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ON THE SOLUTION OF THE GENERALIZED AIRFOIL EQUATION

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1. Introduction. In the present paper we consider the singular integral equation

$$\frac{1}{\pi}\int_{-1}^{1}\frac{f(y)}{x-y}\,dy+\frac{m(x)}{\pi}\int_{-1}^{1}f(y)\ln|x-y|\,dy + \frac{1}{\pi}\int_{-1}^{1}k(x,y)f(y)\,dy=g(x),\qquad -1<x<1, \tag{1.1}$$

where m, k and g are given functions, f is an unknown solution, and the first integral has to be interpreted in the Cauchy principal value sense. Equation (1.1) arises from the two-dimensional oscillating airfoil in a wind tunnel with subsonic flow (see, for example, [**4**]) and has applications in diffraction theory and two-dimensional elasticity theory (see, for example, [**16, 21**]).

The analytical as well as the numerical solutions of equation (1.1) have been studied by many authors [**1**–**3**, **6**–**16**, **18**–**21**, **23**–**27**, **29, 30, 32**]. (Some of these papers only deal with the case $m = 0$.) M. Schleiff [**29**] solved equation (1.1) for $k = 0$ and $m, f \in \mathcal{L}^2_\varrho$, where $\mathcal{L}^2_\varrho$ is the space of square integrable functions on the interval $(-1,1)$ with the Chebyshev weight $\varrho(x) = (1-x^2)^{1/2}$. Using those results, he constructed a Fredholm integral equation of the second kind equivalent to equation (1.1). In the present paper we extend Schleiff's results to the cases of spaces $\mathcal{L}^2_w$ and of weighted Sobolev-type spaces with weights $w(x) = (1-x)^\alpha(1+x)^\beta$, where $|\alpha| = |\beta| = 1/2$ (Section 3). These solvability results then give rise to establishing a numerical procedure for which stability and error estimates in a scale of Sobolev-type norms as well as in weighted uniform norms will be proved (Section 4).

2. Preliminaries. Throughout this paper let λ denote the Lebesgue

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measure in the open interval $\Omega = (-1, 1)$. Those functions on Ω which coincide outside a Lebesgue null set will be regarded as equal.

Define functions ϱ and σ on Ω by

$$\varrho(x) = (1-x^2)^{1/2} \quad \text{and} \quad \sigma(x) = (1-x)^{-1/2}(1+x)^{1/2}, \qquad x \in \Omega. \tag{2.1}$$

Let w always stand for any one of the functions ϱ, $1/\varrho$, σ and $1/\sigma$. Let $\pi^{-1}w\lambda$ denote the indefinite integral of the function $\pi^{-1}w$ with respect to λ. As in [**2**], let $\mathcal{L}^2_w$ denote the space $\mathcal{L}^2(\pi^{-1}w\lambda)$ of complex-valued square integrable functions with respect to the measure $\pi^{-1}w\lambda$. Then $\mathcal{L}^2_w$ becomes a Hilbert space with inner product

$$(f|g)_w = \pi^{-1}\int_{-1}^{1} f\bar{g}w\,d\lambda, \qquad f, g \in \mathcal{L}^2_w.$$

The associated norm on $\mathcal{L}^2_w$ is denoted by $\|\cdot\|_w$. The following relationships are then clear:

(i) $\mathcal{L}^2_{1/\varrho} \subset \mathcal{L}^2(\lambda) \subset \mathcal{L}^2_{\varrho}$;

(ii) $\mathcal{L}^2_{1/\varrho} \subset \mathcal{L}^2_{\sigma} \subset \mathcal{L}^2_{\varrho}$; and

(iii) $\mathcal{L}^2_{1/\varrho} \subset \mathcal{L}^2_{1/\sigma} \subset \mathcal{L}^2_{\varrho}$.

Furthermore, we have

$$\mathcal{L}^2_{\varrho} \subset \bigcap_{1<r<4/3} \mathcal{L}^r(\lambda). \tag{2.2}$$

Let $f \in \mathcal{L}^1(\lambda)$. Then the Cauchy principal value

$$Hf(x) = \frac{1}{\pi}\lim_{\varepsilon\to+0}\left(\int_{-1}^{x-\varepsilon} + \int_{x+\varepsilon}^{1}\right)\frac{f(y)}{x-y}\,dy$$

exists for λ-almost every $x \in \Omega$ and the resulting function Hf is λ-measurable (see [**5**, Theorem 8.1.5], for example). So we have a linear operator H from the space $\mathcal{L}^1(\lambda)$ into the space of all λ-measurable functions. The following lemma is a special case of the Khvedelidze theorem, which can be found in [**15**, Theorem 1.2] or [**24**, Theorem II.3.1], for example.

Lemma 2.1. *Let* $w = \varrho$, $1/\varrho$, σ *or* $1/\sigma$. *Then* $H(\mathcal{L}^2_w) \subset \mathcal{L}^2_w$ *and the restriction* H_w *of* H *to the Hilbert space* $\mathcal{L}^2_w$ *is a continuous linear operator from* $\mathcal{L}^2_w$ *into itself. Furthermore,* $(1/w)H(wf) \in \mathcal{L}^2_w$ *for every* $f \in \mathcal{L}^2_w$.

A continuous linear operator S from a Banach space X into X is called a *Noether* (*Fredholm*) *operator* if its range $\mathcal{R}(S) = S(X)$ is closed and if both the dimension of its null space $\mathcal{N}(S) = S^{-1}(\{0\})$ and the co-dimension of $\mathcal{R}(S)$ in X are finite. The index $\operatorname{ind}(S)$ of such an operator S is defined as

$$\operatorname{ind}(S) = \dim \mathcal{N}(S) - \operatorname{codim} \mathcal{R}(S).$$

Lemma 2.2. *The following statements hold.*

(i) *The operator* $H_\varrho : L^2_\varrho \to \mathcal{L}^2_\varrho$ *is a surjection with null space*

$$\mathcal{N}(H_\varrho) = \{c/\varrho : c \in \mathbf{C}\},$$

and

$$H_\varrho^{-1}(\{g\}) = -(1/\varrho)H(\varrho g) + \mathcal{N}(H_\varrho), \qquad g \in \mathcal{L}^2_\varrho.$$

In particular, $\operatorname{ind}(H_\varrho) = 1$.

(ii) *The operator* $H_{1/\varrho} : \mathcal{L}^2_{1/\varrho} \to \mathcal{L}^2_{1/\varrho}$ *is an injection with range*

$$\mathcal{R}(H_{1/\varrho}) = \{g \in \mathcal{L}^2_{1/\varrho} : (g|1)_{1/\varrho} = 0\},$$

and

$$H^{-1}_{1/\varrho} g = -\varrho H(g/\varrho), \qquad g \in \mathcal{R}(H_{1/\varrho}).$$

In particular, $\operatorname{ind}(H_{1/\varrho}) = -1$.

(iii) *Let* $w = \sigma$ *or* $1/\sigma$. *Then the operator* $H_w : \mathcal{L}^2_w \to \mathcal{L}^2_w$ *is a bijective isometry, and*

$$H_w^{-1} g = -(1/w)H(wg), \qquad g \in \mathcal{L}^2_w.$$

In particular, $\operatorname{ind}(H_w) = 0$.

Proof. Statement (i) follows from the fact that the restriction of H to the Banach space $\mathcal{L}^r(\lambda)$, $1 < r < 4/3$, (cf. (2.2)) has the same property as H_ϱ (see [**17**, Theorem 13.9] or [**26**, Proposition 2.4], for example).

Statement (ii) can be proved as in the case of the restriction of H to the Banach space $\mathcal{L}^r(\lambda)$, $2 < r < \infty$, (see [**17**, Theorem 13.9] or [**26**, Proposition 2.6], for example).

Statement (iii) has been shown in [**30**, p. 149] for the case when $w = \sigma$. The case that $w = 1/\sigma$ can be proved similarly. □

Let $s \geq 0$. We shall now define a linear subspace $\mathcal{L}^2_{\varrho,s}$ of $\mathcal{L}^2_\varrho$ as in [**2**, Section 2]. Let

$$u_n(x) = \frac{2^{1/2}\sin[(n+1)\arccos x]}{\sin(\arccos x)}, \qquad x \in \Omega,$$

for each $n = 0, 1, 2, \dots$. Namely, $2^{-1/2}u_n$, $n = 0, 1, 2, \dots$, are the Chebyshev polynomials of the second kind. Then $\{u_n\}_{n=0}^\infty$ is a complete orthonormal sequence in the Hilbert space $\mathcal{L}^2_\varrho$. Now let $\mathcal{L}^2_{\varrho,s}$ denote the linear subspace of $\mathcal{L}^2_\varrho$ consisting of those functions f on Ω such that

$$\sum_{n=0}^\infty (1+n)^{2s}|(f|u_n)_\varrho|^2 < \infty.$$

The vector space $\mathcal{L}^2_{\varrho,s}$ becomes a Hilbert space with the inner product given by

$$(f|g)_{\varrho,s} = \sum_{n=0}^\infty (1+n)^{2s}(f|u_n)_\varrho\overline{(g|u_n)}_\varrho, \qquad f, g \in \mathcal{L}^2_{\varrho,s}.$$

The associated norm on $\mathcal{L}^2_{\varrho,s}$ will be denoted by $\|\cdot\|_{\varrho,s}$. Clearly the Hilbert space $\mathcal{L}^2_{\varrho,s}$ is continuously embedded into $\mathcal{L}^2_\varrho$. It is worth noting that the definition of $\mathcal{L}^2_{\varrho,s}$ is dependent on $\{u_n\}_{n=0}^\infty$ so that another complete orthonormal sequence in $\mathcal{L}^2_\varrho$ may define a different linear subspace of $\mathcal{L}^2_\varrho$.

