JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 7, Number 4, Fall 1995

ON THE INVERSE OF INTEGRAL OPERATORS WITH KERNEL OPERATORS

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ABSTRACT. We study the boundedness of integral operators whose kernels are functions of operators $Vf(x) := f(x) + \int k(x,t,L)f(t)\,d\mu(t)$, where $k(x,t,\lambda)$ is an entire function of λ and L is an unbounded self-adjoint operator in $L^2_{d\mu(t)}$. By using Korotkov's theorem we derive a simple necessary condition for V to be a Carleman type operator. We are particularly interested in the cases when the inverse operator exists and has the same form as V. This study provides a new method for the inversion of integral equation of Carleman type.

1. Introduction. We first are interested in the boundedness of operators defined by

$$Vf(x) = f(x) + \int k(x,t,L)f(t)\,d\mu(t), \quad d\mu(x) \text{ a.e.}$$

in the Hilbert space $L^2_{d\mu(x)}$, where $k(x,t,\lambda)$ is an entire function of λ, $d\mu$ measurable in x and t, and L is an unbounded self-adjoint operator acting in the separable Hilbert space $L^2_{d\mu(t)}$. In fact one needs $k(x,t,\lambda)$ to be an analytic function of λ in the neighborhood of the spectrum of L only.

For the sake of simplicity we shall assume that

$$k(x,t,\lambda) := \sum_{n\ge 0} a_n(x,t)\lambda^n$$

and so $Vf(x)$ is defined by

$$(1) \qquad Vf(x) := f(x) + \int \sum_{n\ge 0} a_n(x,t)L^n f(t)\,d\mu(t), \quad d\mu(x) \text{ a.e.}$$

Received by the editors on July 27, 1994, and in revised form on August 6, 1995.
AMS *Mathematics Subject Classifications.* 46.

where $a_n(x,t)$ are $d\mu(t)$-measurable and $\int |a_n(x,t)|^2\, d\mu(t) < \infty$ for all n. In the next section, we shall define the domain of the operator V by using the transform associated with the self-adjoint operator L. In the same way we study the possible extension, to spaces containing eigenfunctionals. In these rigged spaces the representation of the operator V reduces to a one-parameter family of Fredholm operators. Thus we reduce the inversion of a Carleman integral operator to the inversion of a family of Fredholm operators. Here the analyticity of the kernel $k(x,t,\lambda)$ in terms of λ plays a major role. We shall prove the following result:

Theorem. *Assume that*

(i) $\tilde{V}$ *and* $\tilde{V}^{-1}$ *are bounded in* Φ', *and*

(ii) $\int\int |k(x,t,\lambda)|^2\, d\mu(x)\, d\mu(t) < \infty\; d\Gamma(\lambda)$ *almost everywhere for all* $\lambda \in \sigma$.

Then there exists an entire function $h(x,t,\lambda)$ *of* λ *and a self-adjoint operator* P *such that*

$$V^{-1}f := f(x) + \int h(x,t,P)\Pi(P-\lambda_i)^{-n_i} f(t)\, d\mu(t)$$

where $h(x,t,\lambda)/\Pi(\lambda-\lambda_i)^{n_i}$ *is the resolvent kernel of* T_λ^{-1} .

Equations involving operators of this kind occur frequently in applied mathematics, for example the inversion of the Laplace transform on the real line leads to

$$\mathcal{L}^{-1} := \frac{1}{\pi}\mathcal{L}T\cosh(i\pi D)T^{-1}$$

where $\mathcal{L}$ is the Laplace transform, i.e., an integral operator, D is the derivation operator and T is a unitary transformation. It is easy to understand the nature of operators defined by (1), by examining the following simple example:

$$Vf(x) = f(x) + \frac{1}{\pi}\int \frac{1}{1+t^2}\sum_{n\geq 0}\frac{(x-t)^n}{n!}D^n f(t)\, dt, \quad x \in R.$$

It is clear that if $f(x)$ is entire then

$$Vf(x) = 2f(x)$$

and hence is a bounded operator in L^2_{dt}.

In fact the method is applicable any time we can decompose the kernel, i.e.,

$$H(x,t) := \sum L^n a_n(x,t)$$

and under certain conditions it follows

$$\int H(x,t)f(t)\,dt := \int \sum_{n\geq 0} a_n(x,t)L^n f(t)\,dt.$$

2. Notation. Without loss of generality, we can assume the spectrum of L, σ say, to be simple. The concept of rigged spaces will help us describe the different parts of the spectrum and find appropriate spaces for the generalized eigenfunctions, see [**5**]. Recall that if $\lambda \in \sigma$ then there exists a minimizing sequence, sometimes referred to as a Weyl sequence, f_n, such that

$$f_n(t,\lambda) \in D_L \subset L^2_{d\mu(t)}, \qquad \|f_n\| = 1, \qquad \|Lf_n - \lambda f_n\| \overset{L^2_{d\mu(t)}}{\longrightarrow} 0.$$

If the Weyl sequence f_n is compact in $L^2_{d\mu(t)}$, then λ is in the discrete spectrum, i.e., eigenvalue. If f_n is not compact in $L^2_{d\mu(t)}$, then we assume the existence of a countably normed space Φ imbedded in $L^2_{d\mu(t)}$, and so it follows $\Phi \hookrightarrow L^2_{d\mu(t)} \hookrightarrow \Phi'$. For the topology, we shall assume that Φ is densely imbedded in $L^2_{d\mu(t)}$ and invariant under L, i.e., $L\Phi \subset \Phi$. Due to the continuous imbedding $L^2_{d\mu(t)} \hookrightarrow \Phi'$ the Weyl sequence $\{f_n(.,\lambda)\}$ is bounded in Φ' and so compact. Clearly this means $f_n(x,\lambda) \overset{\Phi'}{\to} \varphi(x,\lambda)$ and a simple argument shows that

$$L'\varphi(x,\lambda) = \lambda\varphi(x,\lambda) \quad \text{in } \Phi'. \tag{2}$$

Indeed, for $g \in \Phi$,

$$\begin{aligned}\langle (L-\lambda)g, f_n\rangle_{\Phi\times\Phi'} &\longrightarrow \langle (L-\lambda)g, \varphi(x,\lambda)\rangle_{\Phi\times\Phi'}\\ &= \langle g, (L'-\lambda)\varphi(x,\lambda)\rangle_{\Phi\times\Phi'}.\end{aligned}$$

On the other hand,

$$\langle (L-\lambda)g, f_n\rangle_{\Phi\times\Phi'} = \langle g, Lf_n - \lambda f_n\rangle_{\Phi\times\Phi'} \to 0$$

and so (2) follows.

