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SUPERCONVERGENCE OF PIECEWISE POLYNOMIAL COLLOCATIONS FOR NONLINEAR WEAKLY SINGULAR INTEGRAL EQUATIONS

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ABSTRACT. The piecewise polynomial collocation method is discussed to solve nonlinear weakly singular integral equations. Using special collocation points, error estimates at the collocation points are derived showing a more rapid convergence than the global uniform convergence in the interval of integration available by piecewise polynomials. For instance, using piecewise linear collocation, the convergence rate at collocation points is $O(h^4)$ if the singularity of the kernel is sufficiently mild (the global convergence rate is $O(h^2)$).

1. Introduction. Numerical methods for linear Fredholm integral equations of the second kind have been studied extensively during the last 20 years. More recently, much of this analysis has been extended to nonlinear integral equations, either to Hammerstein equations or to general Urysohn equations. For a comprehensive description of the literature see, for example, a survey paper by Atkinson [**1**].

Special attention has been paid to collocation methods for solving Hammerstein equations. A new-type collocation method was presented by Kumar and Sloan [**8**] and its superconvergence properties were studied by Kumar [**6**]. The connection between Kumar and Sloan's method and the iterated spline collocation method was discussed by Brunner [**3**]. Two discrete collocation methods were presented by Kumar [**7**] and Atkinson and Flores [**2**]. A spline collocation method and a product integration method for the weakly singular Hammerstein equation were studied by Kaneko, Noren and Xu [**4**].

Numerical methods for multidimensional weakly singular integral equations were studied by Vainikko [**14**]. Global convergence estimates have been derived, whereas the superconvergence effect at collocation points has been analyzed only for linear equations.

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The purpose of the present paper is to study the convergence rate of the piecewise polynomial collocation method at the collocation points for nonlinear weakly singular Urysohn equations. The error analysis is based upon the certain regularity properties of solutions of Urysohn equations (Section 2) presented by Vainikko [**14**] and proved to a full extent by Pedas and Vainikko [**9**]. After some preliminaries (Section 3) the main result of the paper is formulated in Section 4 (Theorem 2). Using special collocation points, error estimates at the collocation points are derived that show a more rapid convergence than the global uniform convergence in the interval of integration available by piecewise polynomials. To avoid some inconvenience in the arguments caused by a possible discontinuity of the approximate solution, we make use of the framework of discrete convergence theory. Corresponding notions and results are presented in Section 5. The proof of the main result (Theorem 2) itself is presented in Section 6. In Section 7 we give two numerical examples to illustrate the theoretical estimates.

2. Smoothness of the solution. Consider the integral equation

$$u(x) = \int_0^b K(x, y, u(y))\, dy + f(x), \quad 0 < x < b. \tag{1}$$

The real-valued kernel $K = K(x, y, u)$ is assumed to be m times, $m \geq 1$, continuously differentiable with respect to x, y, u for $x, y \in (0, b)$, $x \neq y$ and $u \in (-\infty, \infty)$. We assume that there exists a real number $\nu \in (-\infty, 1)$ such that, for nonnegative integers i, j and k with $i + j + k \leq m$, the following inequalities hold:

$$\left| \left(\frac{\partial}{\partial x}\right)^i \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^j \left(\frac{\partial}{\partial u}\right)^k K(x, y, u) \right| \leq b_1(|u|) \begin{cases} 1 & \text{if } \nu + i < 0, \\ 1 + |\log|x - y|| & \text{if } \nu + i = 0, \\ |x - y|^{-\nu - i} & \text{if } \nu + i > 0, \end{cases} \tag{2}$$

and

$$
\begin{aligned}
(3) \quad & \left| \left(\frac{\partial}{\partial x}\right)^i \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^j \left(\frac{\partial}{\partial u}\right)^k K(x, y, u_1) \right. \\
& \qquad \left. - \left(\frac{\partial}{\partial x}\right)^i \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^j \left(\frac{\partial}{\partial u}\right)^k K(x, y, u_2) \right| \\
& \leq b_2(\max\{|u_1|, |u_2|\})|u_1 - u_2| \\
& \quad \cdot \begin{cases} 1 & \text{if } \nu + i < 0, \\ 1 + |\log|x-y|| & \text{if } \nu + i = 0, \\ |x-y|^{-\nu-i} & \text{if } \nu + i > 0. \end{cases}
\end{aligned}
$$

The functions $b_1 : [0, \infty) \to [0, \infty)$ and $b_2 : [0, \infty) \to [0, \infty)$ are assumed to be increasing.

We see that the kernel K may have a weak singularity on the diagonal $x = y$, $i = j = k = 0$, $0 \leq \nu < 1$. In the case $\nu < 0$, the kernel K is bounded but its derivatives may be singular.

We remark that the asymmetry of (2) and (3) with respect to x and y is only apparent. Actually, using the equality $\partial/\partial y = (\partial/\partial x + \partial/\partial y) - \partial/\partial x$, we can deduce from (2) a similar estimate for

$$
\left(\frac{\partial}{\partial y}\right)^i \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^j \left(\frac{\partial}{\partial u}\right)^k K(x, y, u):
$$

$$
\begin{aligned}
& \left| \left(\frac{\partial}{\partial y}\right)^i \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^j \left(\frac{\partial}{\partial u}\right)^k K(x, y, u) \right| \\
& \qquad \leq \text{const } b_1(|u|) \begin{cases} 1 & \text{if } \nu + i < 0, \\ 1 + |\log|x-y|| & \text{if } \nu + i = 0, \\ |x-y|^{-\nu-i} & \text{if } \nu + i > 0. \end{cases}
\end{aligned}
$$

Introducing the weight functions,

$$
(4) \qquad w_\lambda(x) = \begin{cases} 1 & \text{if } \lambda < 0, \\ (1 + |\log \rho(x)|)^{-1} & \text{if } \lambda = 0, \\ \rho(x)^\lambda & \text{if } \lambda > 0, \end{cases} \qquad 0 < x < b, \ \lambda \in \mathbf{R},
$$

where

$$\rho(x) = \min\{x, b-x\}, \quad 0 < x < b.$$

Define the space $C^{m,\nu}(0,b)$ as the collection of all m times continuously differentiable functions $u : (0,b) \to \mathbf{R}$ such that

$$\|u\|_{m,\nu} \equiv \sum_{i=0}^{m} \sup_{0<x<b} (w_{i-(1-\nu)}(x)|u^{(i)}(x)|) < \infty. \tag{5}$$

In other words, an m times continuously differentiable function u on $(0,b)$ belongs to $C^{m,\nu}(0,b)$ if the growth of its derivatives near the boundary points 0 and b can be estimated as follows:

$$|u^{(i)}(x)| \le \text{const} \begin{cases} 1 & \text{if } i < 1-\nu, \\ 1 + |\log \rho(x)| & \text{if } i = 1-\nu, \\ \rho(x)^{1-\nu-i} & \text{if } i > 1-\nu, \end{cases} \tag{6}$$
$$0 < x < b, \ 0 \le i \le m.$$

The space $C^{m,\nu}(0,b)$, equipped with the norm (5), is complete (is a Banach space).

Note that $C^m[0,b] \subset C^{m,\nu}(0,b)$. On the other hand, a function $u \in C^{m,\nu}(0,b)$ can be extended up to a continuous function on $[0,b]$; this extension we denote again by u and in the sequel we always assume that this extension is done.

The following result, see [**14**, p. 37; **9**], states the regularity properties of solutions of equation (1).

Lemma 1. *Let $f \in C^{m,\nu}(0,b)$, and let the kernel K satisfy conditions* (2) *and* (3). *If integral equation* (1) *has a solution $u \in L^\infty(0,b)$, then $u \in C^{m,\nu}(0,b)$.*

We remark that similar results for linear weakly singular integral equations of second kind are presented in [**15, 17, 14**]. The regularity properties of a weakly singular Hammerstein equation are investigated in [**4**]. The behavior of tangential and nontangential derivatives of solutions of multidimensional weakly singular Urysohn equations is discussed in [**14, 10**].

