Laplacian on $L^2(M)$, V is the volume of M, A is equal to D (of Lemma 5.1) for $n \geq 5$, while $A = B^{-2}$ (of Corollary 4.3) for $n = 3, 4$.

Proof. Let $f \in L^2(M)$ satisfy $\Delta f = \lambda_1 f$ with $\int_M f = 0$. Suppose first that $n = 3$ or 4. Consider the function $g = \operatorname{sgn}(f)|f|^{n-1}$. Clearly $\int_M \operatorname{sgn}(g)|g|^{1/(n-1)}\, dv(x) = 0$. Therefore, by a variational argument,

$$\|g\|_{L^{(n-1)/n}} = \inf_{a \in \mathbf{R}} \|g - a\|_{L^{n/(n-1)}}.$$

By Corollary 4.3,

$$\begin{aligned}\|f\|_{L^n}^{n(n-1)} = \|g\|_{L^{n/(n-1)}}^n &\leq B^n \|\nabla g\|_{L^1}^n \\ &= (n-1)^n B^n \|f^{n-2}\nabla f\|_{L^1}^n.\end{aligned}$$

Moreover, by Holder's inequality and the eigenfunction property of f,

$$\|f^{n-2}\nabla f\|_{L^1}^n \leq \|f\|_{L^n}^{n(n-1)} V \lambda_1^{n/2}.$$

Hence,

$$\lambda_1 \geq \frac{V^{-2/n} B^{-2}}{(n-1)^2}.$$

Now let $n \geq 5$, and define the function $g = \operatorname{sgn}(f)|f|^{(n-2)/2}$. Clearly $\int_M \operatorname{sgn}(g)|g|^{2/(n-2)}\, dv(x) = 0$. Hence, by Lemma 5.1,

$$(**) \qquad D\|f\|_{L^n}^{n-2} = D\|g\|_{L^{2n/(n-2)}}^2 \leq \|\nabla g\|_{L^2}^2.$$

Moreover,

$$\|\nabla g\|_{L^2}^2 = \int_{M_+} |\nabla f^{(n-2)/2}|^2 + \int_{M_-} |\nabla |f|^{(n-2)/2}|^2,$$

where $M_+ = \{x \in M \mid f(x) > 0\}$ and $M_- = \{x \in M \mid f(x) < 0\}$. Since $f^{(n-2)/n} = 0$ on ∂M_+ and ∂M_-, integration by parts yields

$$\|\nabla g\|_{L^2}^2 = \int_{M_+} f^{(n-2)/2} \Delta(f^{(n-2)/2}) + \int_{M_-} |f|^{(n-2)/2} \Delta |f|^{(n-2)/2}.$$

Now

$$\begin{aligned}\Delta(f^{(n-2)/2}) &= -\frac{(n-2)}{2}\frac{(n-4)}{2} f^{(n-6)/2}|\nabla f|^2 + \frac{(n-2)}{2} f^{(n-4)/2}\Delta f \\ &= \frac{(n-2)}{2}\left[-\frac{(n-4)}{2} f^{(n-6)/2}|\nabla f|^2 + \lambda_1 f^{(n-2)/2}\right].\end{aligned}$$

Therefore,

$$\begin{aligned}\|\nabla g\|_{L^2}^2 &= \int_{M_+} f^{(n-2)/2}\Delta(f^{(n-2)/2}) + \int_{M_-} |f|^{(n-2)/2}\Delta|f^{(n-2)/2}| \\ &= \int_{M_+} f^{(n-2)/2}\bigg\{\frac{(n-2)}{2}\bigg[-\frac{(n-4)}{2}f^{(n-6)/2}|\nabla f|^2 \\ &\qquad\qquad + \lambda_1 f^{(n-2)/2}\bigg]\bigg\} \\ &\quad + \int_{M_-} |f|^{(n-2)/2}\bigg\{\frac{(n-2)}{2}\bigg[-\frac{(n-4)}{2}|f|^{(n-6)/2}|\nabla f|^2 \\ &\qquad\qquad + \lambda_1 |f|^{(n-2)/2}\bigg]\bigg\} \\ &\le \frac{(n-2)}{2}\lambda_1\|f\|_{L^{n-2}}^{n-2}.\end{aligned}$$

By substituting into $(**)$, one obtains

$$\frac{(n-1)}{2}\lambda_1\|f\|_{L^{n-2}}^{n-2} \ge D\|f\|_{L^n}^{n-2}.$$

Finally, by using Holder's inequality, one gets

$$\lambda_1 V^{2/n}\frac{(n-2)}{2} \ge D. \qquad \square$$

6. Eigenvalues estimates. We will first prove a weaker form of a Sobolev inequality and then use it to give an estimate from below of the eigenvalues of the Laplacian. In the manifold case such estimates were proved in [**4**].

Lemma 6.1. *Let M be a closed orientable orbifold of dimension $n \ge 3$. Then a constant K exists such that*

$$\int_M |\nabla f|^2\, dv(x) \ge K\bigg(\int_M |f|^2\bigg)^{(n+2)/n}\bigg(\int_M |f|\bigg)^{-4/n}$$

for any $f \in H_2^1(M)$ such that $\int_M f\, dv(x) = 0$.

