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RESOLVENT ESTIMATES FOR ABEL INTEGRAL OPERATORS AND THE REGULARIZATION OF ASSOCIATED FIRST KIND INTEGRAL EQUATIONS

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ABSTRACT. In this paper resolvent estimates for Abel integral operators are provided. These estimates are applied to deduce regularizing properties of Lavrentiev's m-times iterated method as well as iterative schemes (with the discrepancy principle as corresponding parameter choice or stopping rule, respectively) for solving the corresponding Abel integral equations of the first kind.

1. Introduction.

1.1. *Introductory remarks.* Various applications lead to Abel integral equations of the first kind $Au = f_*$, where

$$(1.1) \qquad (Au)(\xi) = \frac{\beta^{1-\alpha}}{\Gamma(\alpha)} \int_\xi^a \frac{\eta^{\beta-1} u(\eta)}{(\eta^\beta - \xi^\beta)^{1-\alpha}}\, d\eta, \quad \xi \in [0,a],$$

or

$$(1.2) \qquad (Au)(\xi) = \frac{\beta^{1-\alpha}}{\Gamma(\alpha)} \int_0^\xi \frac{\eta^{\beta-1} u(\eta)}{(\xi^\beta - \eta^\beta)^{1-\alpha}}\, d\eta, \quad \xi \in [0,a],$$

(with $0 < \alpha < 1$, $0 < a < \infty$, $0 < \beta$, and with Γ denoting Euler's gamma function), see Subsection 1.2 for one of these applications.

In this paper we provide resolvent estimates for Abel integral operators (1.1) and (1.2) (operating from X into X for the spaces $X = L^p([0,a], \xi^{\beta-1}\, d\xi)$, $p \in [1,\infty]$, and $X = C[0,a]$, respectively), i.e., we provide norm estimates of $(\lambda I + A)^{-1}$ for specific $\lambda \in \mathbf{C}$, with I denoting the identity operator in the underlying space X.

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As a preparation, in Section 2 specific resolvent estimates for the operators

$$(V_1u)(\xi) := \int_\xi^a \eta^{\beta-1}u(\eta)\,d\eta, \quad \xi \in [0,a], \tag{1.3}$$

$$(V_2u)(\xi) := \int_0^\xi \eta^{\beta-1}u(\eta)\,d\eta, \quad \xi \in [0,a], \tag{1.4}$$

are presented (for the mentioned spaces X). In Section 3 we consider for $0<\alpha<1$ the fractional powers V_j^α of V_j, $j=1,2$, and we (a) shall see that they coincide with Abel integral operators (1.1) and (1.2), respectively, and (b) provide the mentioned resolvent estimates for V_j^α, $j=1,2$, respectively, cf., Theorems 3.2 and 3.4.

In Section 4, the derived resolvent estimates then are applied for solving numerically (ill-posed) Abel integral equations of the first kind. More specifically, parametric as well as iterative methods are considered in that section, and for the discrepancy principle as parameter choice strategy or stopping rule, respectively, convergence results are stated. Finally, for specific methods like Lavrentiev's m-times iterated method and an iterative implicit scheme, several numerical experiments are presented.

1.2. *The Radon transform for radially symmetric functions.* In this subsection we present an application where an Abel integral equation arises. This example is taken from Gorenflo and Vessella [**7**].

The two-dimensional Radon transform $\mathbf{R}$ maps a function $\psi : \mathfrak{R}^2 \to \mathfrak{R}$ into the set of integrals of ψ along the lines $L_{\theta,s}$, $\theta \in [0,2\pi]$, $s \geq 0$, i.e.,

$$\begin{aligned}(\mathbf{R}\psi)(\theta,s) &:= \int_{L_{\theta,s}} \psi(x)\,dx \\ &= \int_{-\infty}^{\infty} \psi(sx(\theta)+tx(\theta)^\perp)\,dt,\end{aligned}$$

where

$$L_{\theta,s} := \{sx(\theta)+tx(\theta)^\perp : \ t \in \mathfrak{R}\},$$
$$x(\theta) := (\cos\theta,\sin\theta)^T, \qquad x(\theta)^\perp := (-\sin\theta,\cos\theta)^T,$$

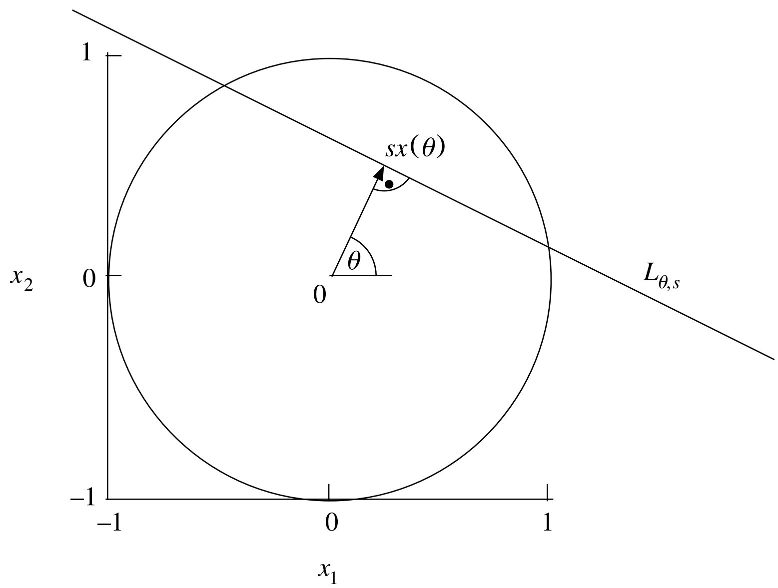


FIGURE 1. Illustration for the Radon transform.

see Figure 1 for an illustration of the situation. If the support of ψ lies in the closed unit disk

$$D := \{x \in \mathcal{R}^2 : \ |x|_2 \le 1\}$$

(with $|\cdot|_2$ denoting the Euclidean norm in $\mathcal{R}^2$), and if ψ moreover is a radially symmetric function, i.e., for some function $u : [0,1] \to \mathcal{R}$ one has

$$\psi(x) = u(|x|_2), \quad x \in D,$$

(this is a realistic assumption for the spectroscopy of cylindrical gas discharges) then $(\mathbf{R}\psi)(\theta, s) = 0$ for $s > 1$, and

$$(\mathbf{R}\psi)(\theta, s) = 2\int_s^1 \frac{ru(r)}{(r^2 - s^2)^{1/2}}\, dr, \quad s \in [0,1].$$

This implies that $g := \mathbf{R}\psi$ is also radially symmetric and that the support of g lies in D, and then for

$$f(s) := g(\theta, s), \quad s \in [0,1],$$

the resulting equation $Au = f$ is an Abel integral equation of the first kind (up to a scalar multiple).

2. Sectorial operators, and integration. In the first part of this section we classify different resolvent conditions for arbitrary linear operators in general Banach spaces. In the second part of this section specific resolvent estimates for the operators V_1 and V_2 (see (1.3) and (1.4), respectively) are presented (with respect to the spaces $X = C[0,a]$ and $X = L^p([0,a],\ \xi^{\beta-1}\,d\xi),\ p \in [1,\infty]$).

2.1. *Sectorial operators.* Throughout this subsection let X be a complex Banach space. For technical reasons in the following definition unbounded operators are admitted, although our main subject are bounded operators (with unbounded inverses).

