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A CYCLIC ELEMENT CHARACTERIZATION OF MONOTONE NORMALITY

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ABSTRACT. A subcontinuum g of a locally connected continuum X is a cyclic element of X provided that g is maximal with respect to the property that no point separates it. In an earlier paper, Cornette showed that a locally connected continuum is the continuous image of an arc if and only if each cyclic element of X is the continuous image of an arc. In this paper we prove the analogous theorem for monotonically normal continua by showing that a locally connected continuum X is monotonically normal if and only if each cyclic element of X is monotonically normal.

Definition. A continuum is a compact connected Hausdorff space. A continuum is called an arc provided that it is a nondegenerate ordered continuum.

Notation. If $S \subset X$, $\mathrm{Int}_X(S)$ will denote the interior of S with respect to X or simply $\mathrm{Int}\,(S)$ if the superspace is clear. Similarly, $\partial_X(S)$ or $\partial(S)$ will denote the boundary of S with respect to X.

Definition. A cyclic element C of a locally connected continuum X is a subcontinuum of X that is maximal with respect to the property that no point separates C. If a cyclic element C of X is nondegenerate, C is said to be a true cyclic element of X. A subset A of X is an A-set of X provided that $X - A = \cup G_i$, where each G_i is open in X, $G_i \cap G_j = \varnothing$ for $i \neq j$, $\partial(G_i)$ contains at most one point, and where if C is an open cover of X then all but a finite number of the G_i lie in some element of C. For any two distinct points a and b of X, the intersection of all A-sets in X containing a and b is called the cyclic chain from a to b and is denoted by $C(a, b)$.

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The reader is referred to Whyburn [**19**] for a complete treatment of the notions in the previous definition.

Definition. A Hausdorff topological space X is said to be monotonically normal (see [**1, 2**] and [**5**]), provided that there exists a function G which assigns, to each point $x \in X$ and each open set U of X containing x, an open set $G(x,U)$ such that

(1) $x \in G(x,U) \subset U$,

(2) if U' is open and $x \in U \subset U'$, then $G(x,U) \subset G(x,U')$,

(3) if x and y are distinct points of X, then $G(x, X-y) \cap G(y, X-x) = \varnothing$.

Such a function G is called a monotone normality operator on X.

Our goal is to prove the following.

Theorem 1. *If X is a locally connected continuum, then X is monotonically normal if and only if each cyclic element of X is monotonically normal.*

In [**4**], the first author has shown the following:

Theorem. *Let X be a locally connected continuum such that each cyclic element of X has a separable G_δ boundary in X. Then X is monotonically normal if and only if each cyclic element of X is monotonically normal.*

Our Theorem 1 is thus an improvement of the above result and is a natural analogue to the following result of Cornette [**3**].

Theorem. *If X is a locally connected continuum, then X is the continuous image of an arc if and only if each cyclic element of X is the continuous image of an arc.*

The interest in the connection between monotone normality and continua which are an IOK (the continuous image of a compact ordered space) stems from results by Heath, Lutzer, and Zenor [**5**] and also a question of Nikiel [**8**] in which he asks if every monotonically normal compactum is an IOK. Some partial results are as follows. In [**9**], Nikiel, Treybig and Tuncali have shown that if X is monotonically normal, or rim-metrizable, or rim-scattered, and for each pair a, b of distinct points of X there is a continuous onto map $f : X \to [c, d]$ so that $f(a) = c$, $f(b) = d$ and $[c, d]$ is a nonmetrizable arc, then X is an IOC (the continuous image of an arc). Also, Rudin has shown that any separable monotonically normal compactum is an IOK ([**11**] and [**12**]) and that any first countable monotonically normal compactum is an IOK [**13**]. Other related results in this area are by Mardesic [**6**], Nikiel [**7**], Ostaszewski [**10**], Simone [**14**], Treybig [**15**] and [**16**], Tymchatyn [**17**] and Ward [**18**].

We now proceed to the proof of Theorem 1.

Suppose first that X is monotonically normal. Heath, Lutzer and Zenor [**5**] have shown that each subspace of a monotonically normal space is monotonically normal; therefore, each cyclic element of X is monotonically normal.

Now suppose that each cyclic element Q of X has a monotone normality operator L_Q. We also select a well-ordering W of X.

Let $a \in X$, and let U be an open set containing a. We proceed to define a monotone normality operator $H(a, U)$.