Since the operator is symmetric, i.e., $L \subset L'$, we shall agree to write

$$L\varphi(x,\lambda) = \lambda\varphi(x,\lambda) \quad \text{in } \Phi'.$$

The associated unitary transform or φ-transform is first defined only in Φ and so denoted by $\hat{f}(\lambda) := \langle f, \varphi(x,\lambda)\rangle_{\Phi\times\Phi'}$. It is then extended by continuity to the whole space $L^2_{d\mu(t)}$ and reads as

$$\forall\, f \in L^2_{d\mu(t)}, \qquad \hat{f}(\lambda) := \int f(t)\varphi(t,\lambda)d\mu(t).$$

The inverse transform of $\hat{f}(\lambda) \in L^2_{d\Gamma}$ is defined by

$$f(t) := \int \hat{f}(\lambda)\overline{\varphi(t,\lambda)}d\Gamma(\lambda)$$

where $\Gamma(\lambda)$ is the spectral function associated with the operator L. Since L is self-adjoint, the set of eigenfunctionals is complete in Φ', that is

$$\tag{3} \begin{gathered} \forall\, f \in \Phi, \qquad \hat{f}(\lambda) = \int f(t)\varphi(t,\lambda)\, d\mu(t) = 0 \\ \forall\, \lambda \in \sigma \quad \Longrightarrow \quad f(t) := 0. \end{gathered}$$

Let us observe that if the spectrum is simple but contains a negative part then the multiplicity of the spectrum of even powers of L will be two.

We now introduce some notations and conditions to be used in the next sections.

Let D_{L^n} denote the domain of the operator L^n in $L^2_{d\mu(x)}$ and $D_{L^\infty} = \bigcap_{n\geq 0} D_{L^n}$. For self-adjoint operators, it is well known that $D_{L^\infty} \neq \varnothing$ and even dense in $L^2_{d\mu(x)}$. Since the kernel contains functions of two variables, by $L_t a(x,t)$ we shall mean the action of L on the variable t. In case we transform functions with several variables, the hat, "ˆ" designates the transformed variable, e.g.,

$$a(x,\hat{\lambda}) := \int a(x,t)\varphi(t,\lambda)d\mu(t).$$

For the sake of simplicity we shall sometimes need to use the following conditions to hold as $|\lambda| \to \infty$:

A) There exists an $a \geq 0$ for all $j > 0$, $|\varphi(t,\lambda)\sum_{n=0}^{n=j} a_n(x,t)\lambda^n| \leq c(x)|\lambda|^q e^{a\lambda^2}$; $d\Gamma(\lambda)$ almost everywhere.

B) $\int \sum_{n\geq 0} |a_n(x,\hat{\lambda})|^2 (n!/a^n) d\Gamma(\lambda) < \infty$, $d\mu(x)$ almost everywhere.

C) There exists a $p > 0$, $(d\Gamma/d\lambda)(\lambda) = O(|\lambda|^p)$ and $\int (1/(1+|\lambda|)^{p+2})\, d\Gamma(\lambda)$.

D) $\int |k(x,t,\lambda)\varphi(t,\lambda)|\, d\mu(t) = O(|\lambda|^s)$, $s \geq 0$.

For the sake of simplicity it is enough to define

$$\Phi := \{f \in L^2_{d\mu(t)}/e^{a\lambda^2}\hat{f}(\lambda) \in S\}$$

where S is the space of rapidly decreasing functions, or Schwartz space. Thus

$$\forall\, f \in \Phi, \quad \exists \psi \in S \quad \text{such that} \quad \hat{f}(\lambda) = e^{-a\lambda^2}\psi(\lambda).$$

The semi-norms associated with Φ are defined as follows

$$\Phi := \{f \in L^2_{d\mu(x)}/ \sup_{k\leq p,x} |(1+|\lambda|)^p D^k(\hat{f}(\lambda)e^{a\lambda^2})| < \infty\}.$$

Under condition C), the space $\Phi \subset D_{L^\infty}$, is invariant under L and perfect. This is a simple consequence of the fact S is a perfect space, see [**5**]. Finally we recall that V is seen as an operator acting in $L^2_{d\mu(x)}$

$$L^2_{d\mu(x)} \xrightarrow{V} L^2_{d\mu(x)} \quad \text{and} \quad D_V = \Phi.$$

The maximal domain of definition of the operator V in $L^2_{d\mu(x)}$ is denoted by

$$D_V^m := \{f \in L^2_{d\mu(x)}/Vf \in L^2_{d\mu(x)}\}.$$

Let us introduce the following Hilbert space indexed by $a > 0$

$$H_a := \{f \in L^2_{d\mu(t)}/||f||_a < \infty\}$$

where $||f||_a^2 := \int e^{a\lambda^2}|\hat{f}(\lambda)|^2 d\Gamma(\lambda)$ and a is a fixed constant.

3. The Operator $\mathcal{K}$. In order to define the operator V, it is sufficient to define the action of the integral part of the operator V, which we denote by $\mathcal{K}$, i.e.,

$$\mathcal{K}f(x) := \int k(x,t,L)f(t)\,d\mu(t) := \int \lim_{k\to\infty} \sum_{n=0}^{n=k} a_n(x,t)L^n f(t)\,d\mu(t).$$

Since the definition of the operator $\mathcal{K}$ involves infinite powers of the self-adjoint operator L, we need to use the spaces H_a and Φ.