Let $t_0 = 1$ and let $t_n(x) = 2^{1/2}\cos(n\arccos x)$ for every $x \in \Omega$ and every $n = 1, 2, \dots$. So t_0, $2^{-1/2}t_1$, $2^{-1/2}t_2, \dots$, are the Chebyshev polynomials of the first kind. Moreover, let

$$p_n(x) = \frac{\cos[(n+2^{-1})\arccos x]}{\cos(2^{-1}\arccos x)}$$

and

$$q_n(x) = \frac{\sin[(n+2^{-1}) \arccos x]}{\sin(2^{-1} \arccos x)}$$

for every $x \in \Omega$ and every $n = 0, 1, 2, \ldots$. The so-defined functions p_n and q_n, $n = 0, 1, 2, \ldots$, are the Chebyshev polynomials of the third and fourth kind, respectively.

If $s \geq 0$ and if $w = 1/\varrho$, σ or $1/\sigma$, then we define the Hilbert space $\mathcal{L}^2_{w,s}$ with inner product $(\cdot|\cdot)_{w,s}$ by using $\{t_n\}_{n=0}^\infty$, $\{p_n\}_{n=0}^\infty$ or $\{q_n\}_{n=0}^\infty$, respectively, as in the definition of $\mathcal{L}^2_{\varrho,s}$.

Observe that $\{u_n\}_{n=0}^\infty$, $\{t_n\}_{n=0}^\infty$, $\{p_n\}_{n=0}^\infty$ and $\{q_n\}_{n=0}^\infty$ are unique complete orthonormal sequences of polynomials, with positive leading coefficients, having the property:

$$\deg u_n = \deg t_n = \deg p_n = \deg q_n = n, \qquad n = 0, 1, 2, \ldots,$$

in the Hilbert spaces $\mathcal{L}^2_\varrho$, $\mathcal{L}^2_{1/\varrho}$, $\mathcal{L}^2_\sigma$ and $\mathcal{L}^2_{1/\sigma}$, respectively.

Given a distribution v on Ω, its derivative in the distribution sense will be denoted by Dv. According to [**2**, pp. 196–197], the space $\mathcal{L}^2_{w,s}$ can be expressed as follows.

Lemma 2.3. *Let $w = \varrho$, $1/\varrho$, σ or $1/\sigma$. Let s be a positive integer. Then a function $f \in \mathcal{L}^2_w$ belongs to $\mathcal{L}^2_{w,s}$ if and only if $\varrho^j D^j f$ is again an element of $\mathcal{L}^2_w$ for every $j = 1, 2, \ldots, s$. Furthermore, the norm $\|\cdot\|_{w,s}$ on $\mathcal{L}^2_{w,s}$ is equivalent to the norm*:

$$f \longmapsto \left(\sum_{j=0}^{s} \|\varrho^j D^j f\|_w^2 \right)^{1/2}, \qquad f \in \mathcal{L}^2_{w,s}.$$

Definition 2.1. Let $w = \varrho$, $1/\varrho$, σ or $1/\sigma$. Let $s > 0$. Define

$$(1/w)\mathcal{L}^2_{1/w,s} = \left\{ \frac{1}{w} f : f \in \mathcal{L}^2_{1/w,s} \right\} \quad (\subset \mathcal{L}^2_w).$$

Equip the vector space $(1/w)\mathcal{L}^2_{1/w,s}$ with the norm so that the linear map $f \mapsto (1/w)f$, $f \in \mathcal{L}^2_{1/w,s}$, from $\mathcal{L}^2_{1/w,s}$ onto $(1/w)\mathcal{L}^2_{1/w,s}$ becomes an isometry; in particular, $(1/w)\mathcal{L}^2_{1/w,s}$ is then a Banach space because so is $\mathcal{L}^2_{1/w,s}$.

The Banach space $(1/w)\mathcal{L}^2_{1/w,s}$ is continuously embedded into $\mathcal{L}^2_w$ and

$$H_w((1/w)\mathcal{L}^2_{1/w,s}) \subset \mathcal{L}^2_{w,s}, \qquad s > 0. \tag{2.3}$$

This inclusion has been shown in [**2**, Lemma 4.1]. Its proof is based on the following result which is a special case of [**33**, (25)].

Lemma 2.4. *The following identities hold*:

(i) $H(\varrho u_n) = t_{n+1}$, $n = 0, 1, 2, \dots$.

(ii) $H(t_0/\varrho) = 0$ *and* $H(t_n/\varrho) = -u_{n-1}$, $n = 1, 2, \dots$.

(iii) $H(\sigma p_n) = -q_n$, $n = 0, 1, 2, \dots$.

(iv) $H(q_n/\sigma) = p_n$, $n = 0, 1, 2, \dots$.

If $w = \varrho$, $1/\varrho$, σ or $1/\sigma$ and $s > 0$, then let

$$H_{w,s} : (1/w)\mathcal{L}^2_{1/w,s} \longrightarrow \mathcal{L}^2_{w,s} \tag{2.4}$$

denote the restriction of H_w to $(1/w)\mathcal{L}^2_{1/w,s}$; see (2.3). The following lemma has essentially been given in [**2**, Lemma 4.2 (ii)] and its proof follows from Lemma 2.4.

Lemma 2.5. *Let* $s > 0$. *Let* $w = \varrho$, $1/\varrho$, σ *or* $1/\sigma$. *Then the linear operator* $H_{w,s}$ *given by* (2.4) *enjoys the same property as* $H_w : \mathcal{L}^2_w \to \mathcal{L}^2_w$ *in* Lemma 2.2.

Let $AC(\Omega)$ denote the space of complex-valued, continuous functions f on Ω for which there is an absolutely continuous function g on the closed interval $[-1, 1]$ such that $f(x) = g(x)$ for every $x \in \Omega$.

Let $w = \varrho$, $1/\varrho$, σ or $1/\sigma$. Let $f \in \mathcal{L}^2_w$. By (2.2), the function Lf defined by

$$(Lf)(x) = \pi^{-1} \int_{-1}^{1} f(y) \ln|y - x| \, dy, \qquad x \in \Omega,$$

belongs to $AC(\Omega)$ and $D(Lf) = Hf$; see [**17**, Section 13], for example. In particular, $Lf \in \mathcal{L}^2_w$ because $AC(\Omega) \subset \mathcal{L}^2_w$.

Lemma 2.6. *The following identities hold.*

(i) $L(t_0/\varrho) = -(\ln 2)t_0 = -(2^{-1/2}\ln 2)u_0$; $L(t_1/\varrho) = -t_1 = -2^{-1}u_1$; *and* $L(t_n/\varrho) = -t_n/n = 2^{-1}(u_{n-2} - u_n)/n$, $n = 2, 3, \dots$.

(ii) $L(\varrho u_0) = -2^{-1}[(\sqrt{2}\ln 2)t_0 - t_2/2]$; *and* $L(\varrho u_n) = -2^{-1}[t_n/n - t_{n+2}/(n+2)]$, $n = 1, 2, \dots$.

(iii) $L(\sigma p_0) = (2^{-1} - \ln 2)q_0 - 2^{-1}q_1$; *and* $L(\sigma p_n) = 2^{-1}[q_{n-1}/n - q_n/n(n+1) - q_{n+1}/(n+1)]$, $n = 1, 2, \dots$.

(iv) $L(q_0/\sigma) = (2^{-1} - \ln 2)p_0 + 2^{-1}p_1$; *and* $L(q_n/\sigma) = -2^{-1}[p_{n-1}/n + p_n/n(n+1) - p_{n+1}/(n+1)]$, $n = 1, 2, \dots$.

Proof. Statement (i) can be found in [**28**, p. 138], for instance. Statement (ii) follows from (i) because $\varrho u_0 = (\sqrt{2}t_0 - t_2)/(2\varrho)$ and $\varrho u_n = (t_n - t_{n+2})/(2\varrho)$ for every $n = 1, 2, \dots$. Statement (iii) has been given in [**1**, Corollary 3.3] and (iv) can be proved similarly. □

Let $L_w : \mathcal{L}^2_w \to \mathcal{L}^2_w$ denote the linear operator which assigns Lf to each function $f \in \mathcal{L}^2_w$, when $w = \varrho$, $1/\varrho$, σ or $1/\sigma$. By [**29**, Satz (Theorem) 2] the operator L_ϱ is continuous. For the remaining cases: $w = 1/\varrho$, σ, $1/\sigma$, the continuity of L_w follows from the closed graph theorem because $\mathcal{L}^2_w \subset \mathcal{L}^2_\varrho$.