Definition 2.1. We call a (possibly unbounded) linear operator $B : X \supset \mathcal{D}(B) \to X$ *weakly sectorial*, if $(0,\infty) \subset \rho(-B)$ and

$$\|(tI + B)^{-1}\| \le M_0/t, \quad t > 0, \tag{2.1}$$

with some $M_0 \ge 1$, and then we use the notation

$$M_0(B) := \inf\{M_0 : (2.1) \text{ is valid for } M_0\}.$$

Here $\rho(-B)$ is the resolvent set of $-B$, i.e., $\rho(-B) = \{\lambda \in \mathbf{C} : \lambda I + B : X \supset \mathcal{D}(B) \to X$ is a one-to-one mapping onto X, $(\lambda I + B)^{-1} \in \mathcal{L}(X)\}$, and $\mathcal{L}(X)$ denotes the space of bounded linear operators in X. Moreover, $\|\cdot\|$ in (2.1) denotes the corresponding operator norm.

It is significant to consider weakly sectorial operators since we can define fractional powers for them, cf., Section 3, and moreover for the numerical solution of equations with an underlying weakly sectorial operator we can consider Lavrentiev's method, cf., Section 4. Weakly sectorial operators coincide with 'weakly positive' operators introduced in Pustyl'nik [**20**]. Our notation is justified by the fact that weakly sectorial operators A fulfill a resolvent condition over a (small) sector, cf., Lemma 2.3. First, however, we introduce the sector $\Sigma_\theta \subset \mathbf{C}$,

$$\Sigma_\theta := \{\lambda = re^{i\varphi} :\ r > 0, |\varphi| \le \theta\}, \quad \theta \in [0,\pi],$$

and moreover we introduce the notation 'sectorial with angle θ_0.'

Definition 2.2. A linear operator $B : X \supset \mathcal{D}(B) \to X$ is *sectorial with angle* $\theta_0 \in (0, \pi]$, if for any $0 \le \theta < \theta_0$ one has $\rho(-B) \supset \Sigma_\theta$ as well as the estimate

$$\|(\lambda I + B)^{-1}\| \le M_\theta / |\lambda|, \quad \lambda \in \Sigma_\theta, \tag{2.2}$$

for some $M_\theta \ge 1$. Then we use the notation

$$M_\theta(B) := \inf\{M_\theta : (2.2) \text{ is valid for } M_\theta\}.$$

If the linear operator B is sectorial with angle θ_0 and moreover has a dense domain, then B is of type $(\pi - \theta_0, M_0(B))$ (in the sense of Tanabe [**24**, Definition 2.3.1]).

As mentioned above, weakly sectorial operators are sectorial with some angle (this is, e.g., Lemma 6.4.1 in Fattorini [**3**]):

Lemma 2.3. *Let the linear operator $B : X \supset \mathcal{D}(B) \to X$ be weakly sectorial. Then B is sectorial with angle $\theta_0 := \arcsin(1/M_0(B))$.*

For the proof of Lemma 2.3 we refer to [**3**]. In the proofs of Propositions 2.7 and 2.8, respectively, we shall use the following lemma and the corresponding Corollary 2.6.

Lemma 2.4. *Suppose that $A \in \mathcal{L}(X)$ has a trivial nullspace $\mathcal{N}(A)$, and let $0 < \theta_0 \le \pi$. Then A is sectorial with angle θ_0 if and only if A^{-1} is sectorial with angle θ_0, and then $M_\theta(A) \le M_\theta(A^{-1}) + 1$, $0 \le \theta < \theta_0$.*

Proof. If A^{-1} is sectorial with angle θ_0, then for any $0 \le \theta < \theta_0$ and any $\lambda \in \Sigma_\theta$, $\lambda I + A$ is a one-to-one mapping onto X, and

$$(\lambda I + A)^{-1} = \frac{1}{\lambda} I - \frac{1}{\lambda^2}\left(A^{-1} + \frac{1}{\lambda} I\right)^{-1}, \tag{2.3}$$

and then it follows immediately that A is sectorial with angle θ_0 (and $M_\theta(A) \le M_\theta(A^{-1}) + 1,\ 0 \le \theta < \theta_0$). The reverse implication follows similarly. $\square$

Definition 2.5. We call a (possibly unbounded) linear operator $B : X \supset \mathcal{D}(B) \to X$ *strictly sectorial*, if there is an $0 < \varepsilon \le \pi/2$ such that B is sectorial with angle $\pi/2 + \varepsilon$.

If the linear operator B is strictly sectorial and has a dense domain, then $-B$ is the infinitesimal generator of a semigroup $\{T(t)\}_{t\ge 0} \subset \mathcal{L}(X)$ that can be extended on a sector Σ_ε (for a small $\varepsilon > 0$) to an analytical, uniformly bounded semigroup, cf., Tanabe [**24**, Theorem 3.3.1]).

We shall see in Section 3 that Abel integral operators, cf., (1.1) and (1.2) are strictly sectorial (with respect to various function spaces), and we shall moreover see that for the numerical solution of ill-posed equations in general spaces with an underlying strictly sectorial operator one can apply iterative methods, cf., Section 4.

As an immediate consequence of Lemma 2.4 we have:

Corollary 2.6. *Suppose that $A \in \mathcal{L}(X)$ has a trivial nullspace $\mathcal{N}(A)$. Then A^{-1} is strictly sectorial if and only if A is strictly sectorial.*

2.2. *Integration operator, and a modification.* As a preparation for Section 3, in this subsection specific resolvent estimates for the integration operators V_1 and V_2 are presented (for the spaces $X = C[0,a]$ and $X = L^p([0,a],\ \xi^{\beta-1}\, d\xi)$, $p \in [1,\infty]$).

2.2.1. *The case $X = C[0,a]$.* For $0 < a < \infty$, let $X = C[0,a]$ be the complex space of complex-valued continuous functions on $[0,a]$, supplied with the maximum norm $\|\cdot\|_\infty$. The latter symbol then is used also for the corresponding operator norm.

Proposition 2.7 (Integration in $C[0,a]$). *Let $\beta > 0$ be real, and let $X = C[0,a]$. Then the Volterra integral operators $V_1 \in \mathcal{L}(X)$ and $V_2 \in \mathcal{L}(X)$ (defined by* (1.3) *and* (1.4)*, respectively) are sectorial with angle $\pi/2$, and this angle $\pi/2$ is best possible, i.e., V_j is not strictly sectorial for $j = 1,2$. One has $M_0(V_j) = 2$, $j = 1,2$, and in fact*

$$\lim_{t\to 0+} \|t(tI + V_j)^{-1}\|_\infty = 2, \quad j = 1,2. \tag{2.4}$$

Proof. We present the proof for V_1 only, since the same technique applies to prove the assertion for V_2 (this proof in fact is carried out in [**19**]). V_1 obviously is well-defined and in $\mathcal{L}(X)$, with $\|V_1\|_\infty = a^\beta/\beta$. Moreover, V_1 is inverse to the (unbounded) operator $B : X \supset \mathcal{D}(B) \to X$ defined by

$$(2.5) \qquad (Bf)(\xi) := -\xi^{-(\beta-1)} f'(\xi), \quad \xi \in [0,a], \quad f \in \mathcal{D}(B),$$

$$(2.6) \qquad \mathcal{D}(B) := \{f \in X : \ f(a) = 0,$$
$$f \text{ is absolutely continuous on } [0,a],$$
$$\xi \longmapsto \xi^{-(\beta-1)} f'(\xi) \in X\}.$$

We observe moreover that for $\lambda \in \mathbf{C}$ and $u \in X$ the equation

$$(2.7) \qquad (\lambda I + B) f = u$$

has the unique solution

$$(2.8) \qquad f(\xi) = \int_\xi^a \eta^{\beta-1} e^{-\lambda(\eta^\beta - \xi^\beta)/\beta} u(\eta)\, d\eta, \quad \xi \in [0,a].$$

We next show that B is weakly sectorial with $M_0(B) \leq 1$, i.e., $(0,\infty) \subset \rho(-B)$ and

$$(2.9) \qquad \|(tI+B)^{-1}\|_\infty \leq 1/t, \quad t > 0.$$

To this end we fix $t > 0$ and take any $u \in X$ and $f \in \mathcal{D}(B)$ such that (2.7) is valid (with $\lambda = t$). For $\xi \in [0,a]$, we then obtain (see (2.8) with $\lambda = t$)

$$\begin{aligned} t|f(\xi)| &\leq \left(t \int_\xi^a \eta^{\beta-1} e^{-t(\eta^\beta - \xi^\beta)/\beta}\, d\eta \right) \cdot \|u\|_\infty \\ &= (1 - e^{-t(a^\beta - \xi^\beta)/\beta}) \cdot \|u\|_\infty \\ &\leq \|u\|_\infty \\ &= \|(tI+B)f\|_\infty, \end{aligned}$$

and taking the supremum over $\xi \in [0,a]$ yields (2.9).