Proof. By Lemma 5.2, a constant $E > 0$ exists such that

$$\|\nabla f\|_{L^2}^2 \geq E[\|f\|_{L^{2n/(n-2)}}^2 - V^{-2/n}\|f\|_{L^2}^2],$$

where V is the volume of M. Since $\int_M f\,dv(x) = 0$, by Theorem 3.1

$$\|\nabla f\|_{L^2}^2 \geq \lambda_1 \|f\|_{L^2}^2,$$

where λ_1 is the first nonzero eigenvalue of the Laplacian. Moreover, by Proposition 5.3,

$$\lambda_1 \geq M V^{-2/n},$$

for some $M > 0$. Therefore, by combining the above inequalities, we get

$$\begin{aligned}\|\nabla f\|_{L^2}^2 &\geq E[\|f\|_{L^{2n/(n-2)}}^2 - V\lambda_1^{-1}\|\nabla f\|_{L^2}^2] \\ &\geq E[\|f\|_{L^{2n/(n-2)}}^2 - V^{1+(2/n)}M^{-1}\|\nabla f\|_{L^2}^2].\end{aligned}$$

That is,

$$\|\nabla f\|_{L^2}^2 \geq K\|f\|_{L^{2n/(n-2)}},$$

for some $K > 0$. By Holder's inequality, with $p = (n-2)/(n+2)$ and $q = -(n-2)/4$, we get

$$\begin{aligned}\|f\|_{L^{2n/(n-2)}} &= \left(\int_M |f|^{(2n+4)/(n-2)-(4/(n-2))}\right)^{(n-2)/(2n)} \\ &\geq \left(\int_M |f|^2\,dv(x)\right)^{(n+2)/(2n)} \left(\int_M |f|\right)^{-(2/n)}. \qquad \square\end{aligned}$$

Theorem 6.2. *Let M be a closed orientable orbifold of dimension $n \geq 2$. Let λ_k be the k-th nonzero eigenvalue of the Laplacian. Then a constant α exists such that*

$$\lambda_k \geq \alpha K \frac{k}{V},$$

where K is as in Lemma 6.1.

Proof. Let $H(x,y,t)$ be the fundamental solution of the heat equation. Then [**5**]

$$H(x,y,t) = \sum_{i=0}^{+\infty} e^{-\lambda_i t}\phi_i(x)\phi_i(y),$$

where $\{\phi_j\}$ is an eigenfunction orthonormal base for $L^2(M)$. Since $\lambda_0 = 0$ with $\phi_0 = V^{-1/2}$, we can define the function $G(x, y, t)$ by

$$G(x,y,t) = H(x,y,t) - \frac{1}{V} = \sum_{i=1}^{+\infty} e^{-\lambda_i t}\phi_i(x)\phi_i(y).$$

By [**5**, Proposition 4.1], H and G converge uniformly together with their derivatives of all orders for $t > 0$. H and G are also C^∞ functions in all their arguments. Clearly G satisfies the semigroup property

$$G(x,y,s+t) = \int_M G(x,z,s)G(z,y,t)\,dv(z).$$

Moreover, as $\int_M \phi_j(y)\,dv(y) = 0$ for $i \geq 1$, we have $\int_M G(x,y,t)\,dv(y) = 0$. Also

$$\int_M |G(x,y,t)|\,dv(y) \leq \int_M (|H(x,y,t)| + V^{-1})\,dv(y) < +\infty,$$

as $\int_M |H| < +\infty$. By the semigroup property,

$$\begin{aligned} G'(x,x,t) &= \int_M G'(x,y,t/2)G(x,y,t/2)\,dv(y) \\ &= \int_M \Delta_y G(x,y,t/2)G(x,y,t/2)\,dv(y). \end{aligned}$$

Integration by parts yields

$$G'(x,x,t) = -\int_M |\nabla G(x,y,t/2)|^2\,dv(y).$$

By Lemma 6.1, we obtain

$$\begin{aligned} -G'(x,x,t) &\geq K\bigg(\int_M |G|^2\bigg)^{(n+2)/n}\bigg(\int_M |G|\bigg)^{-4/n} \\ &\geq KQ^{-4/n}\bigg(\int_M |G(x,y,t/2)|^2\,dv(y)\bigg)^{(n+2)/n}, \end{aligned}$$

for some $Q > 0$. Now the semigroup property of G implies

$$-G'(x,x,t)(G(x,x,t))^{-(n+2)/n} \geq Q^{-4/n}K.$$

Integrating both sides of the above inequality and using the fact that $G(x,x,t) \to \infty$ as $t \to 0$, we have

$$(n/2)G^{-2/n}(x,x,t) \geq Q^{-4/n}Kt.$$

Hence,

$$G(x,x,t) \leq \left(\frac{2}{n}Q^{-4/n}Kt\right)^{-n/2} = Q^2K^{-n/2}t^{-n/2}\left(\frac{n}{2}\right)^{n/2}.$$

Integrating both sides with respect to x and using the expansion formula for G, we have

$$\sum_{i=0}^{+\infty} e^{-\lambda_i t} \leq \alpha(Kt)^{-n/2}V.$$

Let $t = 1/\lambda_k$. Since $\lambda_i/\lambda_k \leq 1$ for $i \leq k$, we have

$$\alpha\left(\frac{K}{\lambda_k}\right)^{-n/2} V \geq \sum_{i=1}^{+\infty} e^{-\lambda_i/\lambda_k} \geq \sum_{i=1}^{k} e^{-\lambda_i/\lambda_k} \geq ke^{-1},$$

which proves the theorem. $\square$

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