3. Error estimates for the piecewise polynomial interpolation. We introduce the following $2N+1$ grid points, $\mathbf{N} \ni N \geq 1$, in the interval $[0,b]$:

$$x_j \equiv x_j^{(N)} = \frac{b}{2}\left(\frac{j}{N}\right)^r, \quad j = 0,1,\dots,N, \tag{7}$$
$$x_{N+j} \equiv x_{N+j}^{(N)} = b - x_{N-j}^{(N)}, \quad j = 1,\dots,N.$$

Here $r \in \mathbf{R}$, $r \geq 1$, characterizes the degree of the nonuniformity of the grid. If $r = 1$, then the grid points (7) are uniformly located; if $r > 1$, then the grid points (7) are more densely located towards the end points of the interval $[0,b]$. We see also that $x_0 = 0$, $x_N = b/2$ and the grid points (7) are located symmetrically with respect to $x_N = b/2$. Note that another analogous partition considered in [**5**] is possible. We refer also to Rice [**11**], who seems to have been the first to study graded grids for approximation of functions with singularities.

In the standard interval $[-1,1]$ we choose m interpolation points $\xi_1,\dots,\xi_m$:

$$-1 \leq \xi_1 < \cdots < \xi_m \leq 1. \tag{8}$$

By the transformation

$$\xi_{ji} = x_j + \left(\frac{\xi_i + 1}{2}\right)(x_{j+1} - x_j), \tag{9}$$
$$i = 1,\dots,m,\ j = 0,1,\dots,2N-1,$$

we transfer the points $\xi_1,\dots,\xi_m$ into the intervals $[x_j, x_{j+1}]$, $j = 0,1,\dots,2N-1$.

To a function $u : [0,b] \to \mathbf{R}$ we assign a piecewise polynomial interpolation $P_N u : [0,b] \to \mathbf{R}$ as follows: on every interval $[x_j, x_{j+1}]$, $0 \leq j \leq 2N-1$, $P_N u$ is a polynomial of degree $m-1$ and $P_N u$ interpolates u at points $\xi_{j1},\dots,\xi_{jm}$. Thus, $P_N u$ is uniquely defined in each interval $[x_j, x_{j+1}]$ independently and may be discontinuous at points $x = x_j$, $j = 1,\dots,2N-1$. We may treat $P_N u$ as a two-valued function in these points; in the sequel, the uniform error estimates concern both possible values of $P_N u$.

We denote by E_N the range of the interpolation projector P_N, i.e., the set of all piecewise polynomial functions on $[0,b]$ which are real polynomials of degree not exceeding $m-1$ on every interval $[x_j, x_{j+1}]$, $0 \le j \le 2N-1$. We introduce also the notation

$$h = b/N.$$

The approximation properties of $P_N u$ on grid (7) are considered in [**16, 17, 14**] and elsewhere. These results can be summarized as follows, cf., [**14**, p. 115].

Lemma 2. *Assume that* $u \in C^{m,\nu}(0,b)$. *Then the following estimates hold*:

if $m < 1-\nu$, *then*

$$\max_{0\le x\le b} |u(x) - (P_N u)(x)| \le \text{const } h^m \quad \textit{for } r \ge 1; \tag{10}$$

if $m = 1-\nu$, *then*

$$\max_{0\le x\le b} |u(x) - (P_N u)(x)| \le \text{const} \begin{cases} h^m(|\log h| + 1) & \textit{for } r = 1, \\ h^m & \textit{for } r > 1; \end{cases} \tag{11}$$

if $m > 1-\nu$, *then*

$$\max_{0\le x\le b} |u(x) - (P_N u)(x)| \le \text{const} \begin{cases} h^{r(1-\nu)} & \textit{for } 1 \le r \le m/(1-\nu) \\ h^m & \textit{for } r \ge m/(1-\nu) \end{cases} \tag{12}$$

and, for $1 \le p < \infty$,

$$\|u - P_N u\|_{L^p(0,b)} \le \text{const} \begin{cases} h^{r(1-\nu+(1/p))} & \textit{for } 1 \le r < m/(1-\nu+(1/p)), \\ & m > 1-\nu+1/p, \\ h^m(1+|\log h|)^{1/p} & \textit{for } r = m/(1-\nu+(1/p)), \\ & m \ge 1-\nu+1/p, \\ h^m & \textit{for } r > m/(1-\nu+(1/p)), \\ & r \ge 1. \end{cases}$$

We remark that an estimate $\max_{0\le x\le b} |u(x) - (P_N u)(x)| \le \text{const } h^m$ is the best that holds for a smooth function u without singularities.

For $u \in C^{m,\nu}(0,b)$, the influence of singularities of its derivatives can be compensated for using larger values of r.

4. Main results. We look for an approximate solution $u_N \in E_N$ to integral equation (1). We require that $u_N(x)$ should satisfy the equation (1) at the collocation points (9):

$$\left[u_N(x) - \int_0^b K(x,y,u_N(y))\,dy - f(x)\right]_{x=\xi_{ji}} = 0, \quad i = 1,\dots,m;\ j = 0,1,\dots,2N-1. \tag{13}$$

The conditions (13) form a system of equations whose exact form is determined by the choice of a basis in E_N. Some examples for such a choice are given in Section 7.

We denote by ε_N the maximal error of the approximate solution $u_N \in E_N$ at the collocation points (9):

$$\varepsilon_N = \max_{\substack{i=1,\dots,m;\\ j=0,1,\dots,2,N-1}} |u_N(\xi_{ji}) - u(\xi_{ji})|. \tag{14}$$

Since u and $P_N u$ coincide at the points (9), we have

$$\varepsilon_N \le \|u_N - P_N u\|_{L^\infty(0,b)}. \tag{15}$$

The following theorem [**14**, p. 143] states the global convergence rate for the collocation method (13).

Theorem 1. *Let the following conditions be fulfilled*:

1. *The kernel K satisfies* (2) *and* (3).

2. $f \in C^{m,\nu}(0,b)$.

3. *The integral equation* (1) *has a solution $u_0 \in L^\infty(0,b)$ and the linearized integral equation*

$$v(x) = \int_0^b K_0(x,y)v(y)\,dy, \quad K_0(x,y) = [\partial K(x,y,u)/\partial u]_{u=u_0(y)}, \tag{16}$$

has in $L^\infty(0,b)$ *only the trivial solution* $v = 0$.

4. *The collocation points* (9) *are used.*

Then there exist $N_0 > 0$ *and* $\delta_0 > 0$ *such that, for* $N > N_0$, *the collocation method* (13) *defines a unique approximation* $u_N \in E_N$ *to* u_0 *satisfying* $\|u_N - u_0\|_{L^\infty(0,b)} \le \delta_0$. *The following error estimates hold*:

$$(17) \quad \sup_{0<x<b} |u_N(x) - u_0(x)| \le \text{const } h^m \quad \textit{for} \begin{cases} r \ge m/(1-\nu) & \textit{if } 1-\nu < m, \\ r > 1 & \textit{if } 1-\nu = m, \\ r \ge 1 & \textit{if } 1-\nu > m, \end{cases}$$

and

$$(18) \quad \varepsilon_N \le \text{const } h^m \quad \textit{for} \begin{cases} r > m/(2(1-\nu)) & \textit{if } \nu \ge 0, \\ r > m/(2-\nu) & \textit{if } -(m-2) \le \nu < 0, \\ r \ge 1 & \textit{if } \nu < -(m-2),\ \nu < 0. \end{cases}$$

Now we assume that $\xi_1, \dots, \xi_m$, see (8), are the knots of a quadrature formula

$$(19) \quad \int_{-1}^{1} \varphi(\xi)\, d\xi \approx \sum_{i=1}^{m} \tilde{w}_i \varphi(\xi_i), \quad -1 \le \xi_1 < \dots < \xi_m \le 1,$$

which is sharp for polynomials of degree $m+\mu$, $0 \le \mu \le m-1$. Using transformation (9) we obtain the quadrature formula

$$(20) \quad \int_{x_j}^{x_{j+1}} \varphi(x)\, dx \approx \frac{x_{j+1}-x_j}{2} \sum_{i=1}^{m} \tilde{w}_i \varphi(\xi_{ji}), \quad j = 0, 1, \dots, 2N-1,$$

which remains to be sharp for polynomials of degree $m+\mu$, $0 \le \mu \le m-1$. Actually, the weights $\tilde{w}_i$ will not be used in our algorithms. The existence of a quadrature formula (19) which is sharp for polynomials of degree $m+\mu$ is used in the convergence analysis (the proof of

Theorem 2) in Section 6. The case $\mu = m - 1$ corresponds to the Gauss quadrature formula and is of the greatest interest in the following theorem which contains the main result of the paper.

