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A REPRESENTATION FORMULA FOR STRONGLY CONTINUOUS RESOLVENT FAMILIES

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ABSTRACT. We give a representation formula for exponentially bounded strongly continuous resolvent families associated to an abstract Volterra equation of scalar type. As an application we derive a characterization of positive resolvent families defined in an ordered Banach space.

1. Introduction. We consider the following Volterra equation defined on a complex Banach space X

$$u(t) = f(t) + \int_0^t a(t-s)Au(s)\,ds, \quad t \in J \tag{1.1}$$

where A is a closed linear unbounded operator in X with dense domain $D(A)$, $a \in L^1_{\text{loc}}(\mathbf{R}_+)$ is a scalar kernel $\neq 0$ and $f \in C(J, X)$, $J := [0, T]$.

The basic concept concerning (1.1) is that of well-posedness which is the direct extension of the corresponding notion usually employed for the abstract Cauchy problem (of first order)

$$\dot{u}(t) = Au(t), \qquad u(0) = u_0. \tag{1.2}$$

It is well known that well-posedness is equivalent to the existence of a resolvent $\{S(t)\}_{t\geq 0} \subseteq \mathcal{B}(X)$ for (1.1), i.e., a strongly continuous family of bounded linear operators in X which commutes with A and satisfies the resolvent equation

$$S(t)x = x + \int_0^t a(t-s)AS(s)x\,ds, \\ t \geq 0, \quad x \in D(A).$$

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The resolvent is the central object to be studied in the theory of Volterra equations; it corresponds to the strongly continuous semigroup generated by A in the special case $a(t) \equiv 1$, i.e., for (1.2). The importance of the resolvent $S(t)$ is shown by the variation of parameters formula

$$u(t) = S(t)f(0) + \int_0^t S(t-s)\dot{f}(s)\,ds, \quad t \in J$$

where $f \in W^{1,1}(J;X)$.

Due to the time invariance of (1.1), Laplace transform methods can be employed. Suppose (1.1) admits an exponentially bounded resolvent $S(t)$ of type (M,ω), i.e., there are constants $M \geq 1$ and $\omega \in \mathbf{R}$ such that

$$\|S(t)\| \leq Me^{\omega t} \quad \text{for all } t \geq 0.$$

Suppose also that $a \in L^1_{\text{loc}}(\mathbf{R}_+)$ is Laplace transformable. Then the Laplace transform $H(\lambda) = \hat{S}(\lambda)$ of the resolvent exists for $\lambda > \omega$ and is represented by

$$H(\lambda) = (\lambda - \lambda\hat{a}(\lambda)A)^{-1}.$$

Several properties of resolvent families have been recently discussed in [**2, 7, 8, 9, 11**]. See also the recent monograph of J. Prüss [**12**] and the references therein.

The purpose of this note is to obtain a representation formula for an exponentially bounded resolvent for (1.1) in terms of $H(\lambda)$.

Exponential representations are well known for strongly continuous semigroups and cosine families of operators, see [**4**] and [**13**]. For example, if $\{T(t)\}_{t\geq 0}$ is a strongly continuous semigroup in the Banach space X with infinitesimal generator B, then

$$T(t)x = \lim_{n\to\infty}\left(I - \frac{t}{n}B\right)^{-n} x, \qquad x \in X, \quad t \geq 0, \tag{1.3}$$

where the convergence is uniform in bounded t-intervals for each fixed x. The formula (1.3) has important implications for the numerical approximation of the trajectories of $\{T(t)\}_{t\geq 0}$, especially for implicit approximation schemes.

In the next section we give our representation formula and, in Section 3, we prove a characterization concerning positivity of resolvent families defined on an ordered Banach space.

2. A representation formula. In what follows we will always assume that (1.1) admits an exponentially bounded strongly continuous resolvent family of type (M, ω) in a complex Banach space X. We will also assume that $a \in L^1_{\text{loc}}(\mathbf{R}_+)$ satisfies $\int_0^\infty e^{-\omega t} a(t)\, dt < \infty$.

The following formula generalizes (1.3).