We conclude then from (2.9) and Lemma 2.3 that B is sectorial with angle $\pi/2$, and then Lemma 2.4 yields that also $V_1 = B^{-1}$ is sectorial with angle $\pi/2$, and $M_0(V_1) \leq 2$.

We next show that (2.4) holds, i.e., that indeed $M_0(V_1) = 2$. To this end we fix $s > 0$ and observe that for u, $f \in X$, one has $(I + sV_1)u = f$ if and only if

$$u(\xi) = f(\xi) - s \int_\xi^a \eta^{\beta-1} e^{-s(\eta^\beta - \xi^\beta)/\beta} f(\eta)\, d\eta, \quad \xi \in [0, a], \tag{2.10}$$

for a reasoning see (2.3) (with $\lambda = s^{-1}$, and with A replaced by V_1), and see moreover (2.7) and (2.8) (with $\lambda = s$, and with f and u interchanged). Now for small $\varepsilon > 0$ take some $f \in C[0, a]$, $\|f\|_\infty = 1$, such that $f(0) = 1$ and $f(\eta) = -1$, $\eta \in [\varepsilon, a]$, and then for $u = (I + sV_1)^{-1} f$ one has, see (2.10),

$$\begin{aligned} \|(I + sV_1)^{-1}\|_\infty &\geq |u(0)| \\ &\geq 1 + s \int_\varepsilon^a \eta^{\beta-1} e^{-s\eta^\beta/\beta}\, d\eta \\ &\quad - s \int_0^\varepsilon \eta^{\beta-1} e^{-s\eta^\beta/\beta}\, d\eta \\ &= 2e^{-s\varepsilon^\beta/\beta} - e^{-sa^\beta/\beta}. \end{aligned}$$

We find (2.4) then by taking $\varepsilon = \varepsilon(s)$ sufficiently small and letting $s \to \infty$.

We finally show that B (and then also V_1) is not strictly sectorial. To this end, for arbitrary real t, let $u(\eta) := e^{it\eta^\beta/\beta}$, $\eta \in [0, a]$. Then the equation

$$(itI + B)f = u$$

has the unique solution, see (2.7) and (2.8) with $\lambda = it$,

$$f(\xi) = e^{it\xi^\beta/\beta} \cdot \left(\frac{a^\beta}{\beta} - \frac{\xi^\beta}{\beta} \right), \qquad \xi \in [0, a],$$

hence $\|f\|_\infty = a^\beta/\beta$, $\|u\|_\infty = 1$, and this shows that B, and then also V_1, is not strictly sectorial. □

Remarks. 1. Estimate (2.9) in fact means that B in (2.5) is accretive, while (2.4) shows that its inverse V_1 is not accretive, see deLaubenfels [**1**] for a similar example. It is open, however, whether $\sup_{n \geq 0} \|(I + sV_1)^{-n}\|_\infty < \infty$ is valid for any $s > 0$.

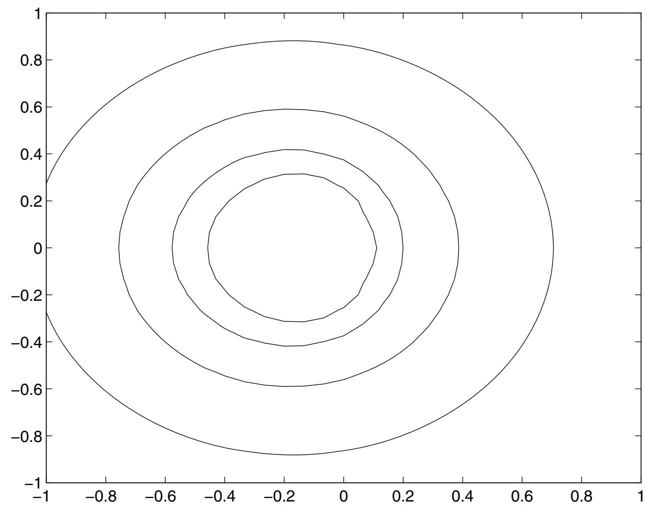


FIGURE 2. Boundary of $\Lambda_\varepsilon(-V_1)$ for $\varepsilon = 0.4,\ 0.2,\ 0.1,\ 0.05$, and for $X = C[0,1]$, $\beta = 1$.

2. It follows, as in the proof of Proposition 2.7, that under the given situation one has $\rho(-V_1) = \mathbf{C}\backslash\{0\}$, and for $\lambda \neq 0$ one gets (for notational convenience let $a = 1$ and $\beta = 1$)

$$\|(\lambda I + V_1)^{-1}\|_\infty = \begin{cases} 1/|\lambda| + (1 - e^{-\operatorname{Re}\lambda/|\lambda|^2})/(\operatorname{Re}\lambda) & \text{if } \operatorname{Re}\lambda \neq 0, \\ 1/|\lambda| + 1/|\lambda|^2 & \text{if } \operatorname{Re}\lambda = 0, \end{cases}$$

so that it is possible to compute the ε-pseudospectra $\Lambda_\varepsilon(-V_1)$, $\varepsilon > 0$. For an arbitrary $A \in \mathcal{L}(X)$ in a general Banach space X those sets $\Lambda_\varepsilon(A)$ are defined by

$$\Lambda_\varepsilon(A) := \{\lambda \in \rho(A) : \ \|(\lambda I - A)^{-1}\| \geq \varepsilon^{-1}\} \cup \sigma(A), \quad \varepsilon > 0,$$

where $\sigma(A) := \mathbf{C}\backslash\rho(A)$ denotes the spectrum of A. ε-pseudospectra are considered in various papers, see, e.g., Reddy, Trefethen [**21**] for very recent results on that topic; the ε-pseudospectra for $-V_1$ are illustrated in Figure 2.

2.2.2. *The case* $X = L^p([0,a],\ \xi^{\beta-1}\,d\xi)$. For real $\beta > 0$ and $1 \leq p < \infty$, let $L^p([0,a],\ \xi^{\beta-1}\,d\xi)$ be the complex space of complex-valued, measurable functions u on $[0,a]$, such that $|u|^p$ is integrable with respect to the measure $\xi^{\beta-1}\,d\xi$, and this space is supplied with the norm

$$\|u\|_p := \left(\int_0^a |u(\xi)|^p \xi^{\beta-1}\,d\xi\right)^{1/p}, \quad u \in L^p([0,a],\ \xi^{\beta-1}\,d\xi).$$

Similarly, by $L^\infty([0,a],\ \xi^{\beta-1}\,d\xi)$ we denote the complex space of complex-valued, measurable functions u on $[0,a]$ which are essentially bounded with respect to the measure $\xi^{\beta-1}\,d\xi$, and then $\|u\|_\infty$ denotes the essential supremum of $|u|$ with respect to the measure $\xi^{\beta-1}\,d\xi$.

If $\beta = 1$ then we use the simplified notation $L^p([0,a])$. We have the following analog of Proposition 2.7.