Theorem 2. *Assume that the following conditions are fulfilled*:

1. *The kernel* $K(x, y, u)$ *and* $\partial K(x, y, u)/\partial u$ *are* $m+\mu+1$ $(\mu \in \mathbf{Z}, 0 \le \mu \le m - 1)$, $m \ge 1$, *times continuously differentiable with respect to* x, y *and* u *for* $x, y \in (0, b)$, $x \ne y$, $u \in \mathbf{R}$, *and satisfy* (2) *and* (3) *with* $i + j + k \le m + \mu + 1$.

2. $f \in C^{m+\mu+1,\nu}(0, b)$.

3. *The integral equation* (1) *has a solution* $u_0 \in L^\infty(0, b)$, *and the linearized equation* (16) *has in* $L^\infty(0, b)$ *only the trivial solution* $v = 0$.

4. *The collocation points* (9) *are generated by the knots* (8) *of a quadrature formula* (19) *which is sharp for polynomials of degree* $m+\mu$, $0 \le \mu \le m - 1$.

5. *The scaling parameter* $r = r(m, \nu, \mu) \ge 1$ *satisfies the conditions exposed in* (17), *but strengthened to the strict inequality* $r > m/(1 - \nu)$ *if* $1 - \nu < \mu + 1$, *and the following additional conditions*:

$$\begin{cases} r \ge (m + 1 - \nu)/(2 - \nu) & \text{if } 1 - \nu < \mu + 1, \\ r > (m + \mu + 1)/(2 - \nu) & \text{if } 1 - \nu \ge \mu + 1. \end{cases} \tag{21}$$

Then

$$\varepsilon_N \le \text{const } h^m \tau_\nu(h) \tag{22}$$

where $h = b/N$ *and*

$$\tau_\nu(h) = \begin{cases} h & \text{if } \nu < 0, \\ h(1 + |\log h|) & \text{if } \nu = 0, \\ h^{1-\nu} & \text{if } \nu > 0. \end{cases} \tag{23}$$

Now assume additionally that

6. $\nu < 0$, $\mu \ge 1$, *and for* $0 \le j \le \min\{\mu - 1, -\nu\}$, $0 \le k \le \min\{\mu - 1, -\nu\}$, *the derivatives*

$$\left(\frac{\partial}{\partial y}\right)^j \left(\frac{\partial}{\partial u}\right)^{k+1} K(x, y, u) \tag{24}$$

are bounded and continuous on $(0,b)\times(0,b)\times(-\rho,\rho)$ *with any* $\rho>0$, *including the diagonal* $x=y$.

Then

$$\varepsilon_N \le \text{const } h^m \sigma_{\nu,\mu}(h) \tag{25}$$

where $h=b/N$ *and*

$$\sigma_{\nu,\mu}(h)=\begin{cases} h^{\mu+1} & \text{if } 1-\nu>\mu+1,\\ h^{\mu+1}(1+|\log h|) & \text{if } 1-\nu=\mu+1,\\ h^{1-\nu} & \text{if } 1-\nu<\mu+1.\end{cases} \tag{26}$$

The proof of Theorem 2 is given in Section 6. Section 5 contains necessary preliminaries for the proof.

Remark 1. (Comment to condition 6 of Theorem 2). For $\nu<0$, Assumption 1 of Theorem 2 guarantees the boundedness and continuity of the derivatives (24) for $j<\min\{\mu-1,-\nu\}$, $j+k\le m+\mu+1$, on any set $(0,b)\times(0,b)\times(-\rho,\rho)$ from which the diagonal $x=y$ is excluded; for $j=-\nu$ with $\nu\in\mathbf{Z}$, $-\nu<\mu-1$, a logarithmical singularity may occur. Condition 6 bans this possible singularity and states the equality of left- and righthand limits of the derivatives of $K(x,y,u)$ at the diagonal $x=y$. It is possible to weaken condition 6 so that the boundedness and continuity of (24) on $(0,b)\times(0,b)\times(-\rho,\rho)$ is assumed only for $j<\min\{\mu-1,-\nu\}$ but then an additional multiplier $1+|\log h|$ appears in the estimates (25) for $\nu\in\mathbf{Z}$, $1-\nu<\mu+1$:

$$\varepsilon_N \le \text{ const } h^m \begin{cases} h^{\mu+1} & \text{if } 1-\nu>\mu+1,\\ h^{1-\nu}(1+|\log h|) & \text{if } 1-\nu\le\mu+1.\end{cases}$$

Notice also that, for all $\nu<1$ and $\mu\ge 0$, we have $\sigma_{\nu,\mu}(h)\le\tau_\nu(h)$ whereby the equality holds in the cases $\nu\ge 0$ and $\mu=0$.

Remark 2. An estimate $\sup_{0<x<b}|u_N(x)-u(x)|=O(h^m)$ is of optimal order even for a function $u\in C^\infty[0,b]$. Theorem 1 shows that, for the collocation method (13), the optimal accuracy $O(h^m)$ can be achieved using sufficiently great values of r. There are possibilities

to reduce r restricting ourselves to uniform estimates at the collocation points only, see (18). Furthermore, Theorem 2 states a convergence rate $\varepsilon_N = o(h^m)$. In other words, the superconvergence phenomenon at collocation points takes place.

Remark 3. Under conditions 1–5 of Theorem 2,

$$\sup_{0<x<b} |\tilde{u}_N(x) - u_0(x)| \le \text{ const } h^m \tau_\nu(h)$$

where

$$\tilde{u}_N(x) = \int_0^b K(x, y, u_N(y))\, dy + f(x).$$

Under conditions 1–6 of Theorem 2,

$$\sup_{0<x<b} |\tilde{u}_N(x) - u_0(x)| \le \text{ const } h^m \sigma_{\nu,\mu}(h).$$

Remark 4. For special Hammerstein equations with $0 \le \nu < 1$, the estimate (22) coincides with the corresponding result from [**5**]. The case $\nu < 0$ is not analyzed in [**5**].