Proposition 2.1. *Let* $s \geq 0$. *Let* $w = \varrho$, $1/\varrho$, σ *or* $1/\sigma$. *Then* L_w *maps the subspace* $(1/w)\mathcal{L}^2_{1/w,s}$ *of* $\mathcal{L}^2_{w,s}$ *into* $\mathcal{L}^2_{w,s+1}$ *and the linear map* $L_{w,s} : (1/w)\mathcal{L}^2_{1/w,s} \to \mathcal{L}^2_{w,s+1}$ *which assigns* $L_w f$ *to each* $f \in (1/w)\mathcal{L}^2_{1/w,s}$ *is continuous.*

Proof. In view of Definition 2.1, the statement is a direct consequence of the following inequalities:

(i) $\|L(f/\varrho)\|^2_{\varrho,s+1} \leq (5/2)\|f\|^2_{1/\varrho,s}$, $f \in \mathcal{L}^2_{1/\varrho,s}$;

(ii) $\|L(\varrho f)\|^2_{1/\varrho,s+1} \leq (3(1+3^{2s})/2)\|f\|^2_{\varrho,s}$, $f \in \mathcal{L}^2_{\varrho,s}$;

(iii) $\|L(f/\sigma)\|^2_{\sigma,s+1} \leq 2(1+2^{2s+1})\|f\|^2_{1/\sigma,s}$, $f \in \mathcal{L}^2_{1/\sigma,s}$; and

(iv) $\|L(\sigma f)\|^2_{1/\sigma,s+1} \leq 2(1+2^{2s+1})\|f\|^2_{\sigma,s}$, $f \in \mathcal{L}^2_{\sigma,s}$.

A routine calculation based on Lemma 2.6 will derive these inequalities. □

Remark 2.1. In the case when $w = \sigma$, the statement of Proposition 2.1 has been given in [**2**, Lemma 5.1 (iv)], without stating constants as above.

Remark 2.2. Let $s \geq 0$. The restriction of L_ϱ to $(1/\varrho)\mathcal{L}^2_{1/\varrho,s}$, defines a continuous linear operator with values in $\mathcal{L}^2_{1/\varrho,s+1}$. In fact,

$$\|L(f/\varrho)\|^2_{1/\varrho,s+1} \leq 4\|f\|^2_{1/\varrho,s}, \qquad f \in \mathcal{L}^2_{1/\varrho,s}.$$

Remark 2.3. The main reason for considering Jacobi weights $w(x) = (1-x)^\alpha(1+x)^\beta$ with $|\alpha| = |\beta| = 1/2$ is that in this case the operator $H_w + mL_w$ behaves like the finite Hilbert transform H_w (see Lemma 2.2, Proposition 3.1 and Theorems 3.1 and 3.2). This allows us in Section 4 to present error estimates for a numerical method based on Chebyshev polynomials and a discretization of the equivalent Fredholm integral equation of the second kind. Note that the aforementioned Jacobi weights are the most frequently used ones in applications.

3. The unperturbed generalized airfoil equation. Let $\Omega = (-1,1)$. Let w stand for any one of the functions ϱ, $1/\varrho$, σ and $1/\sigma$ on Ω as in Section 2. The main aim of this section is to solve, in $\mathcal{L}^2_w$, the singular integral equation

$$(H_w + mL_w)f = g \tag{3.1}$$

for a given $g \in \mathcal{L}^2_w$, when $m \in \mathcal{L}^2_w$.

In the case when $w = \varrho$, the integral equation (3.1) has already been solved by M. Schleiff [**29**]. We shall deduce the remaining cases from his result, by using the fact that $\mathcal{L}^2_w \subset \mathcal{L}^2_\varrho$. Let $m \in \mathcal{L}^2_\varrho$. The Volterra operator V on $\mathcal{L}^2_\varrho$ is defined by

$$(Vf)(x) = \int_{-1}^{x} f\, d\lambda, \qquad x \in \Omega, \tag{3.2}$$

for every $f \in \mathcal{L}^2_\varrho$. Then $V(\mathcal{L}^2_\varrho) \subset AC(\Omega)$. Define a linear operator $M_\varrho : \mathcal{L}^2_\varrho \to \mathcal{L}^2_\varrho$ by

$$M_\varrho f = f + m\left(Vf + \pi^{-1}\int_{-1}^{1} f(y)(-\pi/2 + \arcsin y)\, dy\right)$$

for every $f \in \mathcal{L}^2_\varrho$. Furthermore, define functions a and b on Ω by

$$a(x) = \exp[-(Vm)(x)] \quad \text{and} \quad b(x) = \int_{x}^{1} a/\varrho \, d\lambda$$

for every $x \in \Omega$, respectively. It is clear that M_ϱ is continuous. Moreover, M_ϱ is invertible.

Lemma 3.1 [**29**, pp. 83–84]. *The linear operator $M_\varrho : \mathcal{L}^2_\varrho \to \mathcal{L}^2_\varrho$ is a surjective isomorphism, and its inverse is of the form*

$$M_\varrho^{-1} g = g - (ma)\left[V(g/a) - \left(\int_{-1}^{1} (gb/a)\, d\lambda\right)\left(\int_{-1}^{1} a/\varrho \, d\lambda\right)^{-1}\right]$$

for every $g \in \mathcal{L}^2_\varrho$. In particular,

$$M_\varrho^{-1} m = \pi\left(\int_{-1}^{1} a/\varrho \, d\lambda\right)^{-1} ma.$$

We are now ready to present Schleiff's result in [**29**], which shows that the operator $H_\varrho + mL_\varrho$ behaves like H_ϱ (see Lemma 2.2).

Proposition 3.1. *Let $m \in \mathcal{L}^2_\varrho$. Then the linear operator $H_\varrho + mL_\varrho : \mathcal{L}^2_\varrho \to \mathcal{L}^2_\varrho$ is a continuous surjection such that its null space $\mathcal{N}(H_\varrho + mL_\varrho)$ is spanned by the function Φ defined by*

$$\Phi = [1 - (\ln 2)H(\varrho M_\varrho^{-1} m)](1/\varrho). \tag{3.3}$$

Moreover,

$$(H_\varrho + mL_\varrho)^{-1}(\{g\}) = -(1/\varrho)H(\varrho M_\varrho^{-1} g) + \mathcal{N}(H_\varrho + mL_\varrho) \tag{3.4}$$

for every $g \in \mathcal{L}^2_\varrho$. *In particular,*

$$\operatorname{ind}(H_\varrho + mL_\varrho) = 1.$$

Lemma 3.2. *Let* $w = 1/\varrho$, σ *or* $1/\sigma$ *and* $m \in \mathcal{L}^2_w$. *Then* $\mathcal{L}^2_w$ *is invariant under* M_ϱ *and the restriction* M_w *of* M_ϱ *to* $\mathcal{L}^2_w$ *defines an isomorphism from the Banach space* $\mathcal{L}^2_w$ *onto* $\mathcal{L}^2_w$, *and* M_w^{-1} *coincides with the restriction of* M_ϱ^{-1} *to* $\mathcal{L}^2_w$.

Proof. It is clear that $M_\varrho(\mathcal{L}^2_w) \subset \mathcal{L}^2_w$ because $m \in \mathcal{L}^2_w$. To show that the map $M_w : \mathcal{L}^2_w \to \mathcal{L}^2_w$ is continuous, let $\beta = \|1/w\|_w$. Given $f \in \mathcal{L}^2_w$, it follows from the Cauchy-Schwarz inequality that

$$\int_{-1}^{1} |f|\, d\lambda \le \beta\pi \|f\|_w$$

and hence

$$\|M_w f\|_w \le (1 + 2\beta\pi\|m\|_w)\|f\|_w.$$

Thus M_w is continuous on $\mathcal{L}^2_w$.