Proposition 1. *Let conditions* B) *and* C) *hold, then for $f \in H_a$, where $a > 0$, $\mathcal{K}f(x)$ exists $d\mu(x)$ almost everywhere. If, in addition, $A(x) := \sqrt{\int |\sum_{n\geq 0} |a_n(x,t)|^2 (n!/a^n)|\,d\mu(t)} \in L^2_{d\mu(x)}$, then $H_a \subset D_V^m$.*

Proof.

$$\begin{aligned}
\left|\int k(x,t,L)f(t)\,d\mu(t)\right| &:= \left|\int \sum_{n\geq 0} a_n(x,t)L^n f(t)\,d\mu(t)\right| \\
&\leq \int \left|\sum_{n\geq 0} a_n(x,t)L^n f(t)\right| d\mu(t) \\
&\leq \sqrt{\int \sum_{n\geq 0} |a_n(x,t)|^2 \frac{1}{b_n(t)}\,d\mu(t)} \\
&\quad \times \sqrt{\int \sum_{n\geq 0} |L^n f(t)|^2 b_n(t)|\,d\mu(t)} \\
&\leq \sqrt{\int \sum_{n\geq 0} |a_n(x,t)|^2 \frac{1}{b_n(t)}\,d\mu(t)} \\
&\quad \times \sqrt{|\sum_{n\geq 0} \int |L^n f(t)|^2 b_n(t)\,d\mu(t)} \\
&\leq \sqrt{\int \sum_{n\geq 0} |a_n(x,t)|^2 \frac{1}{b_n(t)}\,d\mu(t)} \\
&\quad \times \sqrt{\int \sum_{n\geq 0} |\lambda^n \hat{f}(\lambda)|^2 b_n(t)\,d\mu(t)}.
\end{aligned}$$

Therefore, if we choose $b_n(t) = a^n/\sqrt{n!}$, we obtain

$$|\mathcal{K}f(x)| \leq A(x)||f||_a$$

where by condition B), $A(x) < \infty$. It is clear that if $A(x) \in L^2_{d\mu(x)}$, then $\mathcal{K}f(x) \in L^2_{d\mu(x)}$ and therefore $H_a \subset D_V$. Hence D_V is not trivial. □

The next result is essential for obtaining an equivalent representation, of the action of V when restricted to a certain space of test functions.

Proposition 2. *Assume that conditions* A), C), *and* D) *hold. Then*

$$\forall\, f \in \Phi \qquad Vf(x) = f(x) + \int R(x,t) f(t)\, d\mu(t)$$

where

$$R(x,t) = \int k(x,t,\lambda)\overline{\varphi(t,\lambda)}\, d\Gamma(\lambda) \tag{4}$$

Proof. Assume that $f \in \Phi$; then there exists $\psi(\lambda) \in S$ such that $\hat{f}(\lambda) = \mathrm{Exp}\,(-a\lambda^2)\psi(\lambda)$ and so $L^n f = \int \lambda^n \hat{f}(\lambda)\overline{\varphi(t,\lambda)} d\Gamma(\lambda)$. Therefore

$$\begin{aligned}
\mathcal{K}f(x) &= \int \sum_{n\geq 0} a_n(x,t) L^n f(t)\, d\mu(t) \\
&= \int \sum_{n\geq 0} a_n(x,t) \int \lambda^n \hat{f}(\lambda)\overline{\varphi(t,\lambda)}\, d\Gamma(\lambda)\, d\mu(t) \\
&= \int \sum_{n\geq 0} a_n(x,t) \int \lambda^n \mathrm{Exp}\,(-a\lambda^2)\psi(\lambda)\overline{\varphi(t,\lambda)}\, d\Gamma(\lambda)\, d\mu(t)
\end{aligned}$$

Recall that since $k(x,t,\lambda)$ is entire, then the series $\sum_{n\geq 0} a_n(x,t)\lambda^n$ converges for all λ and by condition A), as $\lambda \to \infty$ and

$$\left| \sum_{n\geq 0} a_n(x,t)\lambda^n \mathrm{Exp}\,(-a\lambda^2)\psi(\lambda)\overline{\varphi(t,\lambda)} \right| = O(|\lambda|^{-p-2}),$$

which means that the Lebesgue dominated convergence theorem is applicable, and it follows

$$\begin{aligned}\mathcal{K}f(x) &= \iint \sum_{n\geq 0} a_n(x,t)\lambda^n \hat{f}(\lambda)\overline{\varphi(t,\lambda)}\, d\Gamma(\lambda)\, d\mu(t)\\ &= \iint k(x,t,\lambda)\hat{f}(\lambda)\overline{\varphi(t,\lambda)}\, d\Gamma(\lambda)\, d\mu(t).\end{aligned}$$

Since condition C) holds, Fubini's theorem is applicable

$$\begin{aligned}\mathcal{K}f(x) &= \iint k(x,t,\lambda)\overline{\varphi(t,\lambda)}\, d\mu(t)\hat{f}(\lambda)\, d\Gamma(\lambda)\\ &:= \int R(x,\hat{\lambda})\hat{f}(\lambda)\, d\Gamma(\lambda)\\ &= \int R(x,t)f(t)\, d\mu(t).\end{aligned}$$

Hence, the result,

$$\forall\, f \in \Phi \qquad Vf(x) = f(x) + \int R(x,t)f(t)\, d\mu(t). \tag{5}$$

This means that when V is restricted to Φ, it can be represented as an integral operator, and $Vf(x)$ is defined $d\mu(x)$ almost everywhere. We now show that $\mathcal{K}$ admits closure in $L^2_{d\mu(x)}$.

Proposition 3. *If conditions* A), B), *and* D) *hold, then V admits closure in $\mathcal{L}^2_{d\mu(t)}$.*

Proof. We need to show that if $y_k \in \Phi$ such that $y_k \overset{L^2_{d\mu(t)}}{\rightarrow} 0$ and $Vy_k \overset{L^2_{d\mu(t)}}{\rightarrow} c$, then $c = 0$. Since $y_k \in \Phi$, then (5) holds, and by Cauchy Schwartz

$$|Vy_k(x)| \leq |y_k| + ||R(x,.)||\,||y_k||$$

So that as $k \to \infty$ we should have $c(x) = 0$ $d\mu(x)$ almost everywhere as $k \to \infty$.