Proposition 2.8 (Integration in L^p). *Let $\beta > 0$ and let $X = L^p([0,a],\ \xi^{\beta-1}\,d\xi)$ for some $1 \le p \le \infty$. The operators $V_1 \in \mathcal{L}(X)$ and $V_2 \in \mathcal{L}(X)$ (defined by* (1.3) *and* (1.4)*, respectively) are sectorial with angle $\pi/2$, and one has $M_0(V_j) \le 2$, $j = 1, 2$.*

Remark. If $X = L^2([0,a],\ \xi^{\beta-1}\,d\xi)$, then in fact one has $M_0(V_1) = M_0(V_2) = 1$; a reasoning is given in Halmos [**10**, Solution 150] (for the case $a = 1$, $\beta = 1$ and for V_2; the general case follows similarly).

Proof of Proposition 2.8. Throughout the proof, "=" in X means equality almost everywhere. We again give the proof for V_1 only, and to this end we consider

$$\begin{aligned} (Bf)(\xi) &:= -\xi^{-(\beta-1)} f'(\xi), \quad \xi \in [0,a], \quad f \in \mathcal{D}(B), \\ \mathcal{D}(B) &:= \{\ f \in X :\ f \text{ is absolutely continuous on} \\ &\qquad [0,a], f(a) = 0,\ \xi \longmapsto \xi^{-(\beta-1)} f'(\xi) \in X\}. \end{aligned} \tag{2.11}$$

In the sequel we shall show (a) that $V_1 \in \mathcal{L}(X)$ (then it follows immediately that in fact $B = V_1^{-1}$) and (b) that B is weakly sectorial with $M_0(B) \le 1$. To this end for (arbitrary but fixed) $t \ge 0$ and $u \in X$ we consider

$$f(\xi) := \int_\xi^a \eta^{\beta-1} e^{-t(\eta^\beta - \xi^\beta)/\beta} u(\eta)\,d\eta, \quad \xi \in [0,a], \tag{2.12}$$

and we shall estimate $\|f\|_p$ in terms of $\|u\|_p$. For that we define

$$\tilde{f}(\tilde{\xi}) := f((\beta\tilde{\xi})^{1/\beta}), \quad \tilde{\xi} \in [0, \tilde{a}],$$
$$\tilde{u}(\tilde{\eta}) := u((\beta\tilde{\eta})^{1/\beta}), \quad \tilde{\eta} \in [0, \tilde{a}],$$

where

$$\tilde{a} := a^\beta/\beta.$$

Note that

$$\|\tilde{f}\|_{L^p([0,\tilde{a}])} = \|f\|_p, \qquad \|\tilde{u}\|_{L^p([0,\tilde{a}])} = \|u\|_p \tag{2.13}$$

(where $\|\psi\|_{L^p([0,\tilde{a}])}$ denotes the L^p-norm of $\psi : [0, \tilde{a}] \to \mathbf{C}$ with respect to the Lebesgue measure). From the definition of $\tilde{f}$ and $\tilde{u}$ and from (2.12) we formally obtain

$$\tilde{f}(\tilde{\xi}) = \int_{\tilde{\xi}}^{\tilde{a}} e^{-t(\tilde{\eta}-\tilde{\xi})}\tilde{u}(\tilde{\eta})\,d\tilde{\eta}, \quad \tilde{\xi} \in [0, \tilde{a}]. \tag{2.14}$$

An application of Young's inequality for convolutions

$$\begin{gathered}\|k * \psi\|_{L^p(\mathcal{R})} \le \|k\|_{L^1(\mathcal{R})} \cdot \|\psi\|_{L^p(\mathcal{R})}, \\ k \in L^1(\mathcal{R}), \quad \psi \in L^p(\mathcal{R}),\end{gathered} \tag{2.15}$$

(see, e.g., Reed and Simon [**22**, Chapter IX.4]; here, $\|\phi\|_{L^q(\mathcal{R})}$ denotes the L^q-norm of $\phi : \mathcal{R} \to \mathbf{C}$ with respect to the Lebesgue measure), with

$$k(s) := \begin{cases} e^{ts} & \text{if } s \in [-\tilde{a}, 0], \\ 0 & \text{if } s \notin [-\tilde{a}, 0], \end{cases} \tag{2.16}$$

and

$$\psi(\tilde{\eta}) := \begin{cases} \tilde{u}(\tilde{\eta}) & \text{if } \tilde{\eta} \in [0, \tilde{a}], \\ 0 & \text{if } \tilde{\eta} \notin [0, \tilde{a}], \end{cases} \tag{2.17}$$

then yields $\tilde{f} \in L^p([0, \tilde{a}])$. In the case $t = 0$ we find from (2.14)–(2.17) that

$$\|\tilde{f}\|_{L^p([0,\tilde{a}])} \le \tilde{a}\|\tilde{u}\|_{L^p([0,\tilde{a}])},$$

and (2.13) then implies that $V_1 \in \mathcal{L}(X)$, and then it is obvious that $V_1 = B^{-1}$. In the case $t > 0$ it is also obvious that f in (2.12) is the unique solution of the equation

$$(tI + B)f = u, \tag{2.18}$$

and we obtain from (2.14)–(2.17) that

$$t\|\tilde{f}\|_{L^p([0,\tilde{a}])} \le \|\tilde{u}\|_{L^p([0,\tilde{a}])},$$

which in conjunction with (2.13) shows that B is weakly sectorial with $M_0(B) \le 1$.

The rest of the proof is similar to that of Proposition 2.7: Lemma 2.3 and the estimate $M_0(B) \le 1$ imply that B is sectorial with angle $\pi/2$, and then Lemma 2.4 yields that also $V_1 = B^{-1}$ is sectorial with angle $\pi/2$, and $M_0(V_1) \le 2$. □

3. Fractional powers of weakly sectorial operators. We shall consider fractional powers $V^\alpha \in \mathcal{L}(X)$, $\alpha \ge 0$, of weakly sectorial operators $V \in \mathcal{L}(X)$ (with X being an arbitrary complex Banach space) since Abel integral equations are of type $V_j^\alpha u = f_*$ (with V_j, $j = 1, 2$, as in (1.3) and (1.4), respectively, and for specific spaces X and $0 < \alpha < 1$). We moreover consider fractional powers of weakly sectorial operators $A \in \mathcal{L}(X)$ since in Theorem 4.1 (where the regularization properties of certain parameter choices and stopping rules for a class of methods are stated) we can admit then a fractional degree of smoothness for a solution $u_* \in X$ of $Au = f_*$.

3.1. *Properties of fractional powers.* Throughout this subsection let X be a complex Banach space.

Definition 3.1. Let $V \in \mathcal{L}(X)$ be weakly sectorial.

(a) For $0 < \alpha < 1$, *fractional powers* $V^\alpha \in \mathcal{L}(X)$ are defined by

$$V^\alpha u := \frac{\sin \pi\alpha}{\pi} \int_0^\infty s^{\alpha-1}(sI + V)^{-1} V u \, ds, \quad u \in X. \tag{3.1}$$

(b) For arbitrary $\alpha > 0$ we define $V^\alpha \in \mathcal{L}(X)$ by

$$V^\alpha := V^{\alpha - \lfloor \alpha \rfloor} V^{\lfloor \alpha \rfloor},$$

where $\lfloor\alpha\rfloor$ denotes the *greatest integer* $\leq \alpha$.

If $V \in \mathcal{L}(X)$ is weakly sectorial, then in fact the integral in (3.1) exists, and we have also $V^\alpha \in \mathcal{L}(X)$ then. An introduction to fractional powers of operators can be found, e.g., in the monographs by Krein [**12**] and Tanabe [**24**]. They contain also the following classical result (see Subsection 1.5.8 in [**12**], or Theorem 2.3.1 in [**24**]) that provides sufficient conditions for operators to be strictly sectorial, see also Corollary 3.3.