It remains to prove that $M_\varrho^{-1}(\mathcal{L}^2_w) \subset \mathcal{L}^2_w$ and the restriction of M_ϱ^{-1} to $\mathcal{L}^2_w$ is continuous. For this part we mainly follow Schleiff's proof of Lemma 3.1. Let $g \in \mathcal{L}^2_w$ and rewrite $M_\varrho^{-1} g$ in the form

$$(M_\varrho^{-1} g)(x) = g(x) - m(x)a(x)\biggl[\int_{-1}^{x} (g(y)/a(y))\biggl(\int_{-1}^{y} a/\varrho\, d\lambda\biggr)\, dy - \int_{x}^{1} (g(y)/a(y))\biggl(\int_{y}^{1} a/\varrho\, d\lambda\biggr)\, dy\biggr]\biggl(\int_{-1}^{1} a/\varrho\, d\lambda\biggr)^{-1}$$

for every $x \in \Omega$. Then

$$\|M_\varrho^{-1} g\|_w \le \|g\|_w + \|m\|_w \cdot \sup_{x\in\Omega}\biggl[a(x)\biggl(\int_{-1}^{x} |g|/a\, d\lambda + \int_{x}^{1} |g|/a\, d\lambda\biggr)\biggr].$$

Setting

$$\alpha = \Bigl[\sup_{x\in\Omega} a(x)\Bigr] \cdot \Bigl[\inf_{x\in\Omega} a(x)\Bigr]^{-1}$$

and using again the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\|M_\varrho^{-1}g\|_w &\le \|g\|_w + \alpha\|m\|_w \int_{-1}^{1} |g|\,d\lambda \\ &\le (1+\alpha\beta\pi\|m\|_w)\|g\|_w,\end{aligned}$$

which completes the proof. □

In the subsequent Theorem 3.1, we shall adapt the results of Proposition 3.1 to the case when $w=\sigma$. The equalities

$$\begin{aligned}H_\sigma^{-1}h &= -(1/\sigma)H(\sigma h) \\ &= -(1/\varrho)H(\varrho h) + (1/\varrho)\pi^{-1}\int_{-1}^{1}\sigma h\,d\lambda, \qquad h \in \mathcal{L}_\sigma^2,\end{aligned} \tag{3.5}$$

hold in the Banach space $\mathcal{L}_\sigma^2$. In fact, the first equality in (3.5) has already been given in Lemma 2.2 (iii). The second equality follows easily from the fact that

$$-\sigma(y)/\sigma(x) + \varrho(y)/\varrho(x) = (x-y)\sigma(y)/\varrho(x) \tag{3.6}$$

for all $x, y \in \Omega$. By (3.5), we have

$$\begin{aligned}\int_{-1}^{1} H_\sigma^{-1}h\,d\lambda &= -\int_{-1}^{1}(1/\sigma)H(\sigma h)\,d\lambda \\ &= \int_{-1}^{1}\sigma h\,d\lambda, \qquad h \in \mathcal{L}_\sigma^2,\end{aligned} \tag{3.7}$$

because

$$\int_{-1}^{1}(1/\varrho)H(\varrho h)\,d\lambda = 0 \tag{3.8}$$

which is a consequence of the Parseval identity (cf. [**24**, Theorem II.4.4]) and the fact that $H(1/\varrho)=0$ (cf. [**34**, p. 174]). From (3.5) and (3.7) it follows that

$$H_\sigma^{-1}h - (1/\varrho)\pi^{-1}\int_{-1}^{1}H_\sigma^{-1}h\,d\lambda = -(1/\varrho)H(\varrho h), \tag{3.9}$$

for every $h \in \mathcal{L}^2_\sigma$.

Lemma 3.3. *Let $m \in \mathcal{L}^2_\sigma$. The function $\Phi \in \mathcal{L}^2_\varrho$ defined by* (3.3) *belongs to the space $\mathcal{L}^2_\sigma$ if and only if*

$$\pi = (\ln 2) \int_{-1}^{1} H_\sigma^{-1} M_\sigma^{-1} m \, d\lambda, \tag{3.10}$$

in which case $\Phi = (\ln 2) H_\sigma^{-1} M_\sigma^{-1} m$ and $\mathcal{N}(H_\sigma + mL_\sigma) = \operatorname{span}\{\Phi\}$.

Proof. By (3.9) applied to $h = M_\sigma^{-1} m$, we have

$$\Phi = (1/\varrho)\left(1 - \pi^{-1}(\ln 2) \int_{-1}^{1} H_\sigma^{-1} M_\sigma^{-1} m \, d\lambda\right) + (\ln 2) H_\sigma^{-1} M_\sigma^{-1} m.$$

The statement now follows from the facts that

$$1/\varrho \notin \mathcal{L}^2_\sigma \tag{3.11}$$

and

$$\mathcal{N}(H_\sigma + mL_\sigma) = \mathcal{L}^2_\sigma \cap \mathcal{N}(H_\varrho + mL_\varrho). \qquad \square$$

The index of the operator $H_\sigma + mL_\sigma$ is the same as that of H_σ as shown in the following theorem.

Theorem 3.1. *Let m be a nonzero function belonging to the space $\mathcal{L}^2_\sigma$. Then the following statements on the continuous linear operator $H_\sigma + mL_\sigma : \mathcal{L}^2_\sigma \to \mathcal{L}^2_\sigma$ hold.*

(i) *Suppose that* (3.10) *holds. Then*

$$\mathcal{N}(H_\sigma + mL_\sigma) = \operatorname{span}\{H_\sigma^{-1} M_\sigma^{-1} m\}.$$

Furthermore, a function $g \in \mathcal{L}^2_\sigma$ belongs to the range $\mathcal{R}(H_\sigma + mL_\sigma)$ if and only if

$$\int_{-1}^{1} H_\sigma^{-1} M_\sigma^{-1} g \, d\lambda = 0, \tag{3.12}$$

in which case

$$(H_\sigma + mL_\sigma)^{-1}(\{g\}) = H_\sigma^{-1} M_\sigma^{-1} g + \mathcal{N}(H_\sigma + mL_\sigma).$$

(ii) *Suppose that* (3.10) *does not hold. Then* $H_\sigma + mL_\sigma$ *is a bijection, and for a given* $g \in \mathcal{L}^2_\sigma$,

$$(H_\sigma + mL_\sigma)^{-1} g = H_\sigma^{-1} M_\sigma^{-1} [g + (c_g \ln 2) m], \tag{3.13}$$

where c_g *is the constant defined by*

$$c_g = \left(\int_{-1}^{1} \sigma M_\sigma^{-1} g \, d\lambda \right) \left(\pi - (\ln 2) \int_{-1}^{1} \sigma M_\sigma^{-1} m \, d\lambda \right)^{-1}. \tag{3.14}$$

Proof. Recall that Φ is the function given by (3.3) which spans $\mathcal{N}(H_\varrho + mL_\varrho)$; see Proposition 3.1. If $g \in \mathcal{L}^2_\sigma$ and $c \in \mathbf{C}$, then we have

$$\begin{aligned} -\frac{1}{\varrho} H(\varrho M_\varrho^{-1} g) + c\Phi &= -\frac{1}{\varrho} H[\varrho(M_\varrho^{-1} g + (c \ln 2) M_\varrho^{-1} m)] + \frac{c}{\varrho} \\ &= H_\sigma^{-1} M_\sigma^{-1} [g + (c \ln 2) m] \\ &\quad + \frac{1}{\varrho} \left[c\left(1 - \frac{\ln 2}{\pi} \int_{-1}^{1} H_\sigma^{-1} M_\sigma^{-1} m \, d\lambda \right) \right. \\ &\quad \left. - \frac{1}{\pi} \int_{-1}^{1} H_\sigma^{-1} M_\sigma^{-1} g \, d\lambda \right] \end{aligned} \tag{3.15}$$

by applying (3.9) to $h = M_\sigma^{-1}[g + c(\ln 2) m]$. Moreover, observe that

$$(H_\sigma + mL_\sigma)^{-1}(\{g\}) = \mathcal{L}^2_\sigma \cap (H_\varrho + mL_\varrho)^{-1}(\{g\}), \qquad g \in \mathcal{L}^2_\sigma. \tag{3.16}$$

(i) Given $g \in \mathcal{L}^2_\sigma$ and $c \in \mathbf{C}$, it follows from (3.10) and (3.15) that

$$\begin{aligned} -(1/\varrho) H(\varrho M_\varrho^{-1} g) + c\Phi &= H_\sigma^{-1} M_\sigma^{-1} (g + c(\ln 2) m) \\ &\quad - (1/\varrho) \pi^{-1} \int_{-1}^{1} H_\sigma^{-1} M_\sigma^{-1} g \, d\lambda \end{aligned} \tag{3.17}$$

as elements of $\mathcal{L}^2_\varrho$. Hence (3.11) implies that the left-hand side of (3.17) belongs to $\mathcal{L}^2_\sigma$ if and only if (3.12) holds. Accordingly, given $g \in \mathcal{L}^2_\sigma$, it follows from (3.4), (3.16) and (3.17) that

$$(H_\sigma + mL_\sigma)^{-1}(\{g\}) \neq \phi \tag{3.18}$$

if and only if (3.12) holds. Therefore, the second half of statement (i) has been established. The first half of (i) has already been given in Lemma 3.3.