Case 1. Let $a \in \mathrm{Int}\,(K)$, where K is a true cyclic element of X. We define $H(a, U)$ to be $L_K(a, \mathrm{Int}_X(U \cap K)$.

Case 2. Suppose $a \notin \mathrm{Int}\,(K)$ for any true cyclic element K of X.

Before proceeding, we prove the following lemma.

Lemma 1. *If $C = \{C_\alpha : \alpha \in A\}$ is the collection of all components of $X - \{a\}$ which intersect $X - U$, then C is finite.*

Proof. Assume that there exist infinitely many distinct components $C_1, C_2, C_3, \dots$ in C. For each C_i, there exists a connected subset Q_i of C_i so that $a \in \overline{Q_i}$, $Q_i \subset U$, and $\overline{Q_i} \cap \partial U \neq \varnothing$. Let $f : \overline{U} \to [0,1]$ be a continuous map so that $f(a) = 0$ and $f(\partial U) = 1$. For each i, let $x_i \in Q_i \cap f^{-1}((3/8, 5/8))$, and let x denote a limit point of the x_i. There exists a connected open set U' such that $x \in U' \subset f^{-1}((3/8, 5/8))$. Since U' contains two x_i, say x_I and x_J, then it follows that C_I and C_J are subsets of the same components of $X - a$, which is a contradiction. □

Let $Y_1 = \{G_\beta : \beta \in B\}$ be the set of all true cyclic elements of X so that $a \in G_\beta$. Let $Y_2 = \{G_\beta : \beta \in B'\}$ denote those $G_\beta \in Y_1$ so that the component of $X - \{a\}$ containing $G_\beta - \{a\}$ is not a subset of U. By Lemma 1, $Y_2 = \{G_\beta : \beta \in B'\}$ is finite so that the elements thereof may be labeled $G_1, G_2, \dots, G_n$.

For a given $G_i \in Y_2$, $1 \leq i \leq n$, let G_i' denote the set of all z such that there is a component Z of $X - \{z\}$ such that $Z \not\subset U$, $Z \cap G_i = \varnothing$, and $z \in (G_i - \{a\}) \cap U$. It follows from an argument similar to that of Lemma 1 that the only limit points of G_i' are in $\partial(U)$, and that given such a z there are only finitely many such sets Z so that Z is a component of $X - \{z\}$.

Now consider the set S of all cyclic chains $C(a,b)$, where

(1) each $C(a,b)$ is the intersection of all A-sets containing a and b,

(2) $C(a,b) = \{a,b\} \cup E(a,b) \cup (\cup\{E_\alpha^{ab}\})$, where $E(a,b) = \{x \in X : x \text{ separates } a \text{ from } b\}$ and each $\{E_\alpha^{ab}\}$ is a true cyclic element of X having exactly two points common with $\{a,b\} \cup E(a,b)$,

(3) $C(a,b)$ contains no $G_\alpha \in Y_1$ containing a, and

(4) $C(a,b)$ is not a subset of U.

Let $S' = \{C(a,b_1), C(a,b_2), \dots\}$ be a maximal subset of S where $C(a,b_i) \cap C(a,b_j) = \{a\}$ if $i \neq j$. By Lemma 1, S' is finite; so let $S' = \{C(a,b_1), C(a,b_2), \dots, C(a,b_p)\}$.

Now, for each fixed b_i, $1 \leq i \leq p$, let I_{b_i} denote all the elements of S which have a nontrivial segment $[a,t)$, $t \in E(a,b_i)$, common with $C(a,b_i)$. Such a segment $[a,t)$ is the component of $C(a,b_i) - \{t\}$ which contains $\{a\}$. Now suppose that, for each $t \in E(a,b_i)$, there exists an element $C(a,x)$ of I_{b_i} so that $C(a,x) \cap C(a,b_i) \subset [a,t)$. An argument similar to that of Lemma 1 shows that a contradiction results. Therefore, there is a first point w_i in the well-ordering W so that $w_i \in \{a,b_i\} \cup E(a,b_i)$ and $[a,w_i) \subset C(a,b)$ for all $C(a,b)$ in I_{b_i} and $[a,w_i) \subset U$.