5. Approximation of a nonlinear equation. In this section we present some results from [**12, 13**] used in the proof of Theorem 2. Let E and E_h, $0 < h < \bar{h}$, be Banach spaces (all real or all complex), and let $\mathcal{P} = (p_h)_{0<h<\bar{h}}$, $p_h : E \to E_h$, be a family of linear bounded operators (we write $p_h \in \mathcal{L}(E, E_h)$) satisfying the condition

$$\|p_h u\|_{E_h} \longrightarrow \|u\|_E \quad \text{as } h \longrightarrow 0, \quad \forall\, u \in E.$$

A family $(u_h)_{0<h<\bar{h}}$ of elements $u_h \in E_h$ is called discretely converging to an element $u \in E$ (or $\mathcal{P}$-converging) if

$$\|u_h - p_h u\| \longrightarrow 0 \quad \text{as } h \longrightarrow 0;$$

we write $u_h - \to u$. A family $(u_h)_{0<h<\bar{h}}$ of elements $u_h \in E_h$ is called discretely compact if any sequence (u_{h_n}), formed by the elements of the family with $h_n \to 0$, contains a discretely convergent subsequence. A family $(T_h)_{0<h<\bar{h}}$ of linear bounded operators $T_h \in \mathcal{L}(E_h, E_h)$ is called discretely converging to $T \in \mathcal{L}(E, E)$ if the following implication holds:

$$E_h \ni u_h - \longrightarrow u \in E \Longrightarrow T_h u_h - \longrightarrow Tu; \tag{27}$$

we write $T_h - \to T$. Finally, we say that the discrete convergence $T_h - \to T$ is compact, or $T_h - \to T$ compactly, if in addition to (27), the following implication holds:

$$\limsup_{h\to 0} \|u_h\|_{E_h} < \infty \Longrightarrow (T_h u_h) \text{ is discretely compact.} \tag{28}$$

Now we consider the equations

$$u = Tu + f \tag{29}$$

and

$$u_h = T_h u_h + f_h \tag{30}$$

where $f \in E$, $f_h \in E_h$ and $T : \Omega \to E$, $T_h : \Omega_h \to E_h$ are nonlinear operators defined on open sets $\Omega \subseteq E$ and $\Omega_h \subseteq E_h$, respectively. We recall that $T : \Omega \to E$ is called Frechet differentiable at $u^0 \in \Omega$ if there exists a linear operator $T'(u^0) \in \mathcal{L}(E, E)$ such that

$$\|Tu - Tu^0 - T'(u^0)(u - u^0)\|_E \Big/ \|u - u^0\|_E \longrightarrow 0$$
$$\text{as } \|u - u^0\|_E \longrightarrow 0.$$

Lemma 3. *Let the following conditions be fulfilled*:

(i) *Equation* (29) *has a solution* $u^0 \in \Omega$, *and the operator* T *is Frechet differentiable at* u^0.

(ii) *There is a positive* δ *such that the operator* T_h, $0 < h < \bar{h}$, *is Frechet differentiable in the ball* $\|u_h - p_h u^0\|_{E_h} \le \delta$ *which is assumed to be contained in* Ω_h, *and for any* $\varepsilon > 0$ *there is a* δ_ε, $0 < \delta_\varepsilon \le \delta$, *such that, for all* $h \in (0, h)$, $\|T'_h(u_h) - T'_h(p_h u^0)\|_{E_h} \le \varepsilon$ *whenever* $\|u_h - p_h u^0\|_{E_h} \le \delta_\varepsilon$.

(iii) $\|T_h p_h u^0 - p_h T u^0\|_{E_h} \to 0$ *as* $h \to 0$.

(iv) $T'_h(p_h u^0) - \to T'(u^0)$ *compactly whereby* $T'_h(p_h u^0) \in \mathcal{L}(E_h, E_h)$ *is compact and the homogeneous equation* $v = T'(u^0)v$ *has in* E *only the trivial solution.*

(v) $\|f_h - p_h f\|_{E_h} \to 0$ *as* $h \to 0$.

Then there exist $h_0 > 0$ *and* δ_0, $0 < \delta_0 \le \delta$, *such that, for* $0 < h < h_0$, *equation* (30) *has a unique solution* u_h^0 *in the ball* $\|u_h - p_h u^0\| \le \delta_0$. *Thereby* $u_h^0 - \to u^0$ *and the error estimate*

$$c_1 e_h \le \|u_h^0 - p_h u^0\|_{E_h} \le c_2 e_h, \quad 0 < h < h_0, \tag{31}$$

holds where

$$\begin{aligned} e_h &= \|p_h u^0 - T_h p_h u^0 - f_h\|_{E_h} \\ &= \|(p_h T u^0 - T_h p_h u^0) + (p_h f - f_h)\|_{E_h} \end{aligned}$$

and c_1 *and* c_2 *are positive constants independent of* h *and* f.

We refer to [**12, 13**] where this lemma is proved in a more general setting.

6. Proof of Theorem 2. The proof of Theorem 2 is based on Lemmas 1–3. Let conditions 1–5 of Theorem 2 be fulfilled. We denote by $BC(0,b) \subset L^\infty(0,b)$ the space of bounded continuous functions on $(0,b)$,

$$\|u\|_{BC(0,b)} = \sup_{0<x<b} |u(x)| = \|u\|_{L^\infty(0,b)}.$$

Let $u_0 \in L^\infty(0,b)$ be a solution to equation (1). Due to Lemma 1, $u_0 \in C^{m+\mu+1,\nu}(0,b) \subset BC(0,b)$. We consider equation (1) as equation (29) in the space $E = BC(0,b)$,

$$(Tu)(x) = \int_0^b K(x,y,u(y))\,dy.$$

It is clear that the operator $T : E \to E$ is Frechet differentiable at $u_0 \in E$,

$$(T'(u_0)v)(x) = \int_0^b \frac{\partial K(x,y,u_0(y))}{\partial u} v(y)\,dy.$$

Further, the collocation conditions (13) can be represented as the equation $u_N = P_N T u_N + P_N f$ where P_N is the interpolation projector introduced in Section 2. Thus, (13) can be treated as equation (30) whereby $E_h = E_N$ (the space of piecewise polynomials defined in Section 2 and equipped with the supremum-norm), $u_h = u_N$, $p_h = P_N$, $T_h = P_N T$, $f_h = P_N f$, $h = b/N$. It is easy to check, cf., [**14, 17**], that the operators $T : BC(0,b) \to BC(0,b)$ and $P_N T : E_N \to E_N$ satisfy the conditions (ii)–(iv) of Lemma 3. The condition (v) of Lemma 3 is fulfilled too: $\|P_N f - P_N f\|_{E_N} = 0$ as $N \to \infty$. Now estimate (31) yields

$$\begin{aligned}\|u_N - P_N u_0\|_{L^\infty(0,b)} &\le c_2 \|P_N T P_N u_0 - P_N T u_0\|_{L^\infty(0,b)}\\ &\le c_3 \|T P_N u_0 - T u_0\|_{L^\infty(0,b)},\end{aligned}$$

or, using (3), see also (14) and (15),

$$\begin{aligned}\varepsilon_N &\le c_3 \sup_{0<x<b} \left| \int_0^b [K(x,y,(P_N u_0)(y)) - K(x,y,u_0(y))]\, dy \right|\\ &\le c_3 \sup_{0<x<b} \left| \int_0^b \frac{\partial K(x,y,u_0(y))}{\partial u} [(P_N u_0)(y) - u_0(y)]\, dy \right|\\ &\quad + c_4 \|u_0 - P_N u_0\|^2_{L^\infty(0,b)}.\end{aligned} \tag{32}$$

We shall use the notations

$$K_0(x,y,u) = \partial K(x,y,u)/\partial u,$$
$$y_j = (x_j + x_{j+1})/2, \quad j = 0, 1, \dots, 2N-1,$$

and denote by

$$K_{0s}(x,y,u_0(y)) = \sum_{q=0}^{s} \frac{1}{q!} \frac{\partial^q}{\partial y^q} K_0(x,y,u_0(y)) \Big|_{y=y_j} (y - y_j)^q$$

the Taylor expansion of $K_0(x,y,u_0(y))$ with respect to y at the point y_j; the value $s \in \mathbf{Z}_+$ will be chosen later. Fix $x \in (0,b)$. We get from (2)

$$\int_{S(x,h)} |K_0(x,y,u_0(y))|\, dy \le \text{ const } \tau_\nu(h), \tag{33}$$

where $\tau_\nu(h)$ is defined in (23) and

$$S(x,y) = (0,b) \cap \{[0,h] \cup [x-h, x+h] \cup [b-h, b]\}.$$

We have

$$\begin{aligned}(34)\quad &\int_{x_j}^{x_{j+1}} K_0(x,y,u_0(y))[u_0(y) - P_N u_0)(y)]\,dy \\ &= \int_{x_j}^{x_{j+1}} [K_0(x,y,u_0(y)) - K_{0s}(x,y,u_0(y))][u_0(y) - (P_N u_0)(y)]\,dy \\ &+ \sum_{q=0}^{s} \frac{1}{q!}\frac{\partial^q}{\partial y^q} K_0(x,y,u_0(y))\bigg|_{y=y_j} \\ &\cdot \int_{x_j}^{x_{j+1}} (y-y_j)^q [u_0(y) - (P_N^{(q)} u_0)(y)]\,dy\end{aligned}$$

where $0 \le s \le \mu$ and $P_N^{(q)}$ is an interpolation projector similar to P_N but corresponding to the space of piecewise polynomials of degree $m+\mu-q$ and $m+\mu+1-q$ interpolation knots in $[-1,1]$–the knots $\xi_1, \dots, \xi_m$ of the quadrature formula (19) and additional knots $\xi_{m+1}, \dots, \xi_{m+\mu+1-q}$; the choice of the last ones in $[-1,1]$ is arbitrary, but we assume that they are somehow fixed. To establish (34), it suffices to notice (see Assumption 4 of Theorem 2) that

$$\int_{x_j}^{x_{j+1}} (y-y_j)^q [(P_N u_0)(y) - (P_N^{(q)} u_0)(y)]\,dy = 0, \qquad 0 \le q \le s \le \mu.$$