We shall denote the closure of the operator V in $L^2_{d\mu(x)}$ by $\overline{V}$.

Remark. V admits closure in Φ' if and only if $D_{V'} \cap \Phi$ is dense in Φ, see [**3**]. This provides a sufficient condition for V to admit closure in $L^2_{d\mu(t)}$. Recall that a Carleman operator may have an empty domain.

4. Boundedness of $\mathcal{K}$ in $L^2_{d\mu(x)}$. Let us find conditions such that $\mathcal{K}$ is a Hilbert-Schmidt operator. From (4) follows

Proposition 4. *Let conditions* A), B), *and* D) *hold*; *then $\mathcal{K}$ is a Hilbert-Schmidt operator in $L^2_{d\mu(t)}$ if and only if*

$$\int\int |k(x,\hat{\lambda},\lambda)|^2 \, d\Gamma(\lambda)\, d\mu(x) < \infty.$$

Proof. If $f \in \Phi$, then it follows from (4) and Parseval that

$$\mathcal{K}f(x) = \int \sum_{n\geq 0} a_n(x,\hat{\lambda})\lambda^n \hat{f}(\lambda)\, d\Gamma(\lambda);$$

since the φ-transform is a bounded operator, it follows that $\mathcal{K}$ s a Hilbert Schmidt operator if and only if its kernel is square integrable, that is, $k(x,\hat{\lambda},\lambda) \in L^2_{d\mu(x)\, d\mu(t)}$, see [**2**]. □

The next proposition will help us express $R(x,t)$ in terms of the coefficients $a_n(x,t)$.

Proposition 5. *If $a_n(x,t) \in D_{L^n}$ for all n, and conditions* A), B) *and* D) *hold, then*

$$R(x,t) = \sum_{n\geq 0} L_t^n a_n(x,t)$$

$$\forall\, f \in \Phi \qquad \overline{V} f(x) = f(x) + \int \sum_{n\geq 0} L_t^n a_n(x,t) f(t)\, d\mu(t).$$

Proof. Recall that

$$\begin{aligned} R(x,t) &= \int \sum_{n\geq 0} a_n(x,\hat{\lambda})\lambda^n \overline{\varphi(t,\lambda)}\, d\Gamma(\lambda) \\ &= \int \sum_{n\geq 0} L^n \widehat{a_n(x},t)\overline{\varphi(t,\lambda)}\, d\Gamma(\lambda). \end{aligned}$$

The representation (4) will help us obtain sufficient conditions for $\mathcal{K}$ to be a Calerman operator. □

Proposition 6. *Assume that conditions* A), B)*, and* C) *hold and*

$$\int |R(x,t)|^2\, d\mu(t) = \int |k(x,\hat{\lambda},\lambda)|^2\, d\Gamma(\lambda) < \infty,$$

then $\mathcal{K}$ *is a Carleman operator.*

The representation provides an alternative definition for $\overline{V}$, in case condition A) holds and $f \in \Phi$

$$\overline{V} f(x) := f(x) + \int k(x,\hat{\lambda},\lambda)\hat{f}(\lambda)\, d\Gamma(\lambda). \tag{6}$$

$$\overline{V} f(x) := f(x) + \int R(x,t) f(t)\, d\mu(t). \tag{7}$$

where $R(x,t) := \int k(x,\hat{\lambda},\lambda)\overline{\varphi(t,\lambda)}\, d\Gamma(\lambda)$.

5. Extension to Φ'. We would like to see how V can be defined on the set $\{\varphi(x,\lambda)\}_{\lambda\in\sigma} \in \Phi'$. Let us denote by $\widetilde{V}$ the extension of V to the space Φ'.

Proposition 7. *Assume that* V *admits closure in* Φ' *and* $V\varphi(x,\lambda) \in \Phi'$ *for all* $\lambda \in \sigma$*. Then*

$$V\varphi(x,\lambda) = \varphi(x,\lambda) + \overline{R(x,\hat{\lambda})}.$$

Proof. Recall that if $f(x) \in \Phi \subset L^2_{d\mu(x)}$ then $f(x) = \int \hat{f}(\lambda)\varphi(x,\lambda)d\Gamma(\lambda)$. Since V admits closure and $V\varphi(x,\lambda)$ is defined in Φ', it follows that

$$Vf(x) = \int \hat{f}(\lambda)V\overline{\varphi(x,\lambda)}\,d\Gamma(\lambda).$$

On the other hand, from (4) and Parseval equality, we know that

$$\begin{aligned} Vf(x) &= f(x) + \int R(x,\hat{\lambda})\hat{f}(\lambda)\,d\Gamma(\lambda) \\ &= \int \hat{f}(\lambda)\overline{\varphi(x,\lambda)}\,d\Gamma(\lambda) + \int R(x,\hat{\lambda})\hat{f}(\lambda)\,d\Gamma(\lambda) \end{aligned}$$

where $e^{a\lambda^2}\hat{f}(\lambda) \in S$. Hence the result, since the space S is dense in $L^2_{d\Gamma(\lambda)}$. □

In what follows we obtain sufficient conditions such that $V\varphi(x,\lambda)$ is defined for all $\lambda \in \sigma$ by obtaining sufficient conditions for V to extend as a continuous operator in Φ'. This is equivalent to requiring boundedness of the operator V' in the space Φ. Observe that, instead of working in $L^2_{d\mu(x)}$, it will be easier to work in $L^2_{d\Gamma(\lambda)}$ by using the φ transform. In this way we do not have to deal with L directly. From equation (4) we can set

$$W :=\hat{}\ \overline{V}\ \hat{}^{\,-1},$$

which is defined in $L^2_{d\Gamma(\lambda)}$ by

$$W\psi(\lambda) = \psi(\lambda) + \int H(\lambda,\mu)\psi(\mu)\,d\Gamma(\mu) \tag{8}$$

where

$$H(\lambda,\mu) := k(\hat{\lambda},\hat{\mu},\mu) = \int k(x,\hat{\mu},\mu)\varphi(x,\lambda)\,d\mu(x).$$

It follows

$$W'f(\lambda) := f(\lambda) + \int \overline{H(\mu,\lambda)}f(\mu)\,d\Gamma(\mu).$$

By using the semi-norms of the space S, we shall obtain sufficient conditions for the boundedness of the operator W'in the space S.