(ii) By Lemma 3.3, the operator $H_\sigma + mL_\sigma$ is injective. To show its surjectivity, let $g \in \mathcal{L}^2_\sigma$. The left-hand side of (3.15) is an element of $\mathcal{L}^2_\sigma$ if and only if c equals the constant c_g given by (3.14); we have used (3.7) and (3.11). It then follows from (3.4) and (3.16) that $H_\sigma + mL_\sigma$ is surjective and that (3.13) holds. □

Remark 3.1. Let m be a nonzero function belonging to $\mathcal{L}^2_{1/\sigma}$. Then statements (i) and (ii) of Theorem 3.1 hold with replacement of the subscript σ by the subscript $1/\sigma$. The proof will be almost the same if we replace σ by $1/\sigma$. The only exceptional relationships to be modified are (3.5), (3.6) and (3.7). The modified versions are as follows:

$$\begin{aligned} H^{-1}_{1/\sigma} h &= -\sigma H(h/\sigma) \\ &= -(1/\varrho) H(\varrho h) - (1/\varrho)\pi^{-1} \int_{-1}^{1} h/\sigma \, d\lambda, \qquad h \in \mathcal{L}^2_{1/\sigma}; \end{aligned} \tag{3.5*}$$

$$-\sigma(x)/\sigma(y) + \varrho(y)/\varrho(x) = (y-x)/(\varrho(x)\sigma(y)), \qquad x, y \in \Omega; \tag{3.6*}$$

$$\begin{aligned} -\int_{-1}^{1} H^{-1}_{1/\sigma} h \, d\lambda &= -\int_{-1}^{1} \sigma H(h/\sigma)\, d\lambda \\ &= -\int_{-1}^{1} h/\sigma \, d\lambda, \qquad h \in \mathcal{L}^2_{1/\sigma}. \end{aligned} \tag{3.7*}$$

Now we shall consider the case when $w = 1/\varrho$. Of course, we need to apply Proposition 3.1. For our proof, (3.5) will be replaced by

$$\begin{aligned} -\varrho H(h/\varrho) = -(1/\varrho)\Big[& H(\varrho h) - \pi^{-1} \int_{-1}^{1} \mathbf{x} h/\varrho \, d\lambda \\ & - \pi^{-1} \mathbf{x} \int_{-1}^{1} h/\varrho \, d\lambda \Big], \qquad h \in \mathcal{L}^2_{1/\varrho}, \end{aligned} \tag{3.5**}$$

where $\mathbf{x}$ denotes the identity function on Ω. Define continuous linear functionals α and β on the Banach space $\mathcal{L}^2_{1/\varrho}$ by

$$\langle \alpha, h\rangle = \pi^{-1}\int_{-1}^{1}(M_{1/\varrho}^{-1}h)/\varrho\,d\lambda$$

and

$$\langle \beta, h\rangle = \pi^{-1}\int_{-1}^{1}(\mathbf{x}M_{1/\varrho}^{-1}h)/\varrho\,d\lambda$$

for every $h \in \mathcal{L}^2_{1/\varrho}$, respectively. Then the function Φ given by (3.3) has the form

$$\begin{aligned}\Phi = &-(\ln 2)\varrho H[(M_{1/\varrho}^{-1}m)/\varrho]\\ &- [1-(\ln 2)\langle\beta, m\rangle](1/\varrho) - (\ln 2)\langle\alpha, m\rangle(\mathbf{x}/\varrho).\end{aligned}$$

Since $\varrho H[(M_{1/\varrho}^{-1}m)/\varrho] \in \mathcal{L}^2_{1/\varrho}$, the function Φ belongs to $\mathcal{L}^2_{1/\varrho}$ if and only if

$$1-(\ln 2)\langle\beta, m\rangle = 0 = \langle\alpha, m\rangle \tag{3.19}$$

by using the fact that neither $1/\varrho$ nor $\mathbf{x}/\varrho$ is an element of $\mathcal{L}^2_{1/\varrho}$. Now if $g \in \mathcal{L}^2_{1/\varrho}$ and $c \in \mathbf{C}$, then

$$\begin{aligned}-(1/\varrho)H(\varrho M_\varrho^{-1}g) + c\Phi = &-\varrho H[(1/\varrho)M_{1/\varrho}^{-1}(g + c(\ln 2)m)]\\ &+ [c - \langle\beta, g + c(\ln 2)m\rangle](1/\varrho)\\ &- \langle\alpha, g + c(\ln 2)m\rangle(\mathbf{x}/\varrho).\end{aligned}$$

With the above observations, the proof of the following theorem is straightforward, and we shall leave the details with the reader.

Theorem 3.2. *Let m be a nonzero function in the Banach space $\mathcal{L}^2_{1/\varrho}$. Then the following statements on the continuous linear operator $H_{1/\varrho} + mL_{1/\varrho} : \mathcal{L}^2_{1/\varrho} \to \mathcal{L}^2_{1/\varrho}$ hold.*

(i) *Suppose that* (3.19) *holds. Then $H_{1/\varrho}+mL_{1/\varrho}$ has the null space of dimension one given by*

$$\mathcal{N}(H_{1/\varrho} + mL_{1/\varrho}) = \operatorname{span}\{\varrho H[(M_{1/\varrho}^{-1}m)/\varrho]\}.$$

A function $g \in \mathcal{L}^2_{1/\varrho}$ *belongs to* $\mathcal{R}(H_{1/\varrho} + mL_{1/\varrho})$ *if and only if* $\langle \alpha, g\rangle = 0 = \langle \beta, g\rangle$, *in which case*

$$(H_{1/\varrho} + mL_{1/\varrho})^{-1}(\{g\}) = -\varrho H[(M^{-1}_{1/\varrho} g)/\varrho] + \mathcal{N}(H_{1/\varrho} + mL_{1/\varrho})$$

so that $\operatorname{codim} \mathcal{R}(H_{1/\varrho} + mL_{1/\varrho}) = 2$.

(ii) *Suppose that* (3.19) *does not hold. Then* $H_{1/\varrho}+mL_{1/\varrho}$ *is injective and its range consists of those functions* $g \in \mathcal{L}^2_{1/\varrho}$ *such that*

$$(1 - (\ln 2)\langle \beta, m\rangle)\langle \alpha, g\rangle + (\ln 2)\langle \alpha, m\rangle\langle \beta, g\rangle = 0.$$

For such a function g,

$$(H_{1/\varrho} + mL_{1/\varrho})^{-1} g = -\varrho H[(1/\varrho)M^{-1}_{1/\varrho}(g + c_g(\ln 2)m)],$$

where c_g *is the constant determined by the two identities*:

$$c_g(\ln 2)\langle \alpha, m\rangle + \langle \alpha, g\rangle = 0 = c_g(1 - (\ln 2)\langle \beta, m\rangle) - \langle \beta, g\rangle.$$

From the above theorem we can see that

$$\operatorname{ind}(H_{1/\varrho}) = -1 = \operatorname{ind}(H_{1/\varrho} + mL_{1/\varrho}),$$

for all $m \in \mathcal{L}^2_{1/\varrho}$.

Finally we shall show that $H_{\sigma,s} + mL_{\sigma,s}$ has the same properties as $H_\sigma + mL_\sigma$ for every $s > 0$, when m is smooth. Our arguments can easily be adapted to the remaining cases: $w = \varrho$, $1/\varrho$, $1/\sigma$; so we shall not discuss them here.

Let us fix a positive number s and let r be the smallest positive integer such that $r \geq s$. The Hilbert space $\mathcal{L}^2_{\sigma,s}$ is an intermediate space between $\mathcal{L}^2_{\sigma,r-1}$ and $\mathcal{L}^2_{\sigma,r}$. In fact, let Λ be the linear operator in the Hilbert space $\mathcal{L}^2_{\sigma,r-1}$, with domain $\mathcal{D}(\Lambda) = \mathcal{L}^2_{\sigma,r}$, defined by

$$\Lambda f = \sum_{n=0}^{\infty} (1+n)^r (f|p_n)_\sigma p_n, \qquad f \in \mathcal{D}(\Lambda).$$

Then the operator Λ is self-adjoint, positive and unbounded in $\mathcal{L}^2_{\sigma,r-1}$. Moreover, if $0 < \theta < 1$, then the intermediate space $[\mathcal{L}^2_{\sigma,r-1}, \mathcal{L}^2_{\sigma,r}]_\theta$ is defined as the domain of the linear operator $\Lambda^{1-\theta}$ in $\mathcal{L}^2_{\sigma,r-1}$. Of course, $\Lambda^{1-\theta}$ has the form

$$\Lambda^{1-\theta}(f) = \sum_{n=0}^{\infty} (1+n)^{r(1-\theta)} (f|p_n)_\sigma p_n, \qquad f \in \mathcal{D}(\Lambda),$$

(see [**22**, Section 2.1 in Chapter 1]). Hence, $[\mathcal{L}^2_{\sigma,r-1}, \mathcal{L}^2_{\sigma,r}]_\theta = \mathcal{L}^2_{\sigma,r(1-\theta)}$ whenever $0 < \theta < 1$. In particular,

$$[\mathcal{L}^2_{\sigma,r-1}, \mathcal{L}^2_{\sigma,r}]_{1-s/r} = \mathcal{L}^2_{\sigma,s}. \tag{3.20}$$

By $C^r_\varrho([-1,1])$ we denote the space of all r times differentiable functions $u : \Omega \to \mathbf{C}$ such that $\varrho^k D^k u$ has a continuous extension to the closed interval $[-1,1]$ for each $k = 0, 1, \ldots, r$. Furthermore, define a norm on $C^r_\varrho([-1,1])$ by

$$\|u\|_{C^r_\varrho} = \sum_{k=0}^{r} \|\varrho^k D^k u\|_\infty, \qquad u \in C^r_\varrho([-1,1]),$$

where $\|\cdot\|_\infty$ denotes the uniform norm. Given $m \in C^r_\varrho([-1,1])$, let

$$m\mathcal{L}^2_{\sigma,\delta} = \{mf : f \in \mathcal{L}^2_{\sigma,\delta}\},$$

which is a linear subspace of $\mathcal{L}^2_{\sigma,r-1}$ for every $\delta \in [r-1, r]$.