With the help of (34) we have

$$(35)\quad \int_0^b K_0(x,y,u_0(y))[u_0(y) - (P_N u_0)(y)]\,dy = \gamma_1(x) + \gamma_2(x) + \gamma_3(x)$$

where

$$\gamma_1(x) = \sum_{j:[x_j,x_{j+1}]\cap S(x,h)\neq\varnothing} \int_{x_j}^{x_{j+1}} K_0(x,y,u_0(y)) \cdot [u_0(y) - (P_N u_0)(y)]\, dy,$$

$$\gamma_2(x) = \sum_{j:[x_j,x_{j+1}]\cap S(x,y)=\varnothing} \int_{x_j}^{x_{j+1}} [K_0(x,y,u_0(y)) - K_{0s}(x,y,u_0(y))] \cdot [u_0(y) - (P_N u_0)(y)]\, dy,$$

$$\gamma_3(x) = \sum_{j:[x_j,x_{j+1}]\cap S(x,h)=\varnothing} \sum_{q=0}^{s} \frac{1}{q!} \int_{x_j}^{x_{j+1}} \frac{\partial^q}{\partial y^q} K_0(x,y,u_0(y))\Big|_{y=y_j} \cdot (y-y_j)^q [u_0(y) - (P_N^{(q)} u_0)(y)]\, dy.$$

Due to (33),

$$(36) \qquad |\gamma_1(x)| \leq \text{ const } \tau_\nu(h) \|u_0 - P_N u_0\|_{L^\infty(0,b)}, \quad x \in (0,b).$$

Let us consider $\gamma_2(x)$. We have, for $q \in \mathbf{N}$, cf. [**9**],

(37)

$$\begin{aligned}\frac{\partial^q K_0(x,y,u_0(y))}{\partial y^q} &= \frac{\partial^q}{\partial y^q} K_0(x,y,u)\Big|_{u=u_0(y)} \\ &+ \binom{q}{1} u_0'(y) \frac{\partial^{q-1}}{\partial y^{q-1}} \frac{\partial}{\partial u} K_0(x,y,u)\Big|_{u=u_0(y)} \\ &+ \binom{q}{2} \Big\{ u_0''(y) \frac{\partial^{q-2}}{\partial y^{q-2}} \frac{\partial}{\partial u} K_0(x,y,u)\Big|_{u=u_0(y)} \\ &\qquad + (u_0'(y))^2 \frac{\partial^{q-2}}{\partial y^{q-2}} \frac{\partial^2}{\partial u^2} K_0(x,y,u)\Big|_{u=u_0(y)} \Big\} \\ &+ \binom{q}{3} \Big\{ u_0'''(y) \frac{\partial^{q-3}}{\partial y^{q-3}} \frac{\partial}{\partial u} K_0(x,y,u)\Big|_{u=u_0(y)} \\ &\qquad + 3u_0''(y) u_0'(y) \frac{\partial^{q-3}}{\partial y^{q-3}} \frac{\partial^2}{\partial u^2} K_0(x,y,u)\Big|_{u=u_0(y)} \\ &\qquad + (u_0'(y))^3 \frac{\partial^{q-3}}{\partial y^{q-3}} \frac{\partial^3}{\partial u^3} K_0(x,y,u)\Big|_{u=u_0(y)} \Big\}\end{aligned}$$

$$+\cdots+\binom{q}{q}\left\{u_0^{(q)}(y)\frac{\partial}{\partial u}K_0(x,y,u)\bigg|_{u=u_0(y)}\right.$$
$$\left.+\cdots+(u_0'(y))^q\frac{\partial^q}{\partial u^q}K_0(x,y,u)\bigg|_{u=u_0(y)}\right\}.$$

In order to estimate $|K_0(x,y,u_0(y)) - K_{0s}(x,y,u_0(y))|$ we must estimate (37) for $q = s+1$. As a first step, we note that, for any $u_0 \in C^{m+\mu+1,\nu}(0,b)$, the singularities of the terms $(u_0'(y))^q$, $(u_0'(y))^{q-2}\cdot u_0''(y), \dots, u_0'(y)\cdot u_0^{(q-1)}(y)$ in (37) are weaker than the singularity allowed for $u_0^{(q)}(y)$ by the definition of the space $C^{m+\mu+1,\nu}(0,b)$:

$$|(u_0'(y))^q| \le \text{const}\begin{cases}1 & \text{if } \nu<0,\\ (1+|\log\rho(y)|)^q & \text{if } \nu=0,\\ \rho(y)^{-\nu q} & \text{if } \nu>0,\end{cases}$$

and, for $q\ge 2$,

$$|(u_0'(y))^{q-2}\cdot u_0''(y)| \le \text{const}\begin{cases}1 & \text{if } \nu<-1,\\ 1+|\log\rho(y)| & \text{if } \nu=-1,\\ \rho(y)^{-\nu-1} & \text{if } -1<\nu<0,\\ \rho(y)^{-\nu-1}(1+|\log\rho(y)|)^{q-2} & \text{if } \nu=0,\\ \rho(y)^{(q-1)(1-\nu)-q} & \text{if } \nu>0,\end{cases}$$

and, for $q\ge 3$,

$$|u_0'(y)u_0^{(q-1)}(y)| \le \text{const}\begin{cases}1 & \text{if } \nu<2-q,\\ 1+|\log\rho(y)| & \text{if } \nu=2-q,\\ \rho(y)^{2-\nu-q} & \text{if } 2-q<\nu<0,\\ \rho(y)^{2-\nu-q}(1+|\log\rho(y)|) & \text{if } \nu=0,\\ \rho(y)^{2(1-\nu)-q} & \text{if } \nu>0,\end{cases}$$

and

$$|u_0^{(q)}(y)| \le \text{const}\begin{cases}1 & \text{if } \nu<1-q,\\ 1+|\log\rho(y)| & \text{if } \nu=1-q,\\ \rho(y)^{1-\nu-q} & \text{if } \nu>1-q.\end{cases}$$