Denote by

$$S_p := \{f(\lambda) \in C^\infty / ||f||_p := \sup_{k\le p} M_p(\lambda)|D^k e^{a\lambda^2} f(\lambda)|\},$$

then

$$\begin{aligned}||W'f||_p &\le \|f\|_p \\ &\quad + \sup_{k\le p} M_p(\lambda)\left|D^k \int e^{a\lambda^2}\overline{H(\eta,\lambda)}e^{-a\eta^2}e^{a\eta^2} f(\eta)\, d\Gamma(\eta)\right| \\ &\le \|f\|_p \\ &\quad + \sup_{k\le p} M_p(\lambda)\left|\int \frac{\partial^k}{\partial\lambda^k}\overline{e^{a\lambda^2}H(\eta,\lambda)} f(\eta) e^{-a\eta^2}e^{a\eta^2}\, d\Gamma(\eta)\right| \\ &\le \|f\|_p \\ &\quad + \sup_{k\le p} M_p(\lambda)\left|\int \frac{\partial^k}{\partial\lambda^k}\frac{\overline{e^{a\lambda^2}H(\eta,\lambda)}}{M_q(\eta)} M_q(\eta) f(\eta)\, d\Gamma(\eta)\right| \\ &\le c_{pq}\|f\|_q + \sup_{\lambda\in\sigma} M_p(\lambda)\left|\int \frac{\partial^k}{\partial\lambda^k}\frac{\overline{e^{a\lambda^2}H(\eta,\lambda)}}{M_q(\eta)} e^{-a\eta^2} d\Gamma(\eta)\right| \\ &\quad \sup_{j\le q} M_q(\eta)|D^j e^{a\eta^2} f(\eta)| \\ &\le \left\{c_{pq} + \sup_{\lambda\in\sigma} M_p(\lambda)\left|\ \int \frac{\partial^k}{\partial\lambda^k}\frac{\overline{e^{a\lambda^2}H(\eta,\lambda)}}{M_q(\eta)} e^{-a\eta^2} d\Gamma(\eta)\right.\right\}\|f\|_q\end{aligned}$$

where $M_p(\lambda) := (1+|\lambda|)^p$. Hence if $\sup_{\lambda\in\sigma} M_p(\lambda)|\int(\partial^k/\partial\lambda^k)(\overline{e^{a\lambda^2}H(\eta,\lambda)}/M_q(\eta))e^{-a\eta^2}\, d\Gamma(\eta) < \infty$, then V is bounded in Φ'. Also if $\{\varphi(x,\lambda)\} \in \Phi'_p$, then $V\varphi(x,\lambda) \in \Phi'_q$ and

$$\tilde{V}\varphi(x,\lambda) = \varphi(x,\lambda) + \int \sum a_n(x,t) L^n \varphi(t,\lambda) d\mu(t), \quad \text{in } \Phi'$$

together with the identity $L^n\varphi(x,\lambda) = \lambda^n \varphi(x,\lambda)$ in Φ' would imply the following

Proposition 8. *Let* $\{\varphi(x,\lambda)\} \in \Phi'_p$ *and* $\sup_{\lambda\in R} M_p(\lambda)|\int(\partial^k/\partial\lambda^k)((\overline{e^{a\lambda^2}H(\eta,\lambda))}/(M_q(\eta)e^{a\eta^2}))\, d\Gamma(\eta)| < \infty$ *where* $q \ge p$; *then* $V\Phi' \to \Phi'$ *is continuous and*

$$y(x,\lambda) = \varphi(x,\lambda) + \int k(x,t,\lambda)\varphi(t,\lambda)\, d\mu(t) \quad \text{in } \Phi'.$$

Remark. In case $d\mu(t) = dt$ we do not need to use the space $L^2_{d\Gamma(\lambda)}$. Indeed the growth of eigenfunctionals of any self adjoint operator in the Hilbert space $L^2_{R^n}$ is known; they are precisely in Φ_1' where

$$\Phi_1 := \{f / \sup(1+|x|^{3n/2+\varepsilon})|Df(x)| < \infty\},$$

see [**5**]. This is simply due to the fact that $\varphi(\hat{\nu}, \lambda) = \delta(\nu - \lambda)$ has a simple representation. Once a space containing all eigenfunctionals has been obtained, we can proceed in a similar way.

6. The Operator T_λ. If for $\lambda \in \sigma$, the operator $\int k(x,t,\lambda) f(t)\, d\mu(t)$ is compact, then one can use the existing theory of analytic Fredholm theory, see [**8**]. This is easily achieved if $k(x,t,\lambda)$ is a square integrable function of x and t. Now if we assume that $\tilde{V}$ is bounded in Φ', then it is defined on the set $\varphi(x,\lambda)$ which is outside $L^2_{d\mu(x)}$ and, as shown previously, the following holds in Φ'

$$y(x,\lambda) = \varphi(x,\lambda) + \int k(x,t,\lambda)\varphi(t,\lambda)\, d\mu(t). \tag{9}$$

The question then is under what conditions would there exist a kernel $h(x,t,\lambda)$ such that

$$\varphi(x,\lambda) = y(x,\lambda) + \int h(x,t,\lambda) y(t,\lambda)\, d\mu(t)$$

holds also in Φ' for all $\lambda \in \sigma$. The idea is to fix the parameter λ and then restrict the operator to $L^2_{d\mu(t)}$. This allows us to compute the inverse and then extend the operator back to Φ'. The final operation would be to exchange the λ with a certain self-adjoint operator, P. Now looking at λ as a fixed parameter, let us define

$$T_\lambda : f(x) \longrightarrow f(x) + \int k(x,t,\lambda) f(t)\, d\mu(t) \quad \text{in } L^2_{d\mu(x)}.$$