We now let $C'(a,b_i)$ denote the union of all sets Z so that there exists $z \in (a,w_i)$ so that Z is a component of $X - \{z\}$ which is a subset of U, and which consequently does not intersect any $C(a,b)$ in I_{b_i}. For each G_i, $1 \leq i \leq n$, let A_{a_i} denote the collection of all sets Z such that there exists $z \in \partial(G_i) \cap L_{G_i}(a,(U \cap G_i) - G_i') - \{a\}$ such that Z is a component of $X - \{z\}$ which does not intersect G_i and is a subset of U. Let C_α denote the set of all components of $X - \{a\}$ which are subsets of U.

We now define $H(a,U)$ by

$$H(a,U) = \left(\bigcup_{i=1}^{n} L_{G_i}(a,(U \cap G_i) - G_i')\right) \cup \left(\bigcup_{i=1}^{n} A_{a_i}\right) \cup (\cup C_a)$$
$$\cup\left(\bigcup_{i=1}^{p}[a,w_i)\right) \cup \left(\bigcup_{i=1}^{p} C'(a,b_i)\right).$$

Since each of the sets in the unions above is a subset of U, then clearly $a \in H(a,U) \subset U$. We now show that $H(a,U)$ is open. Suppose that $p \in H(a,U)$ but that p is a limit point of the subset L of $X - H(a,U)$ where, without loss of generality, we assume $L \subset U$.

Case A. Suppose $p \neq a$.

Case A_1. Suppose $p \in Z \in C_a$. Let Q be a connected open set so that $p \in Q \subset U - \{a\}$. Let $l \in (L \cap Q)$. But $l \in Q \subset Z \subset H(a,U)$, which is a contradiction.

Case A_2. Suppose $p \in L_{G_i}(a,(U \cap G_i) - \underline{G_i'})$ for some i. Let V denote a connected open set containing p where $\overline{V}$ contains no point of $G_i' \cup \{a\} \cup (X - U) \cup (\cup_{j \neq i} G_j) \cup (\cup_{j=1}^{p} C(a,b_j))$. There is a component Z of $V - G_i$ which has a limit point z in G_i and contains a point of L. If the component Q of $X - \{z\}$ which contains Z is not a subset of

U, then $z \in G'_i$, which is a contradiction. The fact that $Q \subset U$ implies that $Q \in A_{a_i}$ and that $Q \subset H(a, U)$, a contradiction.

Case A$_3$. Suppose $p \in [a, w_i)$ for some i. Let V denote a connected open set containing p so that $\overline{V} \cap C(a, b_i) \subset [a, w_i)$, $\overline{V} \cap G_i = \varnothing$ for $1 \le i \le n$, $\overline{V} \cap C(a, b_j) = \varnothing$ for $j \ne i$, and $\overline{V} \subset U$. Let $l \in (V \cap L)$, and let Z be the component of $V - [a, b_i)$ containing l. There is a limit point z of Z in (a, w_i). Let Z' be the component of $X - \{z\}$ containing Z. If $Z' \not\subset U$, we obtain another chain in $C(a, x_s)$ in I_{b_i}, contradicting the properties of $[a, b_i)$. If $Z' \subset U$, then $Z' \in C'(a, b_i)$ and $l \in Z' \subset H(a, U)$, a contradiction.

Case B. Suppose $p = a$. Let V denote a connected open set containing a such that

(1) $\overline{V} \subset U$,

(2) $(V \cap G_i) \subset L_{G_i}(a, (U \cap G_i) - G'_i)$ for $1 \le i \le n$, and

(3) $(V \cap C(a, b_i)) \subset [a, w_i)$ for $1 \le i \le p$.

Let $l \in (V \cap L)$, and let C denote an open cover of V such that the closures of the elements of C are connected subsets of V. There is a finite chain $V_1, V_2, \ldots, V_q$ of elements of C so that $l \in V_1$ and V_q contains a point of $(\cup_{i=1}^n G_i) \cup (\cup_{i=1}^p [a, w_i))$. Since $\cup_{i=1}^q \overline{V}_i$ contains no w_i, then the component Z of $(V_1 \cup V_2 \cup \cdots \cup V_q) - ((\cup_{i=1}^n G_i) \cup (\cup_{i=1}^p [a, w_i)))$ that contains l has a limit point z in $(\cup_{i=1}^n G_i) \cup (\cup_{i=1}^p [a, w_i))$. Let Z' be the component of $X - \{z\}$ which contains Z.

Case B$_1$. Suppose $z = a$. If $Z' \subset U$, then $Z' \subset C_a$ and $l \in H(a, U)$, which is a contradiction. If $Z' \not\subset U$, there exists a $C(a, x_i)$ which should be in S', another contradiction.