Therefore, it is sufficient to estimate only the terms

$$\frac{\partial^q}{\partial y^q} K_0(x,y,u)\Big|_{u=u_0(y)}, u_0'(y)\frac{\partial^{q-1}}{\partial y^{q-1}}\frac{\partial}{\partial u}K_0(x,y,u)\Big|_{u=u_0(y)},$$

$$u_0''(y)\frac{\partial^{q-2}}{\partial y^{q-2}}\frac{\partial}{\partial u}K_0(x,y,u)\Big|_{u=u_0(y)}, \dots,$$

$$u_0^{(q)}(y)\frac{\partial}{\partial u}K_0(x,y,u)\Big|_{u=u_0(y)}$$

in (37). Using (2) we obtain for $y \in [x_j, x_{j+1}]$ with $[x_j, x_{j+1}] \cap S(x,h) = \varnothing$ the estimate

$$\begin{aligned}
&|K_0(x,y,u_0(y)) - K_{0s}(x,y,u_0(y))| \\
&\le ch^{s+1} \sup_{z\in(y,y_j)} \left| \frac{\partial^{s+1}K_0(x,y,u_0(y))}{\partial y^{s+1}} \right|_{y=z} \\
&\le c'h^{s+1} \sup_{z\in(y,y_j)} \left[\begin{Bmatrix} 1 & \text{if } \nu+s+1<0 \\ 1+|\log|x-z|| & \text{if } \nu+s+1=0 \\ |x-z|^{-\nu-s-1} & \text{if } \nu+s+1>0 \end{Bmatrix} \right. \\
&\quad + \begin{Bmatrix} 1 & \text{if } \nu+s<0 \\ 1+|\log|x-z|| & \text{if } \nu+s=0 \\ |x-z|^{-\nu-s} if & \nu+s>0 \end{Bmatrix} \begin{Bmatrix} 1 & \text{if } \nu<0 \\ 1+|\log\rho(z)| & \text{if } \nu=0 \\ \rho(z)^{-\nu} & \text{if } \nu>0 \end{Bmatrix} \\
&\quad \left. + \dots + \begin{Bmatrix} 1 & \text{if } \nu<0 \\ 1+|\log|x-z|| & \text{if } \nu=0 \\ |x-z|^{-\nu} & \text{if } \nu>0 \end{Bmatrix} \begin{Bmatrix} 1 & \text{if } \nu+s<0 \\ 1+|\log\rho(z)| & \text{if } \nu+s=0 \\ \rho(z)^{-\nu-s} & \text{if } \nu+s>0 \end{Bmatrix} \right] \\
&\le c''h^{s+1} \left[\begin{Bmatrix} 1 & \text{if } \nu+s+1<0 \\ 1+|\log|x-y|| & \text{if } \nu+s+1=0 \\ |x-y|^{-\nu-s-1} & \text{if } \nu+s+1>0 \end{Bmatrix} \right. \\
&\quad + \begin{Bmatrix} 1 & \text{if } \nu+s<0 \\ 1+|\log|x-y|| & \text{if } \nu+s=0 \\ |x-y|^{-\nu-s} & \text{if } \nu+s>0 \end{Bmatrix} \begin{Bmatrix} 1 & \text{if } \nu>0 \\ 1+|\log h| & \text{if } \nu=0 \\ h^{-\nu} & \text{if } \nu>0 \end{Bmatrix} \\
&\quad \left. + \dots + \begin{Bmatrix} 1 & \text{if } \nu<0 \\ 1+|\log|x-y|| & \text{if } \nu=0 \\ |x-y|^{-\nu} & \text{if } \nu>0 \end{Bmatrix} \begin{Bmatrix} 1 & \text{if } \nu+s<0 \\ 1+|\log h| & \text{if } \nu+s=0 \\ h^{-\nu-s} & \text{if } \nu+s>0 \end{Bmatrix} \right].
\end{aligned}$$

Therefore,

$$\int_{(0,b)\setminus S(x,h)} |K_0(x,y,u_0(y)) - K_{0s}(x,y,u_0(y))|\,dy \le c''' h^{s+1} \begin{cases} 1 & \text{if } \nu+s<0, \\ 1+|\log h| & \text{if } \nu+s=0, \\ h^{-\nu-s} & \text{if } \nu+s>0. \end{cases}$$

This enables us to estimate $\gamma_2(x)$ as follows

$$(38)\quad |\gamma_2(x)| \le \text{const}\, \|u_0 - P_N u_0\|_{L^\infty(0,b)} \left\{ \begin{matrix} h^{s+1} & \text{if } \nu+s<0 \\ h^{s+1}(1+|\log h|) & \text{if } \nu+s=0 \\ h^{1-\nu} & \text{if } \nu+s>0 \end{matrix} \right\}, \quad x \in (0,b).$$

Let us turn to $\gamma_3(x)$. In a similar way as above we obtain for $y \in [x_j, x_{j+1}]$ with $[x_j, x_{j+1}] \cap S(x,y) = \varnothing$ the inequality

$$(39)\quad \left| \frac{\partial^q K_0(x,y_j,u_0(y_j))}{\partial y^q} \right| \le \text{const} \left[\left\{ \begin{matrix} 1 & \text{if } \nu+q<0 \\ 1+|\log|x-y|| & \text{if } \nu+q=0 \\ |x-y|^{-\nu-q} & \text{if } \nu+q>0 \end{matrix} \right\} + \left\{ \begin{matrix} 1 & \text{if } \nu+q-1<0 \\ 1+|\log|x-y|| & \text{if } \nu+q-1=0 \\ |x-y|^{-\nu-(q-1)} & \text{if } \nu+q-1>0 \end{matrix} \right\} \left\{ \begin{matrix} 1 & \text{if } \nu<0 \\ 1+|\log h| & \text{if } \nu=0 \\ h^{-\nu} & \text{if } \nu>0 \end{matrix} \right\} + \cdots + \left\{ \begin{matrix} 1 & \text{if } \nu<0 \\ 1+|\log|x-y|| & \text{if } \nu=0 \\ |x-y|^{-\nu} & \text{if } \nu>0 \end{matrix} \right\} \left\{ \begin{matrix} 1 & \text{if } \nu+q-1<0 \\ 1+|\log h| & \text{if } \nu+q-1=0 \\ h^{1-\nu-q} & \text{if } \nu+q-1>0 \end{matrix} \right\} \right].$$

Using (39) we can estimate $\gamma_3(x)$. Furthermore, now (32) and (35)–(39)

yield

(40)

$$\begin{aligned}
\varepsilon_N &\le c_4\|u_0 - P_N u_0\|^2_{L^\infty(0,b)} \\
&\quad + c_5\|u_0 - P_N u_0\|_{L^\infty(0,b)}\tau_\nu(h) \\
&\quad + c_6\|u_0 - P_N u_0\|_{L^\infty(0,b)} \left\{\begin{matrix} h^{s+1} & \text{if } \nu + s < 0 \\ h^{s+1}(1+|\log h|) & \text{if } \nu + s = 0 \\ h^{1-\nu} & \text{if } \nu + s > 0 \end{matrix}\right\} \\
&\quad + c_7 \sum_{q=0}^{s} h^q \cdot \sup_{0<x<b} \int_{(0,b)\setminus S(x,h)} |u_0(y) - P_N^{(q)} u_0(y)| \\
&\quad \cdot \left[\left\{\begin{matrix} 1 & \text{if } \nu + q < 0 \\ 1+|\log|x-y|| & \text{if } \nu + q = 0 \\ |x-y|^{-\nu-q} & \text{if } \nu + q > 0 \end{matrix}\right\} \right. \\
&\quad + \left\{\begin{matrix} 1 & \text{if } \nu + q - 1 < 0 \\ 1+|\log|x-y|| & \text{if } \nu + q - 1 = 0 \\ |x-y|^{-\nu-(q-1)} & \text{if } \nu + q - 1 > 0 \end{matrix}\right\} \\
&\quad \cdot \left\{\begin{matrix} 1 & \text{if } \nu < 0 \\ 1+|\log h| & \text{if } \nu = 0 \\ h^{-\nu} & \text{if } \nu > 0 \end{matrix}\right\} + \cdots + \left\{\begin{matrix} 1 & \text{if } \nu < 0 \\ 1+|\log|x-y|| & \text{if } \nu = 0 \\ |x-y|^{-\nu} & \text{if } \nu > 0 \end{matrix}\right\} \\
&\quad \left. \cdot \left\{\begin{matrix} 1 & \text{if } \nu + q - 1 < 0 \\ 1+|\log h| & \text{if } \nu + q - 1 = 0 \\ h^{1-\nu-q} & \text{if } \nu + q - 1 > 0 \end{matrix}\right\} \right] dy.
\end{aligned}$$