Clearly V can be seen as an extension of the whole family T_λ to the space Φ'. During the extension the kernel will remain unchanged and (9) will hold in the weak sense, i.e., in Φ'. Hence the question becomes, when would the inverse of T_λ be of the same nature as T_λ and have an extension to Φ' or at least be defined on the set of $V\varphi(x,\lambda)$? In some cases we can answer the question in a precise manner. For example, if T_λ is a family of Fredholm operators then it is known, see [**7**], that the resolvent either exists and is a rational function of λ or does not exist at all. It is clear that, in the event of the existence of a resolvent

$$T_\lambda^{-1} : f(x) \longrightarrow f(x) + \int \frac{\sum b_n(x,t)\lambda^n}{\sum c_n \lambda^n} f(t)\, d\mu(t) \tag{10}$$

where $\sum c_n \lambda^n$ is the Fredholm determinant. Thus, even if one succeeds in extending T_λ^{-1} to Φ', then it follows from (10)

$$\varphi(x,\lambda) := V^{-1} y(x,\lambda) = y(x,\lambda) + \int \frac{\sum b_n(x,t)\lambda^n}{\sum c_n \lambda^n} y(t,\lambda)\, d\mu(t). \tag{11}$$

Thus, at first sight the inverse is not of the same type as V. It is readily seen that we need the Fredholm determinant, $\sum c_n \lambda^n$, to have no zeros in σ, see [**8**]. We now obtain a sufficient condition for the existence of V^{-1} in $L^2_{d\mu(x)}$.

Proposition 9. *Assume that $\tilde{V}$ is closed in Φ'. Then $\overline{V}$ is invertible in $L^2_{d\mu(x)}$ if and only if*

$$\int \hat{f}(\lambda)\overline{y(x,\lambda)} d\Gamma(\lambda) = 0 \quad \Longrightarrow \quad f = 0.$$

Proof. The inverse of $\overline{V}$ exists if and only if

$$\overline{V} f = 0 \quad \Longrightarrow \quad f = 0.$$

Since $f \in L^2_{d\mu(x)}$, $f(x) = \int \hat{f}(\lambda)\overline{\varphi(x,\lambda)}\, d\Gamma(\lambda)$, and $\tilde{V}$ is a closed operator,

$$\begin{aligned} \overline{V} f = \overline{V} \int \hat{f}(\lambda)\overline{\varphi(x,\lambda)}\, d\Gamma(\lambda) \\ = \int \hat{f}(\lambda)\tilde{V}\overline{\varphi(x,\lambda)}\, d\Gamma(\lambda) \\ = \int \hat{f}(\lambda)\overline{y(x,\lambda)}\, d\Gamma(\lambda). \end{aligned}$$

Proposition 10 (Existence of the operator P).

i) *If* $\overline{V}^{-1}$ *exists, then there exists a self-adjoint operator* P *acting in* $L^2_{d\mu(x)}$ *and defined by* $P := \overline{V}L\overline{V}^{-1}$, *and*

ii) *If* $\tilde{V}$ *and* $\tilde{V}^{-1}$ *are bounded operators in* Φ', *then* $\tilde{P} := \tilde{V}L\tilde{V}^{-1}$ *is an extension of* P *to* Φ', *and* $\tilde{P}y(x,\lambda) = \lambda y(x,\lambda)$ *in* Φ' *where* $y(x,\lambda) = \tilde{V}\varphi(x,\lambda)$.

Proof. The existence of $\overline{V}^{-1}$ allows us to consider the operator $P = \overline{V}L\overline{V}^{-1}$ in $L^2_{d\mu(x)}$. For the second part it is sufficient to observe that, since the inverse $\tilde{V}^{-1}$ exists in Φ', it follows that

$$\tilde{P} := \tilde{V}L\tilde{V}^{-1}.$$

Indeed, we obviously have

$$\tilde{P}y(x,\lambda) = \tilde{V}L\varphi(x,\lambda) = \lambda\tilde{V}\varphi(x,\lambda) = \lambda y(x,\lambda) \quad \text{in } \Phi'.$$

The following properties can easily be shown.

a) if $\overline{V}^{-1}$ is bounded in $L^2_{d\mu}$, then P is densely defined in $L^2_{d\mu}$,

b) P has a simple spectrum

c) $y(x,\lambda)$ form a complete set of eigenfunctionals

d) there exists a spectral function $\Gamma_2(\lambda)$ such that

$$f(x) = \int \hat{f}^2(\lambda)\overline{y(x,\lambda)}\, d\Gamma_2(\lambda)$$

where $\hat{f}^2(\lambda) := \int f(x)y(x,\lambda)\, d\mu(x)$.

Hence we can replace (9) by

$$\varphi(x,\lambda) := T_\lambda^{-1}y(x,\lambda) = y(x,\lambda) + \int h(x,t,P)y(t,\lambda)\, d\mu(t).$$

We now can examine the case when T_λ is a Fredholm operator.

Theorem. *Assume that*

(i) $\tilde{V}$ *and* $\tilde{V}^{-1}$ *are bounded in* Φ', *and*

(ii) $\int\int |k(x,t,\lambda)|^2 \, d\mu(x) \, d\mu(t) < \infty$, $d\Gamma(\lambda)$ *almost everywhere.*

That is, T_λ *is a Fredholm operator for all* $\lambda \in \sigma$. *Then there exists a function* $h(x,t,\lambda)$ *and a self-adjoint operator* P *such that*

$$V^{-1}f(x) := f(x) + \int h(x,t,P)\Pi(P-\lambda_i)^{-n_i} f(t) \, d\mu(t)$$

where $h(x,t,\lambda)/\Pi(\lambda-\lambda_i)^{n_i}$ *is the resolvent kernel of* T_λ^{-1}.

Proof. The assumption on the existence of $\tilde{V}^{-1}$ and its boundedness in Φ' will help simplify the proof. From (ii) we know from the classical Fredholm theory, see [**7**], that the inverse T_λ^{-1} exists and its kernel will be a rational function of λ.

$$T_\lambda^{-1}f(x) = f(x) + \int \frac{h(x,t,\lambda)}{\Pi(\lambda-\lambda_i)^{n_i}} f(t) \, d\mu(t).$$

From (i) we have $y(x,\lambda) = T_\lambda(\varphi(x,\lambda))$, and it follows

$$T_\lambda^{-1}y(x,\lambda) = \varphi(x,\lambda) = y(x,\lambda) + \int \frac{h(x,t,\lambda)}{\Pi(\lambda-\lambda_i)^{n_i}} y(t,\lambda) \, d\mu(t). \qquad \square$$

Lemma. $\tilde{V}^{-1}$*bounded* $\Rightarrow \lambda_i \notin \sigma$.