Case B$_2$. Suppose $z \ne a$. If $z \in G_i$ for some i, then $z \notin G'_i$. Therefore, $Z' \subset U$, $Z' \in A_{a_i}$ and $l \in H(A, U)$, which is a contradiction. If $z \in [a, w_i)$ for some i, then $Z' \subset U$ implies $Z' \in C'(a, b_i)$ and $l \in Z' \subset H(a, U)$, which is a contradiction. If, in this case, $Z' \not\subset U$, there exists a chain $C(a, x_i)$ whose intersection with $C(a, b_i)$ is $[a, z)$, which contradicts the definition of $[a, w_i)$.

Thus $H(a,U)$ is open in X.

Now suppose that U' is also an open set such that $a \in U \subset U'$. We show that $H(a,U) \subset H(a,U')$, i.e., that H is monotone.

Case A. Let $a \in \mathrm{Int}\,(K)$ where K is a true cyclic element of X. Then, by our definitions above, $H(a,U) = L_K(a, \mathrm{Int}_X(U \cap K)) \subset L_K(a, \mathrm{Int}_X(U' \cap K)) = H(a,U')$.

Case B. Suppose $a \notin \mathrm{Int}\,(K)$ for any true cyclic element K of X. Without loss of generality, let $G_1, G_2, \ldots, G_n$ be labeled $G_1, G_2, \ldots, G_{n_0}, \ldots, G_n$, where the component of $X - \{a\}$ containing any one of $G_1 - \{a\}, \ldots, G_{n_0} - \{a\}$ is not a subset of U' and the component of $X - \{a\}$ containing any one of $G_{n_0+1} - \{a\}, \ldots, G_n - \{a\}$ is a subset of U'. For each i, $1 \leq i \leq n_0$, let G''_i denote the set of all $z \in G_i - \{a\}$ such that there is a component Z of $X - \{z\}$ which does not contain $G_i - \{z\}$ and is not a subset of U'. Let A'_{a_i} denote the set of all sets Y such that there exists $y \in \partial(G_i) \cap L_{G_i}(a, (U' \cap G_i) - G''_i) - \{a\}$ where Y is a component of $X - \{y\}$ which does not intersect G_i and is a subset of U'. We let T denote the set of all $C(a,b)$ in S such that $C(a,b) \not\subset U'$, and suppose that for some q, $1 \leq q \leq p$, that $(\cup I_{b_i}) \not\subset U'$ for $1 \leq i \leq q$ and $(\cup I_{b_i}) \subset U'$ for $(q+1) \leq i \leq p$. Also, without loss of generality, we may assume that $C(a,b_1), \ldots, C(a,b_q)$ are not subsets of U'. For each i with $1 \leq i \leq q$, we let I'_{b_i} denote the set of all $C(a,x)$ in T so that $C(a,x)$ has a nontrivial segment $[a,t)$ common with $C(a,b_i)$, $t \in E(a,b_i)$. As in the case of w_i, there is a first point w'_i in the well-ordering W of X so that $w'_i \in E(a,b_i)$, $[a,w'_i) \subset C(a,b)$ for all $C(a,b)$ in I'_{b_i}, and $[a,w'_i) \subset U'$. For each $i = 1, \ldots, q$, let $C''(a,b_i)$ denote the union of all sets Z so that there exists $z \in (a,w'_i)$ so that Z is a component of $X - \{z\}$ which is a subset of U' and does not intersect any $C(a,b)$ in I'_{b_i}. We let C'_a denote the set of all components of $X - \{a\}$ which are subsets of U'.

We therefore find that

$$H(a,U') = \left(\bigcup_{i=1}^{n_0} L_{G_i}(a, (U' \cap G_i) - G''_i)\right) \cup \left(\bigcup_{i=1}^{n} A'_{a_i}\right) \cup (\cup C'_a)$$
$$\cup \left(\bigcup_{i=1}^{p} [a, w'_i)\right) \cup \left(\bigcup_{i=1}^{p} C''(a,b_i)\right).$$

Clearly, $(\cup_{i=1}^{n_0} L_{G_i}(a,(U\cap G_i)-G_i'))\subset(\cup_{i=1}^{n_0} L_{G_i}(a,(U'\cap G_i)-G_i''))$ and $(\cup_{i=n_0+1}^{n} L_{G_i}(a,(U\cap G_i)-G_i')-G_i'))\subset(\cup C_a')$. Also, each component of $X-\{a\}$ which is a subset of U is also a subset of U', so $(\cup C_a)\subset(\cup C_a')$. We also have that $(\cup_{i=1}^{q}[a,w_i))\subset(\cup_{i=1}^{q}[a,w_i'))$ and $(\cup_{i=q+1}^{n}[a,w_i))\subset(\cup C_a')$. Finally, if $z\in C'(a,b_i)$, then $z\in C''(a,b_i)$ or $Z\subset(\cup C_a')$.