Lemma 3.4. *Let $m \in C^r_\varrho([-1,1])$. Then the following statements hold.*

(i) *Let $\delta = r-1$ or r. Then $m\mathcal{L}^2_{\sigma,\delta} \subset \mathcal{L}^2_{\sigma,\delta}$ and the $\mathcal{L}^2_{\sigma,\delta}$-valued linear operator*:

$$f \longmapsto mf, \qquad f \in \mathcal{L}^2_{\sigma,\delta},$$

is continuous. Moreover,

$$\|mf\|_{\sigma,\delta} \le \text{const} \cdot \|m\|_{C^r_\varrho} \|f\|_{\sigma,\delta}.$$

(ii) *It follows that $m\mathcal{L}^2_{\sigma,s} \subset \mathcal{L}^2_{\sigma,s}$ and the $\mathcal{L}^2_{\sigma,s}$-valued linear operator*:

$$f \longmapsto mf, \qquad f \in \mathcal{L}^2_{\sigma,s},$$

is continuous.

Proof. Statement (i) follows from Lemma 2.3 together with Leibnitz's formula. Statement (ii) is a consequence of the interpolation theorem [**22**, Theorem 5.1 in Chapter 1] because of (i) and (3.20). □

Lemma 3.5. *The Hilbert space $\mathcal{L}^2_{\sigma,s}$ is invariant under the Volterra operator V (see (3.2)) and the restriction of V to $\mathcal{L}^2_{\sigma,s}$ is a continuous linear operator from $\mathcal{L}^2_{\sigma,s}$ into $\mathcal{L}^2_{\sigma,s+1}$.*

Proof. The proof is analogous to that of Lemma 3.4 (ii) because of (3.20). □

Corollary 3.1. *Let $m \in C^r_\varrho([-1,1])$. The restriction $M_{\sigma,s}$ of M_σ to $\mathcal{L}^2_{\sigma,s}$ is an isomorphism onto $\mathcal{L}^2_{\sigma,s}$.*

Proof. The proof follows from Lemmas 3.4 and 3.5. □

Let $m \in C^r_\varrho([-1,1])$. It then follows from Proposition 2.1 and Lemma 3.4 that $mL_{\sigma,s}$ can be regarded as a continuous linear operator from $(1/\sigma)\mathcal{L}^2_{1/\sigma,s}(\subset \mathcal{L}^2_{\sigma,s})$ into $\mathcal{L}^2_{\sigma,s}$ because $\mathcal{L}^2_{\sigma,s+1}$ is continuously embedded into $\mathcal{L}^2_{\sigma,s}$. We are now ready to present the main result which follows immediately from Theorem 3.1 in view of Corollary 3.1.

Theorem 3.3. *Let $s > 0$. Let r be the smallest positive integer such that $r \geq s$. Suppose that $m \in C^r_\varrho([-1,1])$ is a nonzero function. Then the linear operator $H_{\sigma,s} + mL_{\sigma,s} : (1/\sigma)\mathcal{L}^2_{1/\sigma,s} \to \mathcal{L}^2_{\sigma,s}$ is continuous, and statements* (i) *and* (ii) *of Theorem* 3.1 *hold with replacement of the subscript σ by the subscripts σ, s.*

4. A numerical procedure. The results of Section 3 allow us to consider a numerical procedure for solving singular integral equations of the form

$$(H_\varrho + mL_\varrho + K)f = g \tag{4.1}$$

where $g \in \mathcal{L}^2_\varrho$ and $m \in \mathcal{L}^2_\varrho$ are given functions and K is a given compact linear integral operator acting on $\mathcal{L}^2_\varrho$.

Fix a positive integer n. Let $w = \varrho$ or $1/\varrho$, and let h_n be the corresponding polynomial u_n or t_n. Let $y_{n,i}$, $i = 1, \dots, n$, be the zeros of h_n, which are known to be distinct and belong to the open interval $(-1, 1)$. Define the Lagrangian fundamental polynomials $l^w_{n,i}$, $i = 1, 2, \dots, n$, by

$$l^w_{n,i}(y) = \frac{h_n(y)}{(y - y_{n,i})h'_n(y_{n,i})} = \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{y - y_{n,j}}{y_{n,i} - y_{n,j}}, \qquad y \in (-1, 1).$$

The Lagrangian interpolation projector $\mathbf{L}^w_n$ is defined by

$$\mathbf{L}^w_n : u \longmapsto \sum_{i=1}^{n} u(y_{n,i}) l^w_{n,i}$$

for an arbitrary continuous function $u : (-1, 1) \to \mathbf{C}$.

Assume that the operator K has the form

$$Ku(x) = \frac{1}{\pi} \int_{-1}^{1} k(x, y)u(y)\, dy, \qquad x \in (-1, 1), \quad u \in \mathcal{L}^2_\varrho.$$

It is well known that K is a compact operator in $\mathcal{L}^2_\varrho$ if the kernel function k satisfies the condition

$$\int_{-1}^{1} \int_{-1}^{1} |k(x, y)|^2 \varrho(x)/\varrho(y)\, dy\, dx < \infty.$$

In the sequel we make the following assumptions about the smoothness of k. Assume that $k(\cdot, y) \in \mathcal{L}^2_{\varrho,s}$ uniformly with respect to $y \in (-1, 1)$, and $k(x, \cdot) \in \mathcal{L}^2_{1/\varrho,r}$ uniformly with respect to $x \in (-1, 1)$, with some positive real numbers s and r to be specified later; in other words, there are constants C_1 and C_2 (independent of both x and y) such that

$$\|k(\cdot, y)\|_{\varrho,s} \le C_1 \quad \text{and} \quad \|k(x, \cdot)\|_{1/\varrho,r} \le C_2 \tag{4.2}$$

for all $x, y \in (-1, 1)$. Under the above conditions, the operator $K : \mathcal{L}^2_\varrho \to \mathcal{L}^2_{\varrho,t}$ is continuous for all $t \le s$ and compact for all $t < s$ (see, for example, [**2**, Lemma 4.2]).

In what follows we shall denote by $\|\cdot\|^*_{1/\varrho,s}$ the norm on the Banach space $\tilde{\mathcal{L}}^2_{1/\varrho,s} = (1/\varrho)\mathcal{L}^2_{1/\varrho,s}$ (see Definition 2.1); that is,

$$\|v\|^*_{1/\varrho,s} = \|\varrho v\|_{1/\varrho,s}, \qquad v \in \tilde{\mathcal{L}}^2_{1/\varrho,s}.$$

Suppose that $s > 1/2$ and $r > 1/2$. Introduce the operator K_n by

$$K_n u(x) = \frac{1}{\pi}\int_{-1}^{1} [\mathbf{L}^{1/\varrho}_{n,y} k(x,y)] u(y)\, dy, \qquad x \in (-1,1),$$

for all $u \in \mathcal{L}^2_\varrho$. The subscript y in $\mathbf{L}^{1/\varrho}_{n,y}$ indicates that the interpolation is realized with respect to the variable y. Given $x \in (-1,1)$, the function $k(x,\cdot) \in \mathcal{L}^2_{1/\varrho,r}$ is continuous on $(-1,1)$ by [**2**, Theorem 2.5] because $r > 1/2$, and hence we can define $K_n u(x)$ for each $u \in \mathcal{L}^2_\varrho$. The so-defined function $K_n u$ on $(-1,1)$ is continuous again by [**2**, Theorem 2.5] applied to $k(\cdot,y) \in \mathcal{L}^2_{\varrho,s}$ with $s > 1/2$.

Note that by Lemma 2.2 and Proposition 3.1, each solution $f \in \mathcal{L}^2_\varrho$ of the Fredholm integral equation of the second kind

$$(4.1^*) \qquad f - \frac{1}{\varrho} H(\varrho M_\varrho^{-1} K f) = -\frac{1}{\varrho} H(\varrho M_\varrho^{-1} g)$$

is a solution of (4.1). Conversely, a solution f of (4.1) which satisfies $\int_{-1}^{1} f\, d\lambda = 0$ is a solution of (4.1*) because of (3.4) and (3.8).

Let Π_n denote the space of all polynomials of degree less than or equal to $(n-1)$ with complex coefficients. Let $g \in \mathcal{L}^2_{\varrho,s}$. We shall seek an approximate solution $f_n \in \tilde{\mathcal{L}}^2_{1/\varrho,s}$ of equation (4.1*). In other words, f_n is a solution to the equation

$$(4.3) \qquad f_n - \frac{1}{\varrho} H(\varrho \mathbf{L}^\varrho_n M_\varrho^{-1} \mathbf{L}^\varrho_n K_n f_n) = -\frac{1}{\varrho} H(\varrho \mathbf{L}^\varrho_n M_\varrho^{-1} g).$$

It follows from Lemma 2.4 that f_n is necessarily of the form $f_n = v_n/\varrho$ for some $v_n \in \Pi_n$ and that (4.3) is a fully discretized linear algebraic system relative to the coefficients of the unknown polynomial v_n.

Introduce the space $\mathcal{L}^{2,0}_\varrho$ of all functions $u \in \mathcal{L}^2_\varrho$ satisfying $\int_{-1}^{1} u\, d\lambda = 0$.