Proof. Observe that $T_{\lambda_i}^{-1}$ does not exist since its kernel is not even defined. We shall show that if $\lambda \in \sigma$ then T_λ^{-1} is defined. Indeed, since $\tilde{V}$ and $\tilde{V}^{-1}$ are bounded then $y(x,\lambda) \neq 0$. In other words, $\sigma_P = \sigma$, where σ_P is the spectrum of P. Indeed, if $\lambda \in \sigma - \sigma_p$ then it follows $y(x,\lambda) = 0$, and in Φ' this leads to $y(x,\lambda) = 0 = V\varphi(x,\lambda) \neq 0$. Hence $V^{-1}0 \neq 0$. Thus a contradiction, i.e., $y(x,\lambda) \neq 0$ for all $\lambda \in \sigma$. On the other hand, we know from the previous section that $V\varphi(x,\lambda) = y(x,\lambda)$ is equivalent to $T_\lambda\varphi(x,\lambda) = y(x,\lambda)$, and so it follows that

$$\varphi(x,\lambda) = T_\lambda^{-1}y(x,\lambda).$$

Hence, if $\lambda \in \sigma$, then T_λ^{-1} is well defined and so the Lemma is proven. A simple consequence of the Lemma is that $\lambda_i \notin \sigma$. Therefore

$$\begin{aligned}\varphi(x,\lambda) &= T_\lambda^{-1} y(x,\lambda)\\ &= y(x,\lambda) + \int \frac{h(x,t,\lambda)}{\Pi(\lambda-\lambda_i)^{n_i}} y(t,\lambda)\, d\mu(t).\end{aligned}$$

Then use the fact that

$$\frac{1}{(\lambda-\lambda_i)^{n_i}} y(t,\lambda) = (P-\lambda_i)^{-n_i} y(t,\lambda),$$

and since $\lambda_i \notin \sigma$ then $(P-\lambda_i)^{-n_i}$ is a bounded operator. Also $h(x,t,\lambda)y(t,\lambda) = \sum b_n(x,t)\lambda^n y(t,\lambda) = h(x,t,P)y(t,\lambda)$.

$$\begin{aligned}\frac{h(x,t,\lambda)}{\Pi(\lambda-\lambda_i)^{n_i}} y(t,\lambda) &= \sum b_n(x,t)\lambda^n \Pi(\lambda-\lambda_i)^{-n_i} y(t,\lambda)\\ &= \sum b_n(x,t)\Pi(\lambda-\lambda_i)^{-n_i}\lambda^n y(t,\lambda)\\ &= \sum b_n(x,t)\Pi(\lambda-\lambda_i)^{-n_i} P^n y(t,\lambda)\\ &= \sum b_n(x,t)P^n\Pi(\lambda-\lambda_i)^{-n_i} y(t,\lambda)\\ &= \sum b_n(x,t)P^n\Pi(P-\lambda_i)^{-n_i} y(t,\lambda)\\ &= h(x,t,P)\Pi(P-\lambda_i)^{-n_i} y(t,\lambda)\\ &= h(x,t,P)B(p)y(t,\lambda)\end{aligned}$$

where

$$B(P) = \Pi(P-\lambda_i)^{-n_i} \tag{12}$$

is a bounded operator.

It remains to show that

$$V^{-1} f(x) = f(x) + \int h(x,t,P)B(p)f(t)\, d\mu(t).$$

It is easily verified that

$$\begin{aligned}
VV^{-1}f &= Vf + V\int h(x,t,P)B(p)f(t)\,d\mu(t)\\
&= V\int \hat f^2(\lambda)\left\{\overline{y(x,\lambda)}+\int h(x,t,P)B(p)\overline{y(t,\lambda)}\,d\mu(t)\right\}d\Gamma_2(\lambda)\\
&= V\int \hat f^2(\lambda)\overline{\varphi(x,\lambda)}\,d\Gamma_2(\lambda)\\
&= \int \hat f^2(\lambda)\overline{V\varphi(x,\lambda)}\,d\Gamma_2(\lambda)\\
&= \int \hat f^2(\lambda)\overline{y(x,\lambda)}d\Gamma_2(\lambda)\\
&= f(x).
\end{aligned}$$

One simple answer is provided by Volterra operators, since $\sum c_n\lambda^n = 1$. Let us recall that if

$$T_\lambda f(x) := f(x) + \int_{-|x|}^{|x|} k(x,t,\lambda)f(t)\,d\mu(t)$$

where $k(x,t,\lambda) \in L^{2,\text{ loc}}_{d\mu(t)\,d\mu(x)} < \infty$ then T_λ^{-1} exists and

$$T_\lambda^{-1}f(x) = f(x) + \int_{-|x|}^{|x|} h(x,t,\lambda)f(t)\,d\mu(t)$$

where $h(x,t,\lambda)$ is an entire function of λ . □

Remark. The interval of integration can be chosen according to the support of $d\mu(t)$. Recall that the space of continuous functions provides a simple and easy to use space of functionals, see [**5**] and [**4**].

Proposition 11. *Let*

(i) $k(x,t,\lambda)$ *be a continuous function of* x,t *for* $\lambda\in\sigma$.

(ii) $\varphi(x,\lambda)$ *are continuous in* x.

(iii) $T_\lambda\varphi(x,\lambda) := y(x,\lambda) = \varphi(x,\lambda) + \int_{-|x|}^{|x|} k(x,t,\lambda)\varphi(t,\lambda)\,d\mu(t)$. *Then there exists* $h(x,t,\lambda)$ *such that*

$$\varphi(x,\lambda) := T_\lambda^{-1}y(x,\lambda) = y(x,\lambda) + \int_{-|x|}^{|x|} h(x,t,\lambda)y(t,\lambda)\,d\mu(t). \tag{13}$$

Proof. The first condition ensures that T_λ is a Volterra operator in the space of continuous functions. $\square$

We then extend each T_λ^{-1} to Φ' and then replace the λ appearing in $h(x,t,\lambda)$, see (13), by an operator, for which $y(x,\lambda)$ are eigenfunctionals.