Thus $H(a,U)\subset H(a,U')$ and H is monotone.

Now suppose that a,b are distinct elements of X. We show that $H(a,X-b)\cap H(b,X-a)=\varnothing$. We then have that

$$H(a,X-b)=\left(\bigcup_{i=1}^{n} L_{G_i}(a,((X-b)\cap G_i)-G_i')\right)\cup\left(\bigcup_{i=1}^{n} A_{a_i}\right)\cup(\cup C_a)\\ \cup\left(\bigcup_{i=1}^{p}[a,w_i)\right)\cup\left(\bigcup_{i=1}^{p} C'(a,b_i)\right)$$

and, analogously,

$$H(b,X-a)=\left(\bigcup_{i=1}^{n} L_{K_i}(b,((X-a)\cap K_i)-K_i')\right)\cup\left(\bigcup_{i=1}^{n} A_{b_i}\right)\cup(\cup C_b)\\ \cup\left(\bigcup_{i=1}^{p}[b,z_i)\right)\cup\left(\bigcup_{i=1}^{p} C'(b,c_i)\right).$$

Case A. Suppose $a\in G_i$, $b\in K_j$, where G_i and K_j are true cyclic elements of X.

Case A_1. Suppose $G_i=K_j$.

Case A_{1a}. Suppose $a\in\mathrm{Int}\,(G_i)$ and $b\in\mathrm{Int}\,(K_j)$. Then $H(a,X-b)\cap H(b,X-a)=L_{G_i}(a,\mathrm{Int}_X(X-b)-G_i)\cap L_{G_i}(b,\mathrm{Int}_X(X-a)-G_i)\subset L_{G_i}(a,G_i-b)\cap L_{G_i}(b,G_i-a)=\varnothing$.

Case A_{1b}. Suppose $a\in\mathrm{Int}\,(G_i)$ and $b\in\partial(K_j)$. Although we are in the case that $G_i=K_j$, it should be noted that $K_j'\neq G_i'$. Since the only points of $H(b,X-a)$ which lie in $\mathrm{Int}_X G_i$ also lie in

$L_{G_i}(b,((G_i \cap (X - a)) - K_j'))$, then $H(a, X - b) \cap H(b, X - a) \subset L_{G_i}(a, G_i - b) \cap L_{G_i}(b, G_i - a) = \varnothing$.

Case A_{1c}. Suppose $a \in \partial(G_i)$ and $b \in \partial(K_j)$. In this case we have that $H(a, X - b) = (L_{G_i}(a,((X - b) \cap G_i) - G_i')) \cup (\cup A_{a_i}) \cup (\cup C_a)$ and $H(b, X - a) = (L_{K_j}(b,((X - a) \cap K_j) - K_j')) \cup (\cup A_{b_j}) \cup (\cup C_b)$.

Now suppose $R \in C_a$, $S \in C_{b_j}$, $Z \in A_{a_i}$ and $Z' \in A_b$. Now if any $R \cap Z$, $R \cap Z'$, $R \cap S$, $Z \cap S$, $Z \cap Z'$ or $Z' \cap S$ is nonempty, then $G_i = K_j$ is not a true cyclic element of X, a contradiction. Similarly, none of R, Z, Z', S can meet either of $L_{G_i}(a,((X - b) \cap G_i) - G_i')$ or $L_{K_j}(b,((X - a) \cap K_j) - K_j')$, and since these latter two sets are disjoint we have that $H(a, X - b) \cap H(b, X - a) = \varnothing$.

Case A_2. Suppose $G_i \neq K_j$ and no cyclic element of X contains both a and b.