We put

$$\begin{cases} s = \mu & \text{if } 1-\nu \ge \mu + 1, \\ s = [1-\nu] & \text{if } 1-\nu < \mu + 1, \end{cases}$$

where $[1-\nu]$ is the integer part of $1-\nu$. Therefore, see (26),

$$\left\{\begin{matrix} h^{s+1} & \text{if } \nu + s < 0 \\ h^{s+1}(1+|\log h|) & \text{if } \nu + s = 0 \\ h^{1-\nu} & \text{if } \nu + s > 0 \end{matrix}\right\} \le \text{const } \sigma_{\nu,\mu}(h).$$

Due to (21) and (10)–(12) we have $\|u_0 - P_N u_0\|_{L^\infty(0,b)} \le \text{const } h^m$, and, since $\sigma_{\nu,\mu}(h) \le \tau_\nu(h)$, the first three terms on the right side of

(40) fit into the error estimate (22). The integral terms in (40) can be estimated on the basis of (12) adapted for $P_N^{(q)}$:

$$
(41)\quad \begin{gathered}
\|u_0 - P_N^{(q)} u_0\|_{L^p(0,b)} \le \text{const } h^{r(1+(1/p)-\nu)} \\
\text{for } 1 \le r < \frac{m+\mu+1-q}{1+(1/p)-\nu}, \\
r(1+1/p-\nu) < m+\mu+1-q, \quad 1 \le p < \infty.
\end{gathered}
$$

A detailed argument can be found in [**14**, p. 130] where the following estimate is proved for the main integral term in (40):

$$
\begin{aligned}
h^q \sup_{0<x<b} \int_{(0,b)\setminus S(x,h)} & |u_0(y) - (P_N^{(q)} u_0)(y)| \\
& \cdot \left\{ \begin{matrix} 1 & \text{if } \nu+q<0 \\ 1+|\log|x-y|| & \text{if } \nu+q=0 \\ |x-y|^{-\nu-q} & \text{if } \nu+q>0 \end{matrix} \right\} dy. \\
& \le \text{const } h^m \sigma_{\nu,\mu}(h).
\end{aligned}
$$

It is easy to establish the same estimate also for other integral terms of (40). Since $\sigma_{\nu,\mu}(h) \le \tau_\nu(h)$, we have established the error estimate (22).

Now we assume 1–6. To establish the error estimate (25) we only have to prove that in this case

$$
|\gamma_1(x)| \le \text{const } \sigma_{\mu,\nu}(h), \quad x \in (0,b);
$$

for other terms in (40) we already established estimates of order $\sigma_{\mu,\nu}(h)$ (whereby we did it without Assumption 6). Let us divide $\gamma_1(x)$ into three parts

$$
\begin{aligned}
\gamma_1(x) &= \sum_{j:[x_j,x_{j+1}]\cap S(x,h)\neq\varnothing} \int_{x_j}^{x_{j+1}} K_0(x,y,u_0(y))[u_0(y) - P_N u_0(y)]\, dy \\
&= \delta_1(x) + \delta_2(x) + \delta_3(x), \quad x \in (0,b),
\end{aligned}
$$

where

$$
\begin{gathered}
\delta_k(x) = \sum_{j\in J_k} \int_{x_j}^{x_{j+1}} K_0(x,y,u_0(y))[u_0(y) - (P_N u_0)(y)]\, dy, \\
x \in (0,b),\ k = 1,2,3,
\end{gathered}
$$

$$J_1 = \{j : [x_j, x_{j+1}] \cap [x-h, x+h] \neq \varnothing\},$$
$$J_2 = \{j : [x_j, x_{j+1}] \cap [0, h] \neq \varnothing, j \notin J_1\},$$
$$J_3 = \{j : [x_j, x_{j+1}] \cap [b-h, b] \neq \varnothing, j \notin J_1\}.$$

For $j \in J_2 \cup J_3$ we apply the estimates derived on the basis of expansion (34). For $j \in J_1$ we apply a similar expansion of $K_0(x, y, u_0(y))$ with $y = x$:

$$\int_{x_j}^{x_{j+1}} K_0(x, y, u_0(y))[u_0(y) - P_N u_0)(y)]\, dy$$
$$= \int_{x_j}^{x_{j+1}} [K_0(x, y, u_0(y)) - K'_{0s}(x, y, u_0(y)] \cdot [u_0(y) - (P_N u_0)(y)]\, dy$$
$$+ \sum_{q=0}^{s} \frac{1}{q!} \frac{\partial^q}{\partial y^q} K_0(x, y, u_0(y)) \bigg|_{y=x} \cdot \int_{x_j}^{x_{j+1}} (y-x)^q [u_0(y) - (P_N^{(q)} u_0)(y)]\, dy$$

where s is sufficiently small, so that $(\partial^q / \partial y^q) K_0(x, y, u_0(y))$, $q \leq s$, remain continuous at $x = y$,

$$K'_{0s}(x, y, u_0(y)) = \sum_{q=0}^{s} \frac{1}{q!} \frac{\partial^q}{\partial y^q} K_0(x, y, u_0(y)) \bigg|_{x=y} (y-x)^q.$$

We put

(i) $s = \mu - 1$ if $1 - \nu > \mu + 1$;

(ii) $s = |\nu| - 1$ if $1 - \nu \leq \mu + 1$, $\nu \in \mathbf{Z}$;

(iii) $s = [|\nu|]$ if $1 - \nu \leq \mu + 1$, $\nu \notin \mathbf{Z}$.

Then, respectively, (i) $\nu + s + 1 < 0$, (ii) $\nu + s + 1 = 0$, (iii) $0 < \nu + s + 1 < 1$, in all three cases $s + 1 < 1 - \nu$, and we see from (6) that $|u_0^{(i)}(x)| \leq \text{const}$, $i = 0, 1, \ldots, s+1$. Now (37) and (2) together with Assumption 6 yield

$$\left| \left(\frac{\partial}{\partial y} \right)^q K_0(x, y, u_0(y)) \right| \leq \text{const}, \quad x, y \in (0, b), \ q = 0, 1, \ldots, s; \tag{42}$$

$$\left|\left(\frac{\partial}{\partial y}\right)^{s+1} K_0(x,y,u_0(y))\right| \leq \text{const}\begin{cases} 1 & \text{cases (i) and (ii)} \\ |x-y|^{-\nu-(s+1)} & \text{case (iii)} \end{cases}, \quad x,y \in (0,b), x \neq y. \tag{43}$$

Using the integral form of the remainders

$$K_0(x,y,u_0(y)) - K'_{0s}(x,y,u_0(y)) = \int_0^1 \frac{(1-\xi)^s}{s!}\frac{\partial^{s+1}K_0(x,y,u_0(y))}{\partial y^{s+1}}\bigg|_{y=x+\xi(y-x)} d\xi (y-x)^{s+1},$$

$$K_0(x,y,u_0(y)) - K_{0s}(x,y,u_0(y)) = \int_0^1 \frac{(1-\xi)^s}{s!}\frac{\partial^{s+1}K_0(x,y,u_0(y))}{\partial y^{s+1}}\bigg|_{y=y_j+\xi(y-y_j)} d\xi (y-y_j)^{s+1},$$

estimates (41), (42), (43) and $\|u_0 - P_N u_0\|_{L^\infty(0,b)} \leq \text{const } h^m$, it is easy to check that

$$|\gamma_1(x)| \leq |\delta_1(x)| + |\delta_2(x)| + |\delta_3(x)| \leq \text{const } \sigma_{\nu,\mu}(h), \quad x \in (0,b).$$

We refer to [**14**, p. 131] for details concerning the estimation of δ_1; for δ_2 and δ_3 the argument is similar.

The proof of Theorem 2 is completed.

7. Piecewise linear collocation. In the case $m = 2$ there are two choices of the interpolation points (8) of an interest.