Theorem 3.2. *Let* $V \in \mathcal{L}(X)$ *be sectorial with angle* $\theta_0 \in (0,\pi]$. *Then for any* $0 < \alpha < 1$, V^α *is sectorial with angle* $(1-\alpha)\pi + \alpha\theta_0$, *and*

$$M_0(V^\alpha) \leq M_0(V). \tag{3.2}$$

The proof of Theorem 3.2 is based upon a representation for $(\lambda I + V^\alpha)^{-1}$ that one has on a smaller sector:

$$(\lambda I + V^\alpha)^{-1} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \frac{s^\alpha (sI + V)^{-1}}{s^{2\alpha} + 2\lambda s^\alpha \cos \pi\alpha + \lambda^2}\, ds,$$
$$\lambda \in \operatorname{int} \Sigma_{(1-\alpha)\pi}.$$

Here $\operatorname{int} \Sigma_{(1-\alpha)\pi}$ denotes the interior of $\Sigma_{(1-\alpha)\pi}$. We state the following important corollary which is an immediate consequence of Theorem 3.2.

Corollary 3.3. *If* $V \in \mathcal{L}(X)$ *is sectorial with angle* $\pi/2$, *then* V^α *is strictly sectorial for any* $0 < \alpha < 1$.

3.2. *Fractional integration.* In the following theorem V_1^α and V_2^α are explicitly given (with respect to various spaces), and in fact they are (generalized) Abel integral operators, with the classical case obtained for $\alpha = 1/2$, $\beta = 1$.

Theorem 3.4. *Let* $\beta > 0$. *In* $X = C[0,a]$ *or* $X = L^p([0,a],\ \xi^{\beta-1}\,d\xi)$, $1 \leq p \leq \infty$, *for the operators* V_1 *and* V_2, *defined by* (1.3) *and* (1.4), *respectively, one has for* $0 < \alpha < 1$,

$$(V_1^\alpha u)(\xi) = \frac{\beta^{1-\alpha}}{\Gamma(\alpha)} \int_\xi^a \frac{\eta^{\beta-1} u(\eta)}{(\eta^\beta - \xi^\beta)^{1-\alpha}} \, d\eta,$$
$$(V_2^\alpha u)(\xi) = \frac{\beta^{1-\alpha}}{\Gamma(\alpha)} \int_0^\xi \frac{\eta^{\beta-1} u(\eta)}{(\xi^\beta - \eta^\beta)^{1-\alpha}} \, d\eta,$$
$$\xi \in [0,a], \quad u \in X,$$

where in $X = L^p$, *"=" means equality almost everywhere. Moreover,* V_j^α *is strictly sectorial, and* $M_0(V_j^\alpha) \le 2$, $j = 1, 2$.

Proof. Again we give the proof for V_1 only. The inverse operator of V_1 is $(Bf)(\xi) = -\xi^{-(\beta-1)} f'(\xi)$, $\xi \in [0,a]$, $f \in \mathcal{D}(B)$ (for the domain of definition of B see (2.6) and (2.11), respectively), hence one has for $u \in X$

$$V_1^\alpha u = \frac{\sin \pi\alpha}{\pi} \int_0^\infty s^{\alpha-1} (I + sB)^{-1} u \, ds$$
$$= \frac{\sin \pi\alpha}{\pi} \int_0^\infty t^{-\alpha} (tI + B)^{-1} u \, dt,$$

therefore we get (see (2.7), (2.8) and (2.18), (2.12), respectively)

(3.3)
$$(V_1^\alpha u)(\xi) = \frac{\sin \pi\alpha}{\pi} \int_0^\infty t^{-\alpha} \int_\xi^a e^{-t(\eta^\beta - \xi^\beta)/\beta} \cdot \eta^{\beta-1} u(\eta) \, d\eta \, dt$$
$$= \frac{\sin \pi\alpha}{\pi} \int_\xi^a \left(\frac{\beta}{\eta^\beta - \xi^\beta} \right)^{1-\alpha} \left(\int_0^\infty s^{-\alpha} e^{-s} \, ds \right) \cdot \eta^{\beta-1} u(\eta) \, d\eta$$
$$= \frac{\beta^{1-\alpha}}{\Gamma(\alpha)} \int_\xi^a \frac{\eta^{\beta-1} u(\eta)}{(\eta^\beta - \xi^\beta)^{1-\alpha}} \, d\eta, \quad \xi \in [0,a],$$

where for $X = L^p$, "=" again means equality almost everywhere; and interchanging the order of integration in (3.3) is justified by the integrability condition on u. Finally, it follows from Propositions 2.7 and 2.8 as well as Theorem 3.2 and Corollary 3.3 that V_1^α is strictly sectorial with $M_0(V_1^\alpha) \le 2$. □

Remarks. 1. It is an immediate consequence of Propositions 2.7 and 2.8 as well as Theorem 3.2 that V_1^α and V_2^α in fact are sectorial with angle $(1-\alpha)\pi + \alpha\pi/2$ (for $0 < \alpha < 1$). Similar resolvent estimates for V_2^α (for the specific case $\beta = 1$ and $X = L^2([0,1])$, and in terms of the numerical range) can be found in Gohberg and Krein [**5**, Chapter V.6], and also in Gohberg and Krein [**6**, Appendix, Section 6].

2. A different proof of $M_0(V_2^\alpha) \le 2$ (for $0 < \alpha < 1$, $\beta = 1$ and $X = L^p([0,a])$, $1 \le p \le \infty$) is given in Gorenflo and Yamamoto [**8**].

3. It follows immediately from the remark that follows Proposition 2.8 and from (3.2) that for $X = L^2([0,a],\ \xi^{\beta-1}\,d\xi)$ and $0 < \alpha \le 1$ we have in fact $M_0(V_j^\alpha) = 1$, $j = 1, 2$. This in fact means that V_1^α and V_2^α are accretive, cf., also Gerlach and v.Wolfersdorf [**4**] and the literature cited therein (where $V_2^{1/2}$ for $\beta = 1$ and $a = 1$ is considered).

4. Recently Malamud [**14**] has shown that certain analytic perturbations of the kernel associated with V_2^α lead to integral operators $\tilde{V}_2^\alpha$ that are similar to V_2^α and which are therefore also strictly sectorial.

4. Regularization methods.

4.1. *A class of methods.* A common approach for numerically solving linear ill-posed problems in Hilbert spaces is to regularize the normal equations, see the recent monographs and surveys by Engl [**2**], Groetsch [**9**], Hanke and Hansen [**11**], Louis [**13**] and Murio [**15**] and their bibliographies. The approach in [**18**] (and in [**19**]) is to avoid the normalization process, and also other than L^2-spaces are admitted, and the main tool are resolvent conditions. In the next two subsections we review some basics of the approach in [**18, 19**]; we start with a general setting and consider the equation

$$Au = f_*, \tag{4.1}$$

where $A \in \mathcal{L}(X)$ is some (arbitrary) weakly sectorial operator with respect to a Banach space X. We moreover assume that the right-hand side $f_* \in \mathcal{R}(A)$ in (4.1) is known only approximately; more precisely, $f^\delta \in X$ and a noise level $\delta > 0$ are given, i.e.,

$$f_* \in \mathcal{R}(A), \quad f^\delta \in X, \quad \|f_* - f^\delta\| \le \delta, \tag{4.2}$$

where $\mathcal{R}(A)$ denotes the range of A. For the approximate solution of (4.1) we consider general methods

$$u_r^\delta := G_r f^\delta \quad \text{for } r \ge 0, \tag{4.3}$$

where $G_r \in \mathcal{L}(X)$, $r \geq 0$, is designed to approximate the inverse of A (if it exists) as $r \to \infty$. For iteration methods of type (4.3), r corresponds to the number of iterations, and for parametric methods of type (4.3), $\gamma = r^{-1}$ is a regularization parameter.