Theorem 2.1. *If $x \in X$, then uniformly for t in bounded intervals of $\mathbf{R}_+$ we have*

$$
\begin{aligned}
S(t)x = \lim_{n\to\infty} \sum_{k=0}^{n} \sum \binom{n}{k} \frac{k!m!(-1)^{n+m}}{i!j!\cdots h!n!} \left(\frac{y'(\lambda)}{1!}\right)^{i} \left(\frac{y''(\lambda)}{2!}\right)^{j} \\
\cdots \left(\frac{y^{(l)}(\lambda)}{l!}\right)^{h} y^{-m-1}(\lambda) x^{(n-k)}(\lambda) \\
\cdot \lambda^{n+1}(I - \hat{a}(\lambda)A)^{-m-1} x\Big|_{\lambda=n/t}
\end{aligned} \tag{2.1}
$$

where $x(\lambda) := 1/(\lambda\hat{a}(\lambda))$, $y(\lambda) := 1/\hat{a}(\lambda)$ and the second sum is taken over all positive integer solutions of $i + 2j + \cdots + lh = k$; $i + j + \cdots + h = m$.

Proof. Because $S(t)$ is exponentially bounded, we can apply the Widder-Post formula for the inversion of Laplace transform in Banach spaces, see [**6**], and obtain

$$
S(t)x = \lim_{n\to\infty} \frac{(-1)^n}{n!} \lambda^{n+1} H^{(n)}(\lambda)x\Big|_{\lambda=n/t}, \tag{2.2}
$$

where the convergence is uniform in bounded t-intervals for fixed $x \in X$.

Putting $H(\lambda) = x(\lambda)(y(\lambda) - A)^{-1}$ where $x(\lambda) := 1/(\lambda\hat{a}(\lambda))$, and $y(\lambda) := 1/\hat{a}(\lambda)$ we get by Leibnitz's rule

$$
H^{(n)}(\lambda)x = \sum_{k=0}^{n} \binom{n}{k} x(\lambda)^{(n-k)} \frac{d^k}{d\lambda^k}[(y(\lambda) - A)^{-1}]x. \tag{2.3}
$$

Next, by making use of the chain rule and the product rule for differentiation of composite functions, see, e.g., [**5**, p. 19], we have

$$\frac{d^k}{d\lambda^k}[(y(\lambda)-A)^{-1}]x = \sum \frac{k!}{i!j!\cdots h!}\left(\frac{y'(\lambda)}{1!}\right)^i \left(\frac{y''(\lambda)}{2!}\right)^j \cdots \left(\frac{y^{(l)}(\lambda)}{l!}\right)^h \frac{d^m}{dy^m}[(y(\lambda)-A)^{-1}]x \tag{2.4}$$

where the sum is taken over all positive integer solutions of $i+2j+\cdots+lh=k$; $i+j+\cdots+h=m$.

Note that $(d^m/dy^m)[(y(\lambda)-A)^{-1}]x = (-1)^m m!(y(\lambda)-A)^{-m-1}x$. Therefore, substituting (2.4) and (2.3) into (2.2) we get the representation (2.1). □

We have the following corollary of Theorem 2.1.

Corollary 2.2. *For $p=1,2,\dots,$ let $\{S_p(t)\}_{t\geq 0}$ be a sequence of resolvent families for* (1.1) *with A replaced by A_p. Suppose that there exist constants $M>0$ and $\omega \geq 0$ such that $\|S_p(t)\| \leq Me^{\omega t}$. Let $\lim_{p\to\infty}(I-\hat{a}(\lambda)A_p)^{-1}x = (I-\hat{a}(\lambda)A_0)^{-1}x$ for all $\lambda>\omega$ and $x \in X$. Then*

$$S_0(t)x = \lim_{p\to\infty}\lim_{n\to\infty}\sum_{k=0}^{n}\sum \binom{n}{k}\frac{k!m!(-1)^{n+m}}{i!j!\cdots h!n!}\left(\frac{y'(\lambda)}{1!}\right)^i\left(\frac{y''(\lambda)}{2!}\right)^j \cdots \left(\frac{y^{(l)}(\lambda)}{l!}\right)^h y^{-m-1}(\lambda)x^{(n-k)}(\lambda) \cdot \lambda^{n+1}(I-\hat{a}(\lambda)A_p)^{-m-1}x\Big|_{\lambda=n/t}$$

for all $t\geq 0$, $x\in X$.

Proof. The proof follows immediately using the following result of Lizama [**7**]. Under the hypothesis of Corollary 2.2, $\lim_{p\to\infty} S_p(t)x = S_0(t)x$ for all $t\geq 0$, $x\in X$. □

3. Application to positive resolvents. In this section we apply Theorem 2.1 to obtain a criterion for positivity of a resolvent $S(t)$ for (1.1).

We will assume the following hypothesis:

(H) The solutions $s(\lambda, t)$ and $r(\lambda, t)$ of the convolution equations

$$s(\lambda,t)+\lambda\int_0^t a(t-u)s(\lambda,u)\,du=1 \tag{3.1}$$

and

$$r(\lambda,t)+\lambda\int_0^t a(t-u)r(\lambda,u)\,du=a(t)$$

are both nonnegative for each $\lambda > 0$.