The range of the operator $\tilde{K}_n$ defined by

$$\tilde{K}_n u = \frac{1}{\varrho} H(\varrho \mathbf{L}_n^{\varrho} M_{\varrho}^{-1} \mathbf{L}_n^{\varrho} K_n u), \qquad u \in \mathcal{L}_{\varrho}^2,$$

is contained in $\mathcal{L}_{\varrho}^{2,0}$ because of (3.8). Moreover, it follows from Proposition 3.1 and (3.8) that the operator $H_{\varrho} + mL_{\varrho}$ restricted to $\mathcal{L}_{\varrho}^{2,0}$ is a bijection from $\mathcal{L}_{\varrho}^{2,0}$ onto $\mathcal{L}_{\varrho}^2$. Thus the index of the operator $H_{\varrho} + mL_{\varrho} + K : \mathcal{L}_{\varrho}^{2,0} \to \mathcal{L}_{\varrho}^2$ is 0 because K is compact. So if $H_{\varrho} + mL_{\varrho} + K$ happens to be injective on $\mathcal{L}_{\varrho}^{2,0}$, then it becomes a bijection on $\mathcal{L}_{\varrho}^{2,0}$. For each $t \in [0, s)$, let

$$\tilde{\mathcal{L}}_{1/\varrho,t}^{2,0} = \tilde{\mathcal{L}}_{1/\varrho,t}^{2} \cap \mathcal{L}_{\varrho}^{2,0}.$$

Theorem 4.1. *Let $w = \varrho$. Assume that the kernel k of the compact operator K on $\mathcal{L}_{\varrho}^2$ satisfies (4.2) with $s > 1/2$ and $r \geq s + 1/2$. Let d denote the smallest positive integer such that $d \geq s$, and let $m \in C_{\varrho}^d([-1,1])$. Suppose that the homogeneous equation (4.1) possesses only the trivial solution in $\mathcal{L}_{\varrho}^{2,0}$, that is, $(H_{\varrho}+mL_{\varrho}+K)^{-1}(\{0\}) \cap \mathcal{L}_{\varrho}^{2,0} = \{0\}$, that a function $g \in \mathcal{L}_{\varrho,s}^2$ is given, and that $0 \leq t < s$.*

Then the singular integral equation (4.1) has a unique solution f in $\tilde{\mathcal{L}}_{1/\varrho,s}^{2,0}$. Moreover, for all sufficiently large $n \in \mathbf{N}$, the system (4.3) is uniquely solvable in $\tilde{\mathcal{L}}_{1/\varrho,t}^{2,0}$ and the solution $f_n \in \tilde{\mathcal{L}}_{1/\varrho,t}^{2,0}$ is of the form $f_n = v_n/\varrho$ for some $v_n \in \Pi_n$ and satisfies the error estimate

$$\|f_n - f\|_{1/\varrho,t}^{*} \leq \text{const} \cdot n^{t-s} \|g\|_{\varrho,s}. \tag{4.4}$$

Proof. In this proof the symbol c stands for a positive constant (not always the same) which is independent of $n \in \mathbf{N}$. The identity operators on the various Hilbert spaces to be considered are denoted by I.

Step 1. Let $0 \leq \delta \leq s$. The restriction of $H_{\varrho,\delta} + mL_{\varrho,\delta}$ to $\tilde{\mathcal{L}}_{1/\varrho,\delta}^{2,0}$ is denoted also by $H_{\varrho,\delta} + mL_{\varrho,\delta}$ for simplicity. Since $m \in C_{\varrho}^d([-1,1])$ and $d \geq s \geq \delta$, the operator $H_{\varrho,\delta} + mL_{\varrho,\delta}$ is an isomorphism from $\tilde{\mathcal{L}}_{1/\varrho,\delta}^{2,0}$ onto $\mathcal{L}_{\varrho,\delta}^2$, which can be proved as Theorem 3.3.

Step 2. We shall show that (4.1) has a unique solution f in $\tilde{\mathcal{L}}^{2,0}_{1/\varrho,s}$. The natural embedding $Z : \tilde{\mathcal{L}}^2_{1/\varrho,s} \to \mathcal{L}^2_\varrho$ is compact because by [**2**, Conclusion 2.3] the natural embedding from $\mathcal{L}^2_{1/\varrho,s}$ into $\mathcal{L}^2_{1/\varrho}$ is compact. Since $K : \mathcal{L}^2_\varrho \to \mathcal{L}^2_{\varrho,s}$ is continuous by the assumption (4.2), the map KZ is compact, and hence its restriction to $\tilde{\mathcal{L}}^{2,0}_{1/\varrho,s}$ is an $\mathcal{L}^2_{\varrho,s}$-valued compact operator.

Now

$$\operatorname{ind}(H_{\varrho,s} + mL_{\varrho,s} + K) = \operatorname{ind}(H_{\varrho,s} + mL_{\varrho,s}) = 0.$$

The operator

$$H_{\varrho,s} + mL_{\varrho,s} + K : \tilde{\mathcal{L}}^{2,0}_{1/\varrho,s} \to \mathcal{L}^2_{\varrho,s},$$

which is injective by assumption, is a surjective isomorphism. That is, (4.1) has a unique solution f in $\tilde{\mathcal{L}}^{2,0}_{1/\varrho,s}$ and

$$\|f\|^*_{1/\varrho,s} \le \|(H_{\varrho,s} + mL_{\varrho,s} + K)^{-1}\| \cdot \|g\|_{\varrho,s}. \tag{4.5}$$

Step 3. In Steps 3 and 4 we shall establish that the operator

$$I - \tilde{K}_n : \tilde{\mathcal{L}}^{2,0}_{1/\varrho,t} \longrightarrow \tilde{\mathcal{L}}^{2,0}_{1/\varrho,t}$$

is invertible, which will imply that (4.3) has a unique solution in $\tilde{\mathcal{L}}^{2,0}_{1/\varrho,t}$.

When $0 \le \delta \le s$, let $M_{\varrho,\delta}$ denote the restriction of M_ϱ to $\mathcal{L}^2_{\varrho,\delta}$; then $M_{\varrho,\delta} : \mathcal{L}^2_{\varrho,\delta} \to \mathcal{L}^2_{\varrho,\delta}$ is a surjective isomorphism, which can be proved as $M_{\sigma,\delta}$ in Corollary 3.1 by using the assumption: $m \in C^d_\varrho([-1,1])$ and $d \ge s \ge \delta$. Applying Lemma 2.4, define an operator $A : \mathcal{L}^2_{\varrho,t} \to \mathcal{L}^2_{\varrho,t}$ by

$$Av = (1/\varrho)H(\varrho v), \qquad v \in \mathcal{L}^2_{\varrho,t}.$$

Let $W : \tilde{\mathcal{L}}^{2,0}_{1/\varrho,t} \to \tilde{\mathcal{L}}^{2,0}_{1/\varrho,t}$ be the operator given by

$$Wu = AM^{-1}_{\varrho,t}Ku, \qquad u \in \tilde{\mathcal{L}}^{2,0}_{1/\varrho,t}.$$

We claim that the operator $I - W$ is a surjective isomorphism on $\tilde{\mathcal{L}}^{2,0}_{1/\varrho,t}$. In fact, $H_{\varrho,t} + mL_{\varrho,t}$ is an isomorphism from $\tilde{\mathcal{L}}^{2,0}_{1/\varrho,t}$ onto $\mathcal{L}^2_{\varrho,t}$

by Step 1. Since $K : \mathcal{L}^2_\varrho \to \mathcal{L}^2_{\varrho,t}$ is compact by the assumption (4.2), the operator $H_{\varrho,t} + mL_{\varrho,t} + K$ is an isomorphism from $\tilde{\mathcal{L}}^{2,0}_{1/\varrho,t}$ onto $\mathcal{L}^2_{\varrho,t}$, which can be proved as in the second half of Step 2. Now the identity

$$H_{\varrho,t} + mL_{\varrho,t} + K = (H_{\varrho,t} + mL_{\varrho,t})(I - W)$$

on $\tilde{\mathcal{L}}^{2,0}_{\varrho,t}$ establishes our claim.

The operator $I - \tilde{K}_n = (I - W) + (W - \tilde{K}_n)$ becomes invertible for a large $n \in \mathbf{N}$ once we show that

$$\|(W - \tilde{K}_n)u\|^*_{1/\varrho,t} \leq cn^{t-s}\|u\|^*_{1/\varrho,t} \tag{4.6}$$

for every $u \in \tilde{\mathcal{L}}^{2,0}_{1/\varrho,t}$ and $n \in \mathbf{N}$. We shall then have

$$\|(I - \tilde{K}_n)^{-1}\| \leq (1 - \|(I - W)^{-1}\| \cdot \|W - \tilde{K}_n\|)^{-1}$$

provided $\|(I - W)^{-1}\| \cdot \|W - \tilde{K}_n\| < 1$. This is a consequence of the usual Neumann series argument.