Case A_{2a}. Suppose that $a \in \mathrm{Int}\,(G_i)$ and/or $b \in \mathrm{Int}\,(K_j)$. Without loss of generality, assume that $a \in \mathrm{Int}\,(G_i)$. Then $H(a, X - b) = L_{G_i}(a, \mathrm{Int}_X(X - b) \cap G_i)$ and therefore $H(a, X - b)$ meets none of the set used in the construction of $H(b, X - a)$.

Throughout the remaining cases, we let $C(a, b)$ denote the cyclic chain from a to b and note that $E(a, b) \neq \varnothing$. We let w denote the first element of the well-ordering W of X which lies in $E(a, b)$. We also note that if $G_i \subset C(a, b)$ and $K_j \subset C(a, b)$ and F is a true cyclic element of X distinct from G_i and K_j which contains a (respectively b), then $F - \{a\}$ (respectively $F - \{b\}$) determines an element of C_a (respectively C_b).

Case A_{2b}. Suppose $a \in \partial(G_i)$, $b \in \partial K_j$, and both of G_i and K_j are contained in $C(a, b)$. In this case we have $H(a, X - b) = (L_{G_i}(a,((X - b) \cap G_i) - G_i')) \cup (\cup A_{a_i}) \cup (\cup C_a)$ and $H(b, X - a) = (L_{K_j}(b,((X - a) \cap K_j) - K_j')) \cup (\cup A_{b_j}) \cup (\cup C_b)$.

Note that $\partial_{C(a,b)}(C(a, b) - G_i) \in G_i'$ and $\partial_{C(a,b)}(C(a, b) - K_j) \in K_j'$. Therefore, $(L_{G_i}(a,((X - b) \cap G_i) - G_i')) \cap (L_{K_j}(b,((X - a) \cap K_j) - K_j')) = \varnothing$.

Now let $R_a \in C_a$, $R_b \in C_b$, $S_a \in A_{a_i}$ and $S_b \in A_{b_j}$. If one of R_a, S_a meet one of R_b, S_b, then we find that G_i (respectively K_j) is not a cyclic element of X, a contradiction. Also, since $(S_a \cup R_a) \cap K_j = \varnothing$ and $(S_b \cup R_b) \cap G_i = \varnothing$, we obtain $H(a, X - b) \cap H(b, X - a) = \varnothing$.

Case B. Suppose $a \in G_i$ and $b \in K_j$ where G_i and K_j are cyclic elements of X but are not necessarily true cyclic elements of X.

Case B_1. Suppose $a \in G_i \subset C(a, b)$ and b is contained in no true cyclic element of X which is contained in $C(a, b)$. In this case we have that $H(a, X - b) = (L_{G_i}(a, ((X - b) \cap G_i) - G_i')) \cup (\cup A_{a_i}) \cup (C_a)$ and $H(b, X - a) = (\cup C_b) \cup [b, z_j) \cup (\cup C'(b, a))$.

Now let $R_a \in C_a$, $S_a \in A_{a_i}$, $R_b \in C_b$ and $T_b \in C'(b, a)$. If one of S_a, R_a meets one of T_b, R_b, then G_i is not a cyclic element of X, a contradiction. Also, $G_i \cap [b, z_j) = \varnothing$ and neither G_i nor $[b, z_j)$ meets any one of S_a, R_a, T_b and R_b. Thus we obtain $H(a, X - b) \cap H(b, X - a) = \varnothing$.

Case B_2. Suppose $b \in K_j \subset C(a, b)$ and a is contained in no true cyclic element of X which is contained in $C(a, b)$. This case clearly follows from an argument similar to that of the preceding case.

Case B_3. Neither a nor b is contained in a true cyclic element of X which is contained in $C(a, b)$. We then have that $H(a, X - b) = (\cup C_a) \cup [a, w_i) \cup (\cup C'(a, b))$ and $H(b, X - a) = (\cup C_b) \cup [b, z_j) \cup (\cup C'(b, a))$.

Now let $R_a \in C_a$, $R_b \in C_b$, $T_a \in C'(a, b)$ and $T_b \in C'(b, a)$.

If one of R_a, T_a meets one of R_b, T_b, then w does not separate a from b in X, a contradiction. Also, $[a, w_i) \cap [b, z_j) = \varnothing$ and neither $[a, w_i)$ nor $[b, z_j)$ meets any of R_a, T_a, R_b and T_b. Thus, we obtain $H(a, X - b) \cap H(b, X - a) = \varnothing$.

This completes the proof that X is monotonically normal.

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