In the next subsection we shall propose a strategy to find an $r_\delta \geq 0$ such that $u^\delta_{r_\delta} \approx u_*$ (for some solution $u_* \in X$ of (4.1)). First, however, we introduce

$$H_r := I - G_r A, \qquad r \geq 0, \tag{4.4}$$

and impose the following conditions on G_r and H_r, $r \geq 0$:

$$AG_r = G_r A \qquad \text{for } r \geq 0, \tag{4.5}$$

$$\|H_r A^p\| \leq \gamma_p r^{-p} \qquad \text{for } r > 0, \quad 0 \leq p \leq p_0, \tag{4.6}$$

$$\|G_r\| \leq \gamma_* r \qquad \text{for } r \geq 0, \tag{4.7}$$

with constants γ_p and γ_*, and with 'qualification' $p_0 > 0$ (its value can be $p_0 = \infty$; and p in (4.6) takes (finite) real values). We next present three examples for methods of type (4.3) that fulfill (4.5)–(4.7).

4.1.1. *The iterated method of Lavrentiev.* For a weakly sectorial $A \in \mathcal{L}(X)$ (see (2.1) for the definition) we consider (for fixed integer m) Lavrentiev's m-times iterated method which for

$$\gamma = r^{-1}$$

generates a $u^\delta_r \in X$ by

$$\begin{aligned} (A + \gamma I)v_n &= \gamma v_{n-1} + f^\delta, \quad n = 1, 2, \dots, m \\ u^\delta_r &:= v_m \end{aligned} \tag{4.8}$$

with $v_0 = 0$. It is easy to see that $u^\delta_r \in X$ in (4.8) is of the form (4.3) with

$$G_r = r \sum_{j=1}^{m} (I + rA)^{-j},$$

and H_r in (4.4) takes the form

$$H_r = (I + rA)^{-m}.$$

Relations (4.5) and (4.7) quite obviously are fulfilled, and (4.6) with finite qualification $p_0 = m$ is a consequence of

$$\sup_{0\le p\le 1,\ r\ge 0} \|(I+rA)^{-1}(rA)^p\| < \infty,$$

see, e.g., Tanabe [**24**, Lemma 2.3.3] and its proof for the latter estimate.

Lavrentiev's iterated method (for unbounded weakly sectorial operators A in Banach spaces, in general) is considered, e.g., in Schock and Phóng [**23**] where certain a priori parameter choices are provided.

4.1.2. *The Richardson iteration.* For strictly sectorial operators $A \in \mathcal{L}(X)$, a first iteration method of type (4.3) is the Richardson iteration which for $\mu > 0$ (small enough) generates iteratively the sequence

$$u_{r+1}^\delta = u_r^\delta - \mu(Au_r^\delta - f^\delta), \quad r = 0, 1, 2, \dots . \tag{4.9}$$

Here $u_r^\delta \in X$ in (4.9) is of the form (4.3) with

$$G_r = \mu \sum_{j=0}^{r-1} (I - \mu A)^j$$

(for initial vector $u_0^\delta = 0$), and H_r in (4.4) then takes the form

$$H_r = (I - \mu A)^r.$$

Equation (4.5) obviously is valid, and (4.6) with $p_0 = \infty$ as well as (4.7) are easy consequences of Theorems 4.5.4 and 4.9.3 in Nevanlinna [**16**] (for $\mu > 0$ small enough).

4.1.3. *An implicit iteration method.* For strictly sectorial operators $A \in \mathcal{L}(X)$ and for $\mu > 0$ we consider the implicit iteration method

$$(I + \mu A)u_{r+1}^\delta = u_r^\delta + \mu f^\delta, \quad r = 0, 1, 2, \dots . \tag{4.10}$$

Here $u_r^\delta \in X$ in (4.10) is of the form (4.3) with

$$G_r = \mu \sum_{j=1}^{r} (I + \mu A)^{-j} \tag{4.11}$$

(for initial vector $u_0^\delta = 0$), and for H_r in (4.4) one has

$$H_r = (I + \mu A)^{-r}.$$

The commutativity condition (4.5) is satisfied trivially, and (4.6) with $p_0 = \infty$ can be derived by standard results in semigroup theory, see, e.g., Pazy [**17**, Theorem 1.7.7] for the case $p = 0$, and see [**17**, Theorem 2.5.5] for the case $p = 1$; the general case $p > 0$ follows quite similarly as for the case $p = 1$, see [**18**] or [**19**] for the details. Finally, (4.7) follows immediately from (4.6) for $p = 0$ and from (4.11).

4.2. *A discrepancy principle.* We now suppose that $A \in \mathcal{L}(X)$ is weakly sectorial and that (4.2) is valid; we moreover suppose that $\{u_r^\delta\}_r \subset X$ is of type (4.3) such that conditions (4.5)–(4.7) hold (with H_r as in (4.4)). As a rule for choosing r_δ in order to get a good approximation $u_{r_\delta}^\delta \in X$ for some solution $u_* \in X$ of (4.1), we next introduce a discrepancy principle. To this end, let $\Delta_r^\delta \in X$ denote the defect, i.e.,

$$\Delta_r^\delta := A u_r^\delta - f^\delta.$$

Discrepancy principle. Fix a real $b > \gamma_0$ (where γ_0 is as in (4.6)). Moreover, fix some $\theta > 0$ (for iterative methods of type (4.3) let $\theta = 1$, since then r can take integer values only), and set $r(k) = \theta k$. Stop the process of calculating $u_{r(k)}^\delta$, $k = 0, 1, 2, \ldots$, if for the first time

$$\|\Delta_{r(k)}^\delta\| \le b\delta,$$

and let $r_\delta := r(k_\delta)$, where k_δ denotes the stopping index.

The following result theorem can be derived from the results in [**18**]. Convergence result (4.12) shows that the discrepancy principle defines a regularization method, and estimate (4.15) provides, under additional smoothness assumptions, order-optimal convergence rates. Estimates (4.13) and (4.16) provide some information about the efficiency of the underlying algorithm.

Theorem 4.1. *Let X be a Banach space. Let $A \in \mathcal{L}(X)$ be weakly sectorial, and suppose that* (4.2) *is valid. Let $\{u_r^\delta\}_r$ be of type* (4.3), *and let conditions* (4.5)–(4.7) *hold* (*with* H_r *as in* (4.4), *and with qualification* $p_0 > 1$). *Let* $r_\delta \geq 0$ *be chosen by the above described discrepancy principle.*

1. *If* $u_* \in \overline{\mathcal{R}(A)}$ *solves* (4.1), *then*

$$\|u^\delta_{r_\delta} - u_*\| \longrightarrow 0 \quad \text{as } \delta \longrightarrow 0, \tag{4.12}$$

$$r_\delta \delta \longrightarrow 0 \quad \text{as } \delta \to 0. \tag{4.13}$$

2. *If, moreover, for some real* $0 < p \leq p_0 - 1$ *and* $z \in X$,

$$u_* = A^p z, \qquad \varrho := \|z\|, \tag{4.14}$$

then with some constants d_p, $e_p > 0$ *we have the estimates*

$$\|u^\delta_{r_\delta} - u_*\| \leq d_p(\varrho\delta^p)^{1/(p+1)}, \tag{4.15}$$

$$r_\delta \leq e_p(\varrho\delta^{-1})^{1/(p+1)}. \tag{4.16}$$

The constants d_p and e_p depend also on b, and $\overline{\mathcal{R}(A)}$ denotes the closure of $\mathcal{R}(A)$.

Remarks. 1. Theorem 4.1 generalizes results obtained for Hilbert spaces X and symmetric, positive semidefinite operators $A \in \mathcal{L}(X)$, see Vainikko [**25**]. Note that Theorem 4.1 is even important for the numerical solution of nonsymmetric equations $Au = f_*$ in *Hilbert* spaces X since methods of type (4.3) avoid the normalization $A^\star Au = A^\star f_*$, where $A^\star \in \mathcal{L}(X)$ denotes the adjoint operator of A. The smoothness condition (4.14) for nonsymmetric equations, however, then differs from the well-known assumption '$u_* \in \mathcal{R}((A^\star A)^{p/2})$', since one has $\mathcal{R}((A^\star A)^{p/2}) \neq \mathcal{R}(A^p)$.