Our key result in this section is the following theorem.

Theorem 3.1. *Suppose $a \in L^1_{\mathrm{loc}}(\mathbf{R}_+)$ satisfies* (H). *Let X be an ordered Banach space with closed cone K, and suppose that* (1.1) *admits a resolvent family $S(t)$ of type (M,ω). Then $S(t) \geq 0$ if and only if $(I-\hat{a}(\lambda)A)^{-1} \geq 0$ for all $\lambda > \omega$.*

Proof. It follows from

$$(I-\hat{a}(\lambda)A)^{-1}x=\frac{1}{\lambda}\int_0^\infty e^{-\lambda s}S(s)x\,ds, \qquad \lambda>\omega, \quad x\in X$$

that $S(s) \geq 0$ implies $(I-\hat{a}(\lambda)A)^{-1} \geq 0$ for $\lambda > \omega$.

Conversely, observe that $k+m = 2i+3j+\cdots+(l+1)h$ in Theorem 2.1. Hence,

$$\begin{aligned}(-1)^{n+m}&\left(\frac{y'(\lambda)}{1!}\right)^i\left(\frac{y''(\lambda)}{2!}\right)^j\cdots\left(\frac{y^{(l)}(\lambda)}{l!}\right)^h x^{(n-k)}(\lambda)\\ &=\left(\frac{(-1)^2y'(\lambda)}{1!}\right)^i\left(\frac{(-1)^3y''(\lambda)}{2!}\right)^j\\ &\qquad\cdots\left(\frac{(-1)^{l+1}y^{(l)}(\lambda)}{l!}\right)^h(-1)^{n-k}x^{(n-k)}(\lambda).\end{aligned} \tag{3.2}$$

It was shown in [**11**, p. 326] that hypothesis (H) is equivalent to

$$(-1)^n x^{(n)}(\lambda)\geq 0 \quad \text{for all } \lambda>0, n\in N_0 \tag{3.3}$$

and

$$(3.4)\qquad (-1)^n (y')^{(n)}(\lambda) \geq 0 \quad \text{for all } \lambda > 0, n \in N_0.$$

Substituting (3.3) and (3.4) into (3.2), we obtain that the second term in (3.2) is positive. Using Theorem 2.1 we conclude that $S(t) \geq 0$. □

Remark 3.2. i) Theorem 3.1 in essence is already contained in the papers of Clement and Nohel, see the references in [**12**].

ii) Kernels $a(t)$ with the property (H) or, equivalently, satisfying (3.3) and (3.4), are called completely positive by Clement and Nohel, cf. Prüss [**12**].

iii) Observe that the case $a(t) = t$, i.e., the case of the abstract Cauchy problem of second order, is *not* included in the above mentioned class of kernels.

If the cone K is normal and has interior points, we can obtain the following result on existence and positivity of resolvent families.

Theorem 3.3. *Let X be an ordered Banach space with cone K normal and* $\operatorname{int} K \neq \varnothing$. *Suppose $a \in L^1_{\mathrm{loc}}(\mathbf{R}_+)$ satisfies* (H). *The following conditions are equivalent.*

(1) (1.1) *admits a positive resolvent family.*

(2) *A generates a positive C_0 semigroup.*

Proof. First we observe that $\hat{a}(\lambda) \to 0$ as $\lambda \to \infty$. From [**1**, Theorem 2.2.7] we know that A generates a positive C_0 semigroup if and only if the operators $(I - \alpha A)^{-1}$ exist as positive operators for all small $\alpha > 0$. Therefore, the conclusion follows from Theorem 3.1 and [**11**, Theorem 5]. □

Remark 3.4. Let X be an ordered Banach space with cone K normal and $\operatorname{int} K \neq \varnothing$. Suppose (1.1) admits a resolvent family and the kernel a satisfies (H). Then, by [**1**, Proposition 2.14] and Theorem 3.1 we obtain that $S(t)$ is positive if and only if the following property holds:

If

$$x \in D(A) \cap K, \quad x^* \in K^* \quad \text{and} \quad x^*(x) = 0 \quad \text{then } x^*(Ax) \geq 0.$$

In particular, for $a(t) = 1$, we recover a result proved by D. Evans and H. Hanche-Olsen [**3**, Theorem 1] concerning the characterization for generators of norm continuous semigroups of positive operators, see also [**10**, Theorem 1.11].

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