JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 7, Number 1, Winter 1995

A BOUNDARY ELEMENT METHOD FOR A NONLINEAR BOUNDARY VALUE PROBLEM IN STEADY-STATE HEAT TRANSFER

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ABSTRACT. A new boundary variational formulation is presented for the steady-state heat equation with radiation, and a boundary element method is presented for its solution. Furthermore, an optimal error estimate for the boundary element approximation is given.

1. Introduction. We consider the numerical modeling of steady-state heat transfer with radiation. This phenomenon is mathematically represented by the following nonlinear boundary value problem

$$\Delta u = 0, \quad \text{in } \Omega, \tag{1}$$

$$\frac{\partial u}{\partial n} + \gamma |u|^3 u = f(x), \quad \text{on } \Omega, \tag{2}$$

where $\Omega \subset R^2$ is a bounded domain with sufficient smooth boundary Γ, γ is a positive constant associated with the body's emittance [**15**], and f is a given function on Γ. In the engineering literature, the nonlinear boundary value problem (1)–(2) was studied by several authors using boundary element methods [**5, 9, 14**]. Recently Ruotsalainen and Wendland [**13**] have considered the potential problem with boundary condition

$$\frac{\partial u}{\partial n} + g(x, u) = f(x), \quad \text{on } \Gamma, \tag{3}$$

and have given an analysis of a boundary element method for problem (1)–(3). They assume $g : \Gamma \times R \to R$ is a Caratheodory-function, i.e.,

Received by the editors on April 15, 1993, and in revised form on September 14, 1994.
1980 *Mathematics Subject Classification.* (1985 Revision). Primary 65R20.
Key words and phrases. Nonlinear boundary value problem, boundary element method.

$g(\cdot, u)$ is measurable for all $u \in R$, and $g(x, \cdot)$ is continuous for almost all $x \in \Gamma$, and $(\partial/\partial u)g(x, u)$ is Borel measurable satisfying

$$(4) \qquad 0 < l \le \frac{\partial}{\partial u} g(x, u) \le L < \infty, \qquad \forall\, x \in \Gamma,\ u \in R.$$

Ruotsalainen and Wendland reduced problem (1)–(3) to a nonlinear integral equation on the boundary Γ and proved existence and uniqueness of a solution to the problem. Furthermore, the boundary element approximate solution and its optimal error estimate are given under assumption (4) on $g(x, u)$. As pointed out by the authors boundary condition (2) does not satisfy the assumption (4). In 1990 Atkinson and Chandler discussed numerical methods for the nonlinear boundary integral equation given by Ruotsalainen and Wendland. Two numerical methods are proposed and analyzed for the discretizing integral equation [**2**].

In this paper the nonlinear boundary value problem (1)–(2) is reduced to a new boundary variational problem. Furthermore, a boundary element approximation of this problem is given and the rate of convergence of the approximation solution is obtained.

2. An equivalent boundary variational problem. Assume that $u \in H^1(\Omega)$ is the weak solution of problem (1)–(2). Then, in the domain Ω, we have $\Delta u = 0$. Let $p = (\partial u/\partial n)|_\Gamma \in H^{-1/2}(\Gamma)$. As usual, let $W^{m,\rho}(\Omega)$ and $W^{\alpha,\rho}(\Gamma)$ denote the Sobolev spaces on the domain Ω and the closed boundary curve Γ with norms $||\cdot||_{m,\rho,\Omega}$ and $||\cdot||_{\alpha,\rho,\Gamma}$. Note that $W^{m,2}(\Omega) = H^m(\Omega)$, $W^{\alpha,2}(\Gamma) = H^\alpha(\Gamma)$, and $W^{0,\rho}(\Gamma) = L_\rho(\Gamma)$ with integer m, real numbers α and $\rho \ge 1$ [**1**]. By Green's formula, we have

$$(5) \qquad u(x) = \int_\Gamma \frac{\partial G(x, y)}{\partial n_y} u(y)\, ds_y - \int_\Gamma G(x, y) p(y)\, ds_y, \qquad \forall\, x \in \Omega,$$

where $G(x, y) = (1/2\pi) \log |x - y|$ is the corresponding fundamental solution of equation (1); n_y denotes the outward unit normal to boundary Γ at $y \in \Gamma$. From the jump relations of the double-layer potential and normal derivatives of the simple-layer potential as x approaches the boundary Γ, we obtain the following two relationships between p and