2. For Abel integral operators $A = V_1^\alpha \in \mathcal{L}(X)$, where $X = L^2([0,a])$, $0 < \alpha \leq 1$ and $\beta = 1$, we illustrate the statements in Theorem 4.1, and to this end for integer $k \geq 1$ we denote by $W^{k,2}([0,a])$ the Sobolev space of all functions $u : [0,a] \to \mathbf{C}$ such that u and its distributional derivatives $u^{(j)}$ of order $j \leq k$ all belong to $L^2([0,a])$. If

$$u_* \in W^{k,2}([0,a]),$$
$$u_*(a) = u_*'(a) = \cdots = u_*^{(k-1)}(a) = 0,$$

then for $p = k/\alpha$ one has $u_* \in \mathcal{R}(A^p)$, and we thus can expect a convergence rate

$$\|u_{r_\delta}^\delta - u_*\| = \mathcal{O}(\delta^{k/(k+\alpha)}) \quad \text{as } \delta \longrightarrow 0$$

(with the restriction $k \le (m-1)\alpha$ for Lavrentiev's m-times iterated method).

4.3. *Computational experiments.* In this subsection we solve numerically Abel's classical integral equation

$$(4.17) \qquad (Au)(\xi) := \frac{1}{\sqrt{\pi}} \int_0^\xi (\xi - \eta)^{-1/2} u(\eta)\, d\eta = f_*(\xi), \quad \xi \in [0,1].$$

As underlying space we take the complex space

$$X = L^2([0,1])$$

supplied with the inner product

$$\langle u, v \rangle = \int_0^1 u(\eta)\overline{v(\eta)}\, d\eta, \quad u, v \in L^2([0,1]),$$

and the corresponding norm is $\|u\|_2 = \langle u, u \rangle^{1/2}$, $u \in L^2([0,1])$. Then one has $A = V_2^{1/2}$ (for $\beta = 1$ and $a = 1$), and thus A is strictly sectorial (see Theorem 3.4).

We always choose perturbed right-hand sides $f^\delta = f_* + \delta \cdot v$, where $v \in X$ has uniformly distributed random (real) values so that $\|v\|_2 \le 1$, and where

$$\delta = \|f_*\|_2 \cdot \%/100,$$

with % noise $\in \{0.33, 1.00, 2.00, 3.00, 10.00\}$ in our implementations.

We carry out Lavrentiev's (m-times iterated) method as well as the implicit iteration method. For both methods we have $\gamma_0 = 1$ in (4.6) (see the remark that follows Theorem 3.4), thus we can take $b = 1.5$ in the definition of the parameter choice/stopping rule. The iterated method of Lavrentiev is considered with $m = 5$, and the corresponding norm of the defect Δ_r^δ is computed successively for $r = 0, \theta, 2\theta, 3\theta, \ldots, \theta = 1.0$, until it falls below the required level $(3/2)\delta$.

We next present the results of experiments for two different right-hand sides. In our first experiment we choose

$$f_*(\xi) = \frac{\sqrt{\pi}}{2}\xi, \quad \xi \in [0,1],$$

and then the solution of (4.17) is given by, cf., Gorenflo and Vessella [**7**, Chapter 1.1]

$$u_*(\eta) = \sqrt{\eta}, \quad \eta \in [0,1],$$

and thus

$$u_* \in \mathcal{R}(A^p) \quad \text{for all } 0 < p < 2, \tag{4.18}$$

$$u_* \notin \mathcal{R}(A^2). \tag{4.19}$$

The following tables contain the results for Lavrentiev's (5-times iterated) method and the implicit method, respectively. Note that due to (4.19) one cannot derive from Theorem 4.1 that the entries in the third and fifth column, respectively, stay bounded as % of noise decreases. On the other hand, however, due to (4.18), it is no surprise that these entries in our experiments in fact stay bounded.

TABLE 1.

Lavrentiev's (iterated) method					
% noise	$\Vert u^\delta_{r_\delta} - u_*\Vert_2$	$\Vert u^\delta_{r_\delta} - u_*\Vert_2/\delta^{2/3}$	r_δ	$r_\delta\delta^{1/3}$	♯ flops
10.00	0.1343	0.97	1.0	0.37	0.3e+06
3.00	0.0672	1.09	2.0	0.50	0.5e+06
1.00	0.0245	0.83	2.0	0.34	0.5e+06
0.33	0.0109	0.77	3.0	0.35	0.6e+06

TABLE 2.

Implicit iteration method					
% noise	$\Vert u^\delta_{r_\delta} - u_*\Vert_2$	$\Vert u^\delta_{r_\delta} - u_*\Vert_2/\delta^{2/3}$	r_δ	$r_\delta\delta^{1/3}$	♯ flops
10.00	0.1197	0.86	15	5.57	1.1e+06
3.00	0.0501	0.81	24	5.96	1.6e+06
1.00	0.0246	0.83	35	6.03	2.0e+06
0.33	0.0114	0.80	52	6.19	2.8e+06

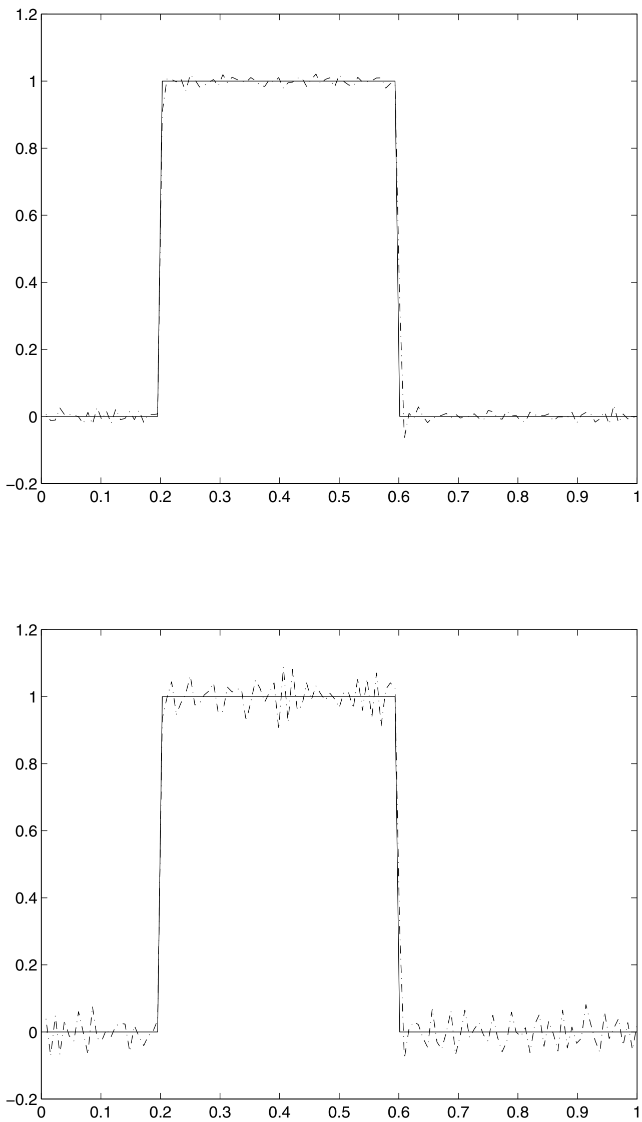
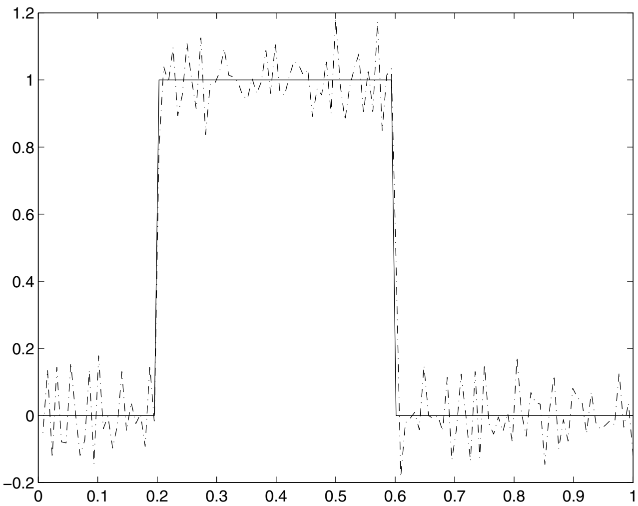


FIGURE 3. Discrepancy principle for Lavrentiev's method ($m = 5$); 0.33% noise (top) and 1.00% noise (bottom).