Step 4. The aim of this step is to prove (4.6). To this end, fix a function $u \in \tilde{\mathcal{L}}^{2,0}_{1/\varrho,t}$ and a positive integer n. Let $J : \tilde{\mathcal{L}}_{1/\varrho,t} \to \mathcal{L}^2_\varrho$ be the natural injection. Define a linear operator $D_n : \mathcal{L}^2_\varrho \to \mathcal{L}^2_{\varrho,t}$ by

$$D_n h = (M_\varrho^{-1} K - \mathbf{L}^\varrho_n M_\varrho^{-1} \mathbf{L}^\varrho_n K_n)h, \qquad h \in \mathcal{L}^2_\varrho.$$

Using the operator A given in Step 3 we have

$$(W - \tilde{K}_n)u = AD_n Ju.$$

Let $h = Ju$. Then

$$D_n h = M^{-1}_{\varrho,t}(K - \mathbf{L}^\varrho_n K_n)h + (I - \mathbf{L}^\varrho_n)(M^{-1}_{\varrho,s} - I)\mathbf{L}^\varrho_n K_n h. \tag{4.8}$$

In view of the assumption $r \geq s + 1/2 > s > 1/2$, apply [**2**, Lemma 4.4] to obtain that, if $0 \leq \delta \leq s$, then

$$\|(K - \mathbf{L}^\varrho_n K_n)h\|_{\varrho,\delta} \leq cn^{\delta - s}\|h\|_\varrho. \tag{4.9}$$

Letting $\delta = t$ in (4.9), we have

$$\|M_{\varrho,t}^{-1}(K - \mathbf{L}_n^{\varrho} K_n)h\|_{\varrho,t} \le cn^{t-s}\|M_{\varrho,t}^{-1}\| \cdot \|h\|_{\varrho}. \tag{4.10}$$

On the other hand, from [**2**, Theorem 3.4] which requires the assumption (4.2) and $s > 1/2$, it follows that

$$\|v - \mathbf{L}_n^{\varrho} v\|_{\varrho,t} \le cn^{t-s}\|v\|_{\varrho,s}, \qquad v \in \mathcal{L}_{\varrho,s}^2. \tag{4.11}$$

From (4.9) with $\delta = s$, we derive

$$\|\mathbf{L}_n^{\varrho} K_n h\|_{\varrho,s} \le c\|h\|_{\varrho}.$$

This together with (4.11) gives

$$\|(I - \mathbf{L}_n^{\varrho})(M_{\varrho,s}^{-1} - I)\mathbf{L}_n^{\varrho} K_n h\|_{\varrho,t} \le cn^{t-s}\|M_{\varrho,s}^{-1} - I\| \cdot \|h\|_{\varrho}. \tag{4.12}$$

It then follows from (4.8), (4.10) and (4.12) that

$$\|D_n h\|_{\varrho,t} \le cn^{t-s}(\|M_{\varrho,t}^{-1}\| + \|M_{\varrho,s}^{-1} - I\|)\|h\|_{\varrho}. \tag{4.13}$$

Therefore we have

$$\begin{aligned}\|(W - \tilde{K}_n)u\|_{1/\varrho,t}^* &\le \|A\| \cdot \|D_n J u\|_{\varrho,t} \\ &\le cn^{t-s}\|A\|(\|M_{\varrho,t}^{-1}\| + \|M_{\varrho,s}^{-1} - I\|)\|J\| \cdot \|u\|_{1/\varrho,t}^*,\end{aligned}$$

which implies (4.6).

Step 5. Since (4.6) holds for every $n \in \mathbf{N}$, there is an $N \in \mathbf{N}$ such that $I - \tilde{K}_n$ is invertible wherever $n \ge N$, as observed in Step 3. Let $b = \sup_{n \ge N} \|(I - \tilde{K}_n)^{-1}\|$ which is finite by (4.6) and (4.7). Fix a positive integer n satisfying $n \ge N$. Let

$$f_n = -(I - \tilde{K}_n)^{-1}\left[\frac{1}{\varrho} H(\varrho \mathbf{L}_n^{\varrho} M_{\varrho,s}^{-1} g)\right]$$

which is the unique solution of (4.3). As noted before, $f_n = v_n/\varrho$ for some $v_n \in \Pi_n$ by applying Lemma 2.4. It is easy to see that

$$\begin{aligned}f_n - f &= (I - \tilde{K}_n)^{-1}\left(-\frac{1}{\varrho} H(\varrho \mathbf{L}_n^{\varrho} M_{\varrho,s}^{-1} g) - (I - \tilde{K}_n) f\right) \\ &= (I - \tilde{K}_n)^{-1} A[(I - \mathbf{L}_n^{\varrho}) M_{\varrho,s}^{-1} g - D_n J f].\end{aligned}$$

By (4.11) we have

$$\|(I-\mathbf{L}_n^{\varrho})M_{\varrho,s}^{-1}g\|_{\varrho,t} \le cn^{t-s}\|M_{\varrho,s}^{-1}g\|_{\varrho,s} \le cn^{t-s}\|M_{\varrho,s}^{-1}\|\cdot\|g\|_{\varrho,s}. \tag{4.14}$$

Substituting Jf for h in (4.13) gives that $\|D_nJf\|_{\varrho,t} \le cn^{t-s}\|Jf\|_{\varrho}$. It then follows from (4.5) that

$$\|D_nJf\|_{\varrho,t} \le cn^{t-s}\|J\|\cdot\|(H_{\varrho,s}+mL_{\varrho,s}+K)^{-1}\|\cdot\|g\|_{\varrho,s}. \tag{4.15}$$

From (4.14) and (4.15) we finally obtain

$$\begin{aligned}\|f_n-f\|_{1/\varrho,t}^{*} &\le \|(I-\tilde{K}_n)^{-1}\|\cdot\|A\|(\|(I-\mathbf{L}_n^{\varrho})M_{\varrho,s}^{-1}g\|_{\varrho,t} \\ &\quad + \|D_nJf\|_{\varrho,t}) \\ &\le b\|A\|(cn^{t-s}\|g\|_{\varrho,s})\end{aligned}$$

wherever $n \ge N$. This establishes the error estimate (4.4). □

Remark 4.1 [**29**, Section 4]. The homogeneous equation (4.1) has a unique solution $f \in \mathcal{L}_{\varrho}^2$ satisfying

$$\int_{-1}^{1} f\,d\lambda = \pi C$$

with given $C \in \mathbf{R}$ if k and m fulfill the estimate

$$B < (1+\alpha\sqrt{\pi}\|m\|_{\varrho})^{-1}. \tag{4.16}$$

Here

$$\begin{aligned}\pi^2B^2 &= \int_{-1}^{1}\int_{-1}^{1} |k(x,y)^2\varrho(x)/\varrho(y)\,dy\,dx \\ &\quad - \int_{-1}^{1}\left[\int_{-1}^{1} k(x,y)/\varrho(y)\,dy\right]^2\varrho(x)\,dx\end{aligned}$$

and

$$\alpha = \left[\sup_{-1<x<1} a(x)\right]\cdot\left[\inf_{-1<x<1} a(x)\right]^{-1}$$

with the function a defined in the beginning of Section 3.

If (4.16) is fulfilled, then the general solution $f \in \mathcal{L}^2_\varrho$ of (4.1) satisfies the estimate

$$\|f\|^2_\varrho \leq [1 - (1 + \alpha\sqrt{\pi}\|m\|_\varrho)B]^{-2}[(1 + \alpha\sqrt{\pi}\|m\|_\varrho)\|g\|_\varrho + |C|(\alpha(\ln 2)\|m\|_\varrho + (1 + \alpha\sqrt{\pi}\|m\|_\varrho)\|h\|_\varrho)]^2 + \pi|C|^2$$

where

$$h(x) = \frac{1}{\pi}\int_{-1}^{1} k(x,y)/\varrho(y)\,dy \qquad x \in (-1,1).$$

Note that if $t > 1/2$ then the following estimate holds:

$$\sup_{-1<x<1} |u(x)|/\varrho(x) \leq \text{const} \cdot \|u\|_{1/\varrho,t} \quad \text{if } u \in \mathcal{L}^2_{1/\varrho,t} \tag{4.17}$$

(see [**25**, Theorem 7] and [**7**, equation (35)]). Thus the estimate of Theorem 4.1 gives an error estimate with respect to the uniform norm. More precisely,

$$\sup_{-1<x<1} |f_n(x) - f(x)| \leq \text{const} \cdot n^{t-s}\|g\|_{\varrho,s} \tag{4.18}$$

for all t in the open interval $(2^{-1}, s)$ under the assumptions of Theorem 4.1. Indeed, for each $t \in (2^{-1}, s)$ the estimate (4.18) follows from (4.4) and (4.17).

We remark that (4.3) can be considered as an alternative numerical scheme to the well-known collocation method where an approximate solution in the form $f_n = v_n/\varrho$ is sought, of equation (4.1), and the unknown polynomial $v_n \in \Pi_n$ is determined by the equation

$$(H_\varrho + \mathbf{L}^\varrho_n m L_\varrho + \mathbf{L}^\varrho_n K_n)f_n = \mathbf{L}^\varrho_n g. \tag{4.19}$$

The statements of Theorem 4.1 hold on replacing equation (4.3) by equation (4.19) (see [**2**] for the case of constant m; for $m \in C^r_\varrho([-1,1])$, the proof is similar to that of Theorem 4.1).

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