$u|_\Gamma$ [**7, 11**]:

(6)
$$\frac{1}{2}u(x) = \int_\Gamma \frac{\partial G(x,y)}{\partial n_y} u(y)\, ds_y - \int_\Gamma G(x,y)p(y)\, ds_y, \qquad \forall\, x \in \Gamma,$$
(7)
$$\frac{1}{2}p(x) = \int_\Gamma \frac{\partial^2 G(x,y)}{\partial n_x \partial n_y} u(y)\, ds_y - \int_\Gamma \frac{\partial G(x,y)}{\partial n_x} p(y)\, ds_y, \qquad \forall\, x \in \Gamma,$$

where

$$\int_\Gamma \frac{\partial^2 G(x,y)}{\partial n_x \partial n_y} u(y)\, ds_y = \frac{d}{ds_x}\int_\Gamma G(x,y)\frac{du(y)}{ds_y}\, ds_y, \qquad \forall\, x \in \Gamma.$$

Let
$$H_*^{-1/2}(\Gamma) = \left\{ q \mid q \in H^{-1/2}(\Gamma) \text{ and } \int_\Gamma q\, ds = 0 \right\}.$$

Multiplying (6) by a function $q \in H_*^{-1/2}(\Gamma)$ and integrating over Γ, we obtain

$$-b(q,u) + a_0(p,q) = 0, \qquad \forall\, q \in H_*^{-1/2}(\Gamma), \tag{8}$$

where

$$a_0(p,q) = -\int_\Gamma \int_\Gamma G(x,y)p(y)q(x)\, ds_y\, ds_x,$$
$$b(q,u) = \frac{1}{2}\int_\Gamma uq\, ds - \int_\Gamma \int_\Gamma \frac{\partial G(x,y)}{\partial n_y} u(y)q(x)\, ds_y\, ds_x.$$

Multiplying (7) by a function $v \in H^{1/2}(\Gamma)$ and integrating by parts, we get

$$a_0\left(\frac{du}{ds}, \frac{dv}{ds}\right) + b(p,v) = \int_\Gamma pv\, ds, \qquad \forall\, v \in H^{1/2}(\Gamma). \tag{9}$$

On the other hand, it is straightforward to check that problem (1)–(2) is equivalent to the following variational problem: Find $u \in H^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v\, dx + a_1(u,v) = \int_\Gamma fv\, ds, \qquad \forall\, v \in H^1(\Omega), \tag{10}$$

where

$$a_1(u,v) = \gamma \int_\Gamma |u|^3 uv\, ds. \tag{11}$$

For any $v = \in H^1(\Omega)$ and u satisfying $\Delta u = 0$ in Ω, we have

$$\int_\Omega \nabla u \cdot \nabla v\, dx = \int_\Gamma pv\, ds.$$

By the equality (9), we have

$$\int_\Omega \nabla u \cdot \nabla v\, dx = a_0\left(\frac{du}{ds}, \frac{dv}{ds}\right) + b(p,v), \qquad \forall\, v \in H^1(\Omega),$$

with u satisfying $\Delta u = 0$ in Ω. Then variational problem (10) is reduced to the following boundary variational problem: Find $(u,p) \in H^{1/2}(\Gamma) \times H_*^{-1/2}(\Gamma)$, such that

$$\begin{aligned} &a_0\left(\frac{du}{ds}, \frac{dv}{ds}\right) + a_1(u,v) + b(p,v) = \int_\Gamma fv\, ds, \qquad \forall\, v \in H^{1/2}(\Gamma), \\ &-b(q,u) + a_0(q,p) = 0, \qquad \forall\, q \in H_*^{-1/2}(\Gamma). \end{aligned} \tag{12}$$

Before we establish the existence and uniqueness of the problem (12), we recall some results on the bilinear forms $a_0(p,q)$ and $b(q,v)$.