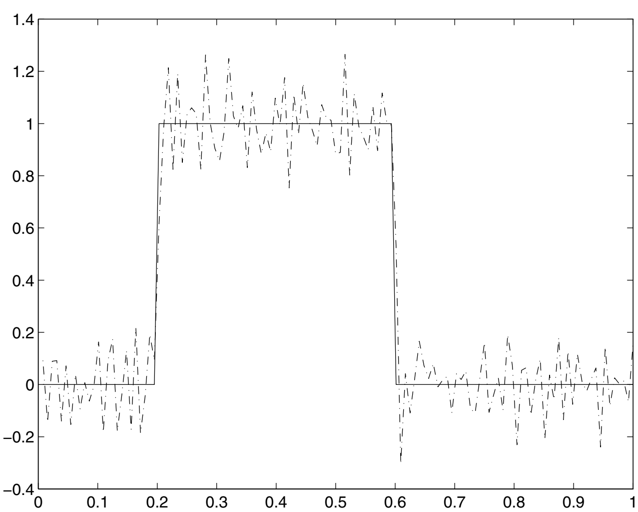


FIGURE 4. Discrepancy principle for Lavrentiev's method ($m = 5$); 2.00% noise (top) and 3.00% noise (bottom).

In the sequel we shall make some comments concerning the underlying discretization (with no discussion of the corresponding discretization error). For the implementations, equation (4.17) has been discretized by a Bubnov-Galerkin method with piecewise constant functions, with $\Psi_j = \chi_{[(j-1)h, jh]}$, $j = 1, \dots, N$, as basis functions. Here $h = 1/N$, and χ_M denotes the characteristic function corresponding to a set M. We take $N = 128$ in fact, and the entries of the corresponding $N \times N$ matrix $A_h = (\langle A\Psi_j, \Psi_i \rangle)$ are computed exactly. All computations are performed in MATLAB on an IBM RISC/6000.

The matrix A_h is in fact a triangular (Toeplitz) matrix, and that means that for any $y \in \mathbf{C}^N$, $(I + rA_h)^{-1}y$ can be computed very fast for *different* values of r; thus for the Volterra integral equation (4.17), the number of operations for finding r_δ for Lavrentiev's method is rather small, see the last column of Table 1. Note that the common approach of normalizing the discretized (nonsymmetric) equation $A_h u_h = f_{*,h}$ destroys the triangular form of the system.

In our next (and final) experiment we choose

$$f_*(\xi) = \frac{2}{\sqrt{\pi}}((\xi - 0.2)^{1/2}\chi_{[0.2,1]}(\xi) - (\xi - 0.6)^{1/2}\chi_{[0.6,1]}(\xi)), \quad \xi \in [0, 1];$$

then

$$u_* = \chi_{[0.2,0.6]}$$

solves (4.17), see again [**7**, Chapter 1.1]. Figures 3 and 4 illustrate the results for the (5-times iterated) method of Lavrentiev. The solid lines correspond to u_*, and the dotted lines correspond to the approximations that are obtained by the same parameter choice as for the first right-hand side.

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REFERENCES

1. R. deLaubenfels, *Inverses of generators*, Proc. Amer. Math. Soc. **104** (1988), 443–448.

2. H.W. Engl, *Regularization methods for the stable solution of inverse problems*, Surveys Math. Indust. **4** (1993), 71–143.

3. H.O. Fattorini, *The Cauchy problem*, Addison-Wesley, Reading, 1st edition, 1983.

4. W. Gerlach and L. v.Wolfersdorf, *On approximate computation of the values of the normal derivative of solutions to linear partial differential equations of second order with applications to Abel's integral equation*, Z. Angew. Math. Meth. **66** (1986), 31–36.

5. I.C. Gohberg and M.G. Krein, *Introduction to the theory of linear nonselfadjoint operators*, 1st edition, American Math. Soc., Providence, Rhode Island, 1969.

6. ———, *Introduction to the theory and applications of Volterra operators in Hilbert Space*, 1st edition, American Math. Soc., Providence, Rhode Island, 1970.

7. R. Gorenflo and S. Vessella, *Abel integral equations*, 1st edition, Springer, New York, 1991.

8. R. Gorenflo and M. Yamamoto, *On regularized inversion of Abel integral equations*, in *Analysis and mechanics of continuous media*, Volume 3 (N.H. Anh, D.M. Duc, B.D. Khanh, P.Q. Khanh and T.D. Van., eds.), Mathematical Society of Ho Chi Minh City, Ho Chi Minh City, VietNam, 1995.

9. C.W. Groetsch, *Inverse problems in the mathematical sciences*, 1st edition, Vieweg, Wiesbaden, 1993.

10. P.R. Halmos, *A Hilbert space problem book*, 1st, reprint edition, Springer, New York, 1978.

11. M. Hanke and P.C. Hansen, *Regularization methods for large-scale problems*, Surveys Math. Indust. **3** (1993), 253–315.

12. M.G. Krein, *Linear differential equations in Banach space*, 1st edition, Amer. Math. Soc., Providence, Rhode Island, 1971.

13. A.K. Louis, *Inverse und schlecht gestellte Probleme*, 1st edition, B.G. Teubner, Stuttgart, 1989.

14. M. Malamud, *Similar Volterra operators and related problems of the theory of differential equations with fractional order*, Publ. Moscow Math. Soc. **55**, (1993), 73–148, in Russian.

15. D.A. Murio, *The mollification method and the numerical solution of ill-posed problems*, 1st edition, Wiley, New York, 1993.

16. O. Nevanlinna, *Convergence of iterations for linear equations*, 1st edition, Birkhäuser, Basel, 1993.

17. T. Pazy, *Semigroups and applications to partial differential operators*, 1st, reprint edition, Springer, New York, 1983.

18. R. Plato, *The discrepancy principle for iterative and parametric methods to solve linear ill-posed equations*, Numer. Math., to appear.

19. R. Plato, *Iterative and parametric methods for linear ill-posed equations*, Habilitationsschrift, Fachbereich Mathematik, TU Berlin, 1995.

20. E.I. Pustyl'nik, *On functions of a positive operator*, Math. USSR Sb. **47** (1982), 27–42.

21. S.C. Reddy and L.N. Trefethen, *Pseudospectra of the convection-diffusion operator*, SIAM J. Appl. Math. **54** (1994), 1634–1649.

22. M. Reed and B. Simon, *Methods of modern mathematical physics* II, 1st edition, Academic Press, New York, 1975.

23. E. Schock and Vũ Quôc Phóng, *Regularization of ill-posed problems involving unbounded operators in Banach spaces*, Hokkaido Math. J. **20** (1991), 559–569.

24. H. Tanabe, *Equations of evolution*, 1st edition, Pitman, London, 1979.

25. G.M. Vainikko, *The discrepancy principle for a class of regularization methods*, U.S.S.R. Comput. Math. and Math. Phys. **22** (1982), 1–19.

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