Remark. In practice the main difficulty one encounters is the choice of rigged spaces Φ. This question has been investigated by several authors. In fact it is possible to use Hilbert spaces only, $H_+ \hookrightarrow H \hookrightarrow H_-$ with compact embedding. In case the solution is known to be smooth then spaces of continuous functions can be used. The operator $B(P)$ defined by (12) and may be unbounded, in case $\lim_{n\to\infty} \lambda_n \in \sigma$.

7. Conclusion. By studying the inverse operator we came up with a new technique for constructing the inverse of an integral operator of "Carleman type of the second kind." The main idea is to expand the Carleman kernel $H(x,t) = \sum_{n\geq 0} L_t^n a_n(x,t)$ whose action, under certain conditions, reduces to

$$\int H(x,t)f(t)\,d\mu(t) = \int k(x,t,L)f(t)\,d\mu(t),$$

and then use the operators T_λ^{-1} and P to reconstruct V^{-1}. The idea of generalizing the Taylor expansion to arbitrary operators goes back to Delsatres, where it was used to define the generalized translation operators.

8. Examples.

Example 1. Consider the following operator

$$Vf(x) = f(x) + \int_0^x \sum c_n(x)a(x,t)L^n f(t)\,dt, \quad x > 0$$

where

$$a(x,t) := \begin{cases} \operatorname{Exp}\left(-1/((x-t)^2t^2)\right) & 0 < t < x \\ 0 & \text{otherwise} \end{cases}$$

and

$$\begin{cases} Lf(t) := (-d^2/dt^2)f(t), \quad t > 0, \\ f(0) = 0 \end{cases}$$

since $a(x,t)$ is smooth and is an exponential

$$L^n a(x,t) = (-1)^n D^{2n} a(x,t) = p_n(x,t)a(x,t),$$

where $p(x,t)$ is a polynomial in $1/(x-t)$ and $1/t$.

We then choose $c_n(t)$ such that

$$a(x,t)\sum_{n\geq 0} c_n(x)p_n(x,t) \quad \text{is smooth.}$$

Since condition C) holds, Proposition 3 implies that the operator V can be written as

$$Vf(x) = f(x) + \int_0^x \left[\sum_{n\geq 0} c_n(x)p_n(x,t)\right] a(x,t)f(t)\,dt.$$

The operator T_λ would be defined by

$$\begin{aligned} y(x,\lambda) &= T_\lambda \cos(x\sqrt{\lambda}) \\ &= \cos(x\sqrt{\lambda}) + \int_0^x a(x,t)\left[\sum c_n(x)\lambda^n\right]\cos(t\sqrt{\lambda})\,dt. \end{aligned}$$

If the set $y(x,\lambda)$ is complete, i.e., V^{-1} exists, then there exists an operator P such that $Py(x,\lambda) = \lambda y(x,\lambda)$ and

$$V^{-1}f(x) = f(x) + \int_0^x s(x,t,P)f(t)\,dt.$$

Example 2. Let

$$Vf(x) = f(x) + \int_{-\infty}^{\infty} k(x-t, -id/dt)f(t)\,dt$$

where

$$k(x-t,\lambda) := \sum_{n\geq 0} a_n(t-x)\lambda^n.$$

The operator $\overline{V}$ is therefore given by $\overline{V}f := f + \int \sum_{n\geq 0} a_n^{(n)}(t - x)(-i)^n f(t)\,dt$ where we assume that $a_n(t) \in C_0^\infty$ and $\sum a_n^{(n)}(x) \in L_R^1$. Clearly, the functionals

$$y(x,\lambda) := Ve^{it\lambda} = [1 + k(\hat{\lambda},\lambda)]e^{it\lambda}$$

are multiples of $e^{it\lambda}$.

On the other hand, $T_\lambda f := f + \int k(x-t,\lambda)f(t)\,dt$. We assume $k(x-t,\lambda) \in L^1$, for all $\lambda \in R$. Clearly, since we have a convolution equation we can compute the inverse. By Wiener's theorem, if $1 + k(\hat{\mu},\lambda) \neq 0$ then there exists a function $h(\hat{\mu},\lambda) \in L_{d\mu}^1$ such that

$$1 + h(\hat{\mu},\lambda) = \frac{1}{1 + k(\hat{\mu},\lambda)} \tag{14}$$

and using the Fourier transform, we end up with

$$T_\lambda^{-1} f(x) := \psi(x) = f(x) + \int h(x-t,\lambda)f(t)\,dt.$$

The extension of the operator to the eigenfunctionals, follows simply from (14),

$$\begin{aligned} e^{ix\lambda} &= T_\lambda^{-1}[1 + k(\hat{\lambda},\lambda)]e^{ix\lambda} \\ &= [1 + k(\hat{\lambda},\lambda)]e^{ix\lambda} \\ &\quad + \int h(x-t,\lambda)[1 + k(\hat{\lambda},\lambda)]e^{it\lambda}\,dt. \end{aligned}$$

Since the operator V^{-1} exists, we obviously obtain an operator P, whose eigenfunctionals are $[1 + k(\hat{\lambda},\lambda)]e^{ix\lambda}$ $P := -id/dt$. Hence, the inverse operator $\tilde{V}^{-1}$ is given by

$$\begin{aligned} e^{ix\lambda} &= T_\lambda^{-1}[1 + k(\hat{\lambda},\lambda)]e^{ix\lambda} \\ &= [1 + k(\hat{\lambda},\lambda)]e^{ix\lambda} \\ &\quad + \int h(x-t,-id/dt)[1 + k(\hat{\lambda},\lambda)]e^{it\lambda}\,dt \end{aligned}$$

$$V^{-1} f(x) := f(x) + \int h(x-t,-id/dt)f(t)\,dt.$$

In general the operator P is different from the operator L.

Acknowledgment. I would like to acknowledge the support of K.F.U.P.M. during my visit to the University of Illinois at Urbana.

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