1) $\xi_1 = -1$, $\xi_2 = 1$. In this case the approximate solution $u_N \in E_N$ to equation (1) can be represented in the form of a continuous piecewise linear function:

$$u_N(x) = \sum_{l=0}^{2N} c_l \varphi_l(x), \quad 0 \leq x \leq b,$$

where $\varphi_l(x)$, $l = 0, 1, \ldots, 2N$, are basic linear splines corresponding to grid points (7), i.e., $\varphi_l(x)$ is linear on every interval $[x_j, x_{j+1}]$,

$j = 0, 1, \ldots, 2N-1$, $\varphi_l(x_k) = 0$ for $l \neq k$ and $\varphi_k(x_k) = 1$, $k = 0, 1, \ldots, 2N$. Collocation method (13) leads to the nonlinear system with respect to $c_k = u_N(x_k)$, $k = 0, 1, \ldots, 2N$:

$$c_k = \int_0^b K\left(x_k, y, \sum_{l=0}^{2N} c_l \varphi_l(y)\right) dy + f(x_k),$$
$$k = 0, 1, \ldots, 2N.$$

The error estimate (18) of Theorem 1 takes the form

$$\varepsilon_N \leq \text{ const } h^2 \quad \text{for } \begin{cases} r > 1/(1-\nu) & \text{if } \nu \geq 0, \\ r \geq 1 & \text{if } \nu < 0, \end{cases} \tag{44}$$

where $h = b/N$ and $\varepsilon_N = \max_{0 \leq k \leq 2N} |c_k - u_0(x_k)|$. Theorem 2 cannot be applied since there exists no quadrature formula $\int_{-1}^1 \varphi(\xi)\, d\xi \approx w_{-1}\varphi(-1) + w_1\varphi(1)$ which is sharp for polynomials of degree 2.

2) $\xi_1 = -1/\sqrt{3}$, $\xi_2 = 1/\sqrt{3}$ (ξ_1 and ξ_2 are the knots of the Gaussian quadrature formula $\int_{-1}^1 \varphi(\xi)\, d\xi \approx \varphi(\xi_1) + \varphi(\xi_2)$). In this case the approximate solution $u_N \in E_N$ can be represented in the form

$$u_N(x) = \sum_{k=1}^{2} c_{lk}\varphi_{lk}(x), \quad x \in [x_l, x_{l+1}], \tag{45}$$

where $\varphi_{lk}(x)$ is on $[x_l, x_{l+1}]$ a linear polynomial satisfying $\varphi_{lk}(\xi_{li}) = \delta_{ki}$, $i = 1, 2$, (δ_{ki} is the Kronecker symbol). By knots $\xi_2 = -\xi_1 = 1/\sqrt{3}$ we find the collocation points (9) ($m = 2$) and the collocation conditions (13) take the form of a nonlinear system to determine the coefficients $c_{lk} = u_N(\xi_{lk})$, $k = 1, 2$; $l = 0, \ldots, 2N-1$. From Theorem 1 we obtain, for the approximate solution (45), the estimate (44) where $h = b/N$ and

$$\varepsilon_N = \max_{k=1,2; l=0,1,\ldots,2N-1} |u_N(\xi_{lk}) - u_0(\xi_{lk})|. \tag{46}$$

From Theorem 2, $m = 2$, $\mu = 1$, we obtain the following estimates for the error (46) (see (22) and (25)):

$$\varepsilon_N \leq \text{ const } h^2 \begin{cases} h & \text{if } \nu < 0, \\ h(1 + |\log h|) & \text{if } \nu = 0, \\ h^{1-\nu} & \text{if } \nu > 0, \end{cases} \tag{47}$$

or (in conditions 1–6 of Theorem 2)

$$\varepsilon_N \le \text{ const } h^2 \begin{cases} h^2 & \text{if } \nu < -1, \\ h^2(1 + |\log h|) & \text{if } \nu = -1, \\ h^{1-\nu} & \text{if } \nu > -1, \end{cases} \tag{48}$$

provided that

$$\begin{cases} r \ge 1 & \text{if } \nu < -2, \\ r > 4/(2-\nu) & \text{if } -2 \le \nu \le -1, \\ r \ge (3-\nu)/(2-\nu) & \text{if } -1 < \nu < 1-\sqrt{2}, \\ r > 2/(1-\nu) & \text{if } 1-\sqrt{2} \le \nu < 1. \end{cases} \tag{49}$$

Now we present two examples.

Example 1. Consider the integral equation

$$u(x) = \int_0^1 |x-y|^{3/2} u^2(y)\, dy + f(x), \quad 0 < x < 1, \tag{50}$$

where

$$\begin{aligned} f(x) = x^{5/2} - \frac{512}{45045} x^{15/2} - (1-x)^{5/2} \\ \cdot \Big[\frac{2}{5} x^5 + \frac{10}{7}(1-x)x^4 + \frac{20}{9}(1-x)^2 x^3 \\ + \frac{20}{11}(1-x)^3 x^2 + \frac{10}{13}(1-x)^4 x + \frac{2}{15}(1-x)^5 \Big]. \end{aligned}$$

It is easy to check that $u_0(x) = x^{5/2}$ is the exact solution to equation (50) and Assumptions 1–3 and 6 of Theorem 2 are fulfilled with $\nu = -3/2$, $m = 2$, $\mu = 1$.

Example 2. Consider the integral equation

$$u(x) = \int_0^1 |x-y|^{-1/2} u^2(y)\, dy + f(x), \quad 0 < x < 1, \tag{51}$$

where

$$f(x) = [x(1-x)]^{1/2} + \frac{16}{15}x^{5/2} + 2x^2(1-x)^{1/2} + \frac{4}{3}x(1-x)^{3/2} + \frac{2}{5}(1-x)^{5/2} - \frac{4}{3}x^{3/2} - 2x(1-x)^{1/2} - \frac{2}{3}(1-x)^{3/2}.$$

It is easy to check that $u_0(x) = [x(1-x)]^{1/2}$ is the exact solution to equation (51) and Assumptions 1–2 of Theorem 2 are fulfilled with $\nu = 1/2$, $m = 2$, $\mu = 1$.

The equations (50) and (51) were solved numerically by the collocation method $\{(45), (13), m = 2\}$. In both of these two cases, the Gaussian points $\xi_2 = -\xi_1 = 1/\sqrt{3}$ were used for determining collocation points (9). In the first case we have chosen $r = 12/10$ ($r > 8/7$, see (49)) and in the second case we have chosen $r = 41/10$ ($r > 4$, see (49)). The coefficients $c_{lk} = u_N(\xi_{lk})$, $k = 1, 2$, $l = 0, 1, \dots, 2N-1$, were calculated from (13) ($m = 2$) by the Newton method. All the integrals, which are needed for the construction of the system (13), were found analytically. Some results of the numerical experiments are presented in the following Table 1 where ε_N is defined in (46). The experiments were carried out on the computer IBM 4381 (in double precision).

From Table 1 we can see that the numerical results are consistent with the theoretical estimation which is $\varepsilon_N = O(h^4)$, see (48), for equation (50) and $\varepsilon_N = O(h^{5/2})$, see (47), for equation (51), $h = 1/N$.

TABLE 1.

N	ε_N for (50), $r=1.2$	$\varepsilon_{N/2}/2^4$	ε_N for (51), $r=4.1$	$\varepsilon_{N/2}/2^{5/2}$
2	0.13E-3		0.69E-2	
4	0.86E-5	0.79E-5	0.16E-2	0.12E-2
8	0.60E-6	0.54E-6	0.23E-3	0.28E-3
16	0.84E-7	0.37E-7	0.29E-4	0.40E-4
32	0.56E-7	0.52E-8	0.71E-5	0.51E-5

Notice that the number of collocation points (the number of unknowns) is $4N$. It is surprising that a high accuracy at collocation points has been achieved already for relatively small N.

For greater r ($r > 12/10$ for equation (50) and $r > 41/10$ for equation (51)) the computations gave the same convergence rates but somewhat worse results. This can be probably explained by larger constants in the error estimates for greater r.

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