Lemma 2.1 [**7, 8, 12**]. (i) *$a_0(p,q)$ is a bounded and coercive bilinear form on $H^{-1/2} * (\Gamma) \times H_*^{-1/2}(\Gamma)$, i.e., there exist two constants $M_0 > 0$ and $\beta_0 > 0$ such that*

$$|a_0(p,q)| \le M_0 ||p||_{-1/2,2,\Gamma} ||q||_{-1/2,2,\Gamma}, \qquad \forall\, p,q \in H_*^{-1/2}(\Gamma), \tag{13}$$

$$a_0(p,p) \ge \beta_0 ||p||^2_{-1/2,2,\Gamma}, \qquad \forall\, p \in H_*^{-1/2}(\Gamma). \tag{14}$$

(ii) *$b(q,v)$ is a bounded bilinear form on $H_*^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$, i.e., there exists a constant $M_1 > 0$ such that*

$$\begin{aligned} &|b(q,v)| \le M_1 ||q||_{-1/2,2,\Gamma} ||v||_{1/2,2,\Gamma}, \\ &\forall\, q \in H_*^{-1/2}(\Gamma),\ v \in H^{1/2}(\Gamma). \end{aligned} \tag{15}$$

For $a_1(u,v)$, we have

Lemma 2.2. (i) *$a_1(u,v)$ is well defined on $H^{1/2}(\Gamma)\times H^{1/2}(\Gamma)$.*

(ii) *The functional $a_1(v,v)$ is strictly convex on $H^{1/2}(\Gamma)$ and*

$$a_1(v,v)=\gamma||v||_{0,5,\Gamma}^5, \qquad \forall\, v\in H^{1/2}(\Gamma). \tag{16}$$

(iii) *For any $u,v,w\in H^{1/2}(\Gamma)$, the following inequalities hold*

$$|a_1(u,w)-a_1(v,w)|\le 4\gamma\int_\Gamma(|u|^3+|v|^3)|u-v||w|\,ds, \tag{17}$$

$$a_1(v,v-w)-a_1(w,v-w)\ge\frac{\gamma}{4}\int_\Gamma(|v|^3+|w|^3)(v-w)^2\,ds. \tag{18}$$

Proof. (i) By Sobolev's imbedding theorem [**1**], we know that for any $v\in H^{1/2}(\Gamma)$ that $v\in L_5(\Gamma)$ and

$$||v||_{0,5,\Gamma}\le C||v||_{1/2,2,\Gamma}.$$

Since

$$|a_1(u,v)|=\gamma\left|\int_\Gamma|u|^3uv\,ds\right|\le\gamma\left(\int_\Gamma|u|^5\,ds\right)^{4/5}\left(\int_\Gamma|v|^5\,ds\right)^{1/5},$$

we conclude that $a_1(u,v)$ is well defined on $H^{1/2}(\Gamma)\times H^{1/2}(\Gamma)$.

(ii) The conclusion follows by making use of the strict convexity of the mapping $t\in R\to|t|^5$.

(iii) By the definition (11) of $a_1(u,v)$, we have

$$a_1(u,w)-a_1(v,w)=\gamma\int_\Gamma(|u|^3u-|v|^3v)w\,ds.$$

Furthermore, we get

$$|(|u|^3u-|v|^3v)w|=\left|4w\int_v^u|\tau|^3\,d\tau\right|\le4(|u|^3+|v|^3)|u-v||w|.$$

On the other hand, we obtain that, for $vw \geq 0$,

$$\begin{aligned}(|v|^3 v - |w|^3 w)(v-w) &= (|v|^4 - |w|^4)(|v| - |w|)\\ &\geq (|v|^3 + |w|^3)(|v| - |w|)^2\\ &= (|v|^3 + |w|^3)(v-w)^2.\end{aligned}$$

and for $vw < 0$,

$$\begin{aligned}(|v|^3 v - |w|^3 w)(v-w) &= (|v|^4 + |w|^4)(|v| + |w|)\\ &\geq \frac{1}{2}(|v|^2 + |w|^2)^2(|v| + |w|)\\ &\geq \frac{1}{4}(|v|^2 + |w|^2)(|v| + |w|)(|v| + |w|)^2\\ &\geq \frac{1}{4}(|v|^3 + |w|^3)(v-w)^2.\end{aligned}$$

Thus the inequalities (17) and (18) are proved.

For any $v \in H^{1/2}(\Gamma)$, let

$$||v||_*^2 = \left\|\frac{dv}{ds}\right\|_{-1/2,2,\Gamma}^2 + ||v||_{0,2,\Gamma}^2. \tag{19}$$

Lemma 2.3. $||\cdot||_*$ *is an equivalent norm on* $H^{1/2}(\Gamma)$*, i.e., there exist two positive constants* C_1 *and* C_2 *such that*

$$C_1||v||_{1/2,2,\Gamma} \leq ||v||_* \leq C_2||v||_{1/2,2,\Gamma}, \qquad \forall\, v \in H^{1/2}(\Gamma). \tag{20}$$

Proof. Let $\theta = 2\pi s/L$, where L denotes the length of the boundary and s denotes the length of the arc from the point $p_0 \in \Gamma$ to a point $p \in \Gamma$ along the boundary Γ. Making a change of variable from s to θ, it is seen that we only need to prove (20) when Γ is the circle with radius 1. On the other hand, we know that the space $C^1(\overline{\Omega})$ is dense in $H^1(\Omega)$ [**1**]. If we can prove (20) for any $v \in C^1(\overline{\Omega})$, then (20) holds for $v \in H^{1/2}(\Gamma)$.

Suppose Γ is the circle with radius 1. For any $v \in C^1(\overline{\Omega})$ on boundary Γ, we have

$$v = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)$$

where

$$a_n = \frac{1}{\pi}\int_0^{2\pi} v(\theta)\cos n\theta\, d\theta, \qquad n = 0, 1, 2, \dots$$
$$b_n = \frac{1}{\pi}\int_0^{2\pi} v(\theta)\sin n\theta\, d\theta, \qquad n = 1, 2, \dots$$
$$||v||_{0,2,\Gamma}^2 = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2),$$
$$||v||_{1/2,2,\Gamma}^2 = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(1+n^2)^{1/2}(a_n^2 + b_n^2),$$
$$\frac{dv}{d\theta} = \sum_{n=1}^{\infty}(nb_n \cos n\theta - na_n \sin n\theta),$$
$$\left\|\frac{dv}{d\theta}\right\|_{-1/2,2,\Gamma}^2 = \sum_{n=1}^{\infty}(1+n^2)^{-1/2}(n^2 b_n^2 + n^2 a_n^2).$$

Furthermore,

$$\left\|\frac{dv}{d\theta}\right\|_{-1/2,2,\Gamma}^2 + ||v||_{0,2,\Gamma}^2 \le 2\left\{\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(1+n^2)^{1/2}(a_n^2 + b_n^2)\right\} \le 2||v||_{1/2,2,\Gamma}^2$$

and

$$\left\|\frac{dv}{d\theta}\right\|_{-1/2,2,\Gamma}^2 + ||v||_{0,2,\Gamma}^2 \ge ||v||_{1/2,2,\Gamma}^2.$$

Thus, the proof is complete. □

Corollary 2.1. *Let* $H_*^{1/2}(\Gamma) = \{v \in H^{1/2}(\Gamma)$ *and* $\int_\Gamma v\, ds = 0\}$, *then there exists a constant* $C_3 > 0$ *such that*

$$||v||_{1/2,2,\Gamma} \le C_3 \left\|\frac{dv}{ds}\right\|_{-1/2,2,\Gamma}, \qquad \forall\, v \in H_*^{1/2}(\Gamma). \tag{21}$$

Lemma 2.4. *There is a constant* $M_2 > 0$ *such that*

$$|b(q,v)| \le M_2 ||q||_{-1/2,2,\Gamma} \left\|\frac{dv}{ds}\right\|_{-1/2,2,\Gamma}, \quad \forall\, q \in H_*^{-1/2}(\Gamma),\ v \in H^{1/2}(\Gamma). \tag{22}$$

Proof. For any $v \in H^{1/2}(\Gamma)$, let

$$\bar{v} = \frac{1}{\text{mes}\,(\Gamma)} \int_\Gamma v\,ds,$$

then $u = \bar{v}$ is a solution of $\Delta u = 0$ and $p = (\partial u/\partial n)|_\Gamma = 0$. Hence $(\bar{v}, p)$ satisfies the equation

$$-b(q, \bar{v}) + a_0(q, p) = 0, \qquad \forall\, q \in H_*^{-1/2}(\Gamma).$$

Since $p = 0$, we get

$$\begin{aligned} |b(q, v)| &= |b(q, v - \bar{v})| \le M_1 ||q||_{-1/2,2,\Gamma} ||v - \bar{v}||_{1/2,2,\Gamma} \\ &\le C_3 M_1 ||q||_{1/2,2,\Gamma} \left\| \frac{dv}{ds} \right\|_{-1/2,2,\Gamma}, \\ &\qquad \forall\, q \in H_*^{-1/2}(\Gamma),\ v \in H^{1/2}(\Gamma). \end{aligned}$$

Thus the conclusion follows with $M_2 = C_3 M_1$. □

We now establish the existence and uniqueness of a solution of the problem (12).

Theorem 2.1. *Suppose that* $f \in H^{-1/2}(\Gamma)$. *Then the problem* (12) *has a unique solution* $(u, p) \in H^{1/2}(\Gamma) \times H_*^{-1/2}(\Gamma)$.

Proof. For any given $v \in H^{1/2}(\Gamma)$ we consider the problem: Find $p \in H_*^{-1/2}(\Gamma)$ such that

$$a_0(q, p) = b(q, v), \qquad \forall\, q \in H_*^{-1/2}(\Gamma). \tag{23}$$

From Lemma 2.1 and the Lax-Milgram theorem [**6**], we know that the problem (23) has a unique solution which is denoted by $Kv \in H_*^{-1/2}(\Gamma)$. Thus, we can eliminate the unknown function p in the problem (12). In fact, for the solution (u, p) of the problem (12), we have $p = Ku$ and

$$b(p, v) = a_0(p, Kv) = a_0(Ku, Kv).$$

Hence the problem (12) is reduced to: Find $u \in H^{1/2}(\Gamma)$ such that

$$(24)\quad a_0\left(\frac{du}{ds}, \frac{dv}{ds}\right) + a_1(u,v) + a_0(Ku, Kv) = (f, v), \qquad \forall\, v \in H^{1/2}(\Gamma),$$

which is a nonlinear variational problem on Γ.

Let

$$\begin{aligned} I(v) = \frac{1}{2}\left[a_0\left(\frac{dv}{ds}, \frac{dv}{ds}\right) + a_0(Kv, Kv)\right] \\ + \frac{1}{5}a_1(v,v) - (f,v), \quad \forall\, v \in H^{1/2}(\Gamma). \end{aligned}$$

It is straightforward to check that the nonlinear variational problem (24) is equivalent to the following minimization problem: Find $u \in H^{1/2}(\Gamma)$ such that

$$(25)\qquad I(u) = \inf_{v \in H^{1/2}(\Gamma)} I(v).$$

By Lemmas 2.1, 2.2 and the imbedding theorem,

$$||v||_{0,2,\Gamma} \le C_0 ||v||_{0,5,\Gamma}, \qquad \forall\, v \in L_5(\Gamma),$$

with a positive constant C_0, we obtain

$$\begin{aligned} I(v) &\ge \frac{1}{2}a_0\left(\frac{dv}{ds}, \frac{dv}{ds}\right) + \frac{1}{5}a_1(v,v) - (f,v) \\ &\ge \frac{\beta_0}{2}\left\|\frac{dv}{ds}\right\|^2_{-1/2,2,\Gamma} + \frac{\gamma}{5}||v||^5_{0,5,\Gamma} - ||f||_{-1/2,2,\Gamma}||v||_{1/2,2,\Gamma} \\ &\ge \frac{\beta_0}{2}\left\|\frac{dv}{ds}\right\|^2_{-1/2,2,\Gamma} + \frac{\gamma}{5C_0^5}||v||^5_{0,2,\Gamma} - ||f||_{-1/2,2,\Gamma}||v||_{1/2,2,\Gamma}. \end{aligned}$$

Applying Lemma 2.3, we conclude that

$$\lim_{||v||_{1/2,2,\Gamma} \to \infty} I(v) = +\infty.$$

Combining the results in Lemma 2.1–2.2, we deduce the strict convexity and the differentiability of the functional $I(v)$ [**6**, p. 3.14]. By the theorem of minimization of convex functionals [**3**, p. 13] we conclude that the minimization problem (25) has a unique solution, $u \in H^{1/2}(\Gamma)$, which is the unique solution of (24). Let $p = Ku$, where u is the unique solution of (25). Then $(u,p) \in H^{1/2}(\Gamma) \times H_*^{-1/2}(\Gamma)$ is the unique solution of the problem (12), which finishes the proof. □

Furthermore, we have the following

Theorem 2.2. *The boundary variational problem (12) is equivalent to the following saddle-point problem: Find* $(u,p) \in H^{1/2}(\Gamma)\times H_*^{-1/2}(\Gamma)$ *such that*

$$
\begin{gathered}
L(u,q) \le L(u,p) \le L(v,p),\\
\forall\, v \in H^{1/2}(\Gamma),\ q \in H_*^{-1/2}(\Gamma),
\end{gathered}
\tag{26}
$$

where

$$
\begin{aligned}
L(v,q) = {}& \frac{1}{2} a_0\left(\frac{dv}{ds}, \frac{dv}{ds}\right)\\
&+ \frac{1}{5} a_1(v,v) + b(q,v) - \frac{1}{2} a_0(q,q) - \int_\Gamma f v \, ds.
\end{aligned}
$$

Proof. Assume that (u,p) is the solution of (26). Then for any $q \in H_*^{-1/2}(\Gamma)$ and real number λ, we have

$$L(u, p+\lambda q) \le L(u,p).$$

That is,

$$\lambda\{b(q,u) - a_0(q,p)\} - \frac{\lambda^2}{2} a_0(q,q) \le 0.$$

Since $a_0(q,q) \ge 0$ and λ is an arbitrary real number, we have

$$-b(q,u) + a_0(q,p) = 0, \qquad \forall\, q \in H_*^{-1/2}(\Gamma).$$

On the other hand, for any $v \in H^{1/2}(\Gamma)$ and any real number λ, we have

$$L(u,p) \le L(u+\lambda v, p), \qquad \forall\, v \in H^{1/2}(\Gamma).$$

A computation shows
(27)
$$L(u+\lambda v,p) = L(u,p) + \lambda\left\{a_0\left(\frac{du}{ds},\frac{dv}{ds}\right) + b(p,v) - \int_\Gamma f v\,ds\right\} + \frac{\lambda^2}{2}a_0\left(\frac{dv}{ds},\frac{dv}{ds}\right) + \frac{1}{5}[a_1(u+\lambda v,u+\lambda v) - a_1(u,u)].$$

Let
$$J(\lambda) = \frac{1}{5}a_1(u+\lambda v,u+\lambda v),$$
then we have
$$J(0) = \frac{1}{5}a_1(u,u)$$
$$\frac{dJ}{d\lambda} = \gamma\int_\Gamma |u+\lambda v|^3(u+\lambda v)v\,ds$$
$$\frac{dJ}{d\lambda}|_{\lambda=0} = \gamma\int_\Gamma |u|^3 uv\,ds = a_1(u,v)$$
$$\frac{d^2J}{d\lambda^2} = 4\gamma\int_\Gamma |u+\lambda v|^3 v^2\,ds \geq 0, \qquad \forall\,\lambda\in R.$$

Hence, we obtain

(28)
$$\frac{1}{5}\{a_1(u+\lambda v,u+\lambda v) - a_1(u,u)\} = J(\lambda) - J(0) = \lambda a_1(u,v) + \frac{\lambda^2}{2}\frac{d^2J}{d\lambda^2}\bigg|_{\lambda=\tilde\lambda},$$

where $\tilde\lambda$ lies between 0 and λ. Combining (27) and (28), we arrive at the following inequality
$$\lambda\left\{a_0\left(\frac{du}{ds},\frac{dv}{ds}\right) + a_1(u,v) + b(q,v) - \int_\Gamma f v\,ds\right\} + \frac{\lambda^2}{2}\left\{a_0\left(\frac{dv}{ds},\frac{dv}{ds}\right) + \frac{d^2J}{d\lambda^2}\bigg|_{\lambda=\tilde\lambda}\right\} \geq 0, \qquad \forall\,v\in H^{1/2}(\Gamma).$$

Since λ is an arbitrary real number, we get
$$a_0\left(\frac{du}{ds},\frac{dv}{ds}\right) + a_1(u,v) + b(p,v) = \int_\Gamma f v\,ds, \qquad \forall\,v\in H^{1/2}(\Gamma).$$

This means that $(u,p) \in H^{1/2}(\Gamma) \times H_*^{-1/2}(\Gamma)$ is the solution of the problem (12). Each of the above steps is reversible, hence we conclude that if (u,p) is the solution of the problem (12), then it is the solution of the saddle-point theorem (26), finishing the proof. □

Remark. For the more general nonlinear problem (1)–(3), we can get a similar result under some suitable assumptions on $g(x,u)$. For example, suppose that $g(x,u)$ and $(\partial/\partial u)g(x,u)$ are two continuous functions on $\Gamma \times R$, and

$$\begin{aligned} |g(x,u)| &\le C(1+|u|^{\beta+1}), \\ \alpha_0|u|^\beta &\le \frac{\partial}{\partial u} g(x,u) \\ &\le \alpha_1 |u|^\beta, \qquad \forall\, x \in \Gamma,\ u \in R; \end{aligned}$$

where C, β, α_0 and α_1 are positive constants. Then the problem (1)–(3) is equivalent to the variational problem (24) with

$$a_1(u,v) = \int_\Gamma g(x,u) v\, dx.$$

3. The boundary element approximations. Here we consider a numerical approximation scheme for finding the approximate solution of the problem (12). Assume that S_h and M_h are two finite-dimensional subspaces of $H^{1/2}(\Gamma)$ and $H_*^{-1/2}(\Gamma)$, respectively. The discrete problem corresponding to (12) is: Find $(u_h, p_h) \in (S_h, M_h)$, such that

$$\begin{aligned} a_0\left(\frac{du_h}{ds}, \frac{dv_h}{ds}\right) + a_1(u_h, v_h) + b(p_h, v_h) &= (f, v_h), \qquad \forall\, v_h \in S_h, \\ -b(q_h, u_h) + a_0(q_h, p_h) &= 0, \qquad \forall\, q_h \in M_h. \end{aligned} \tag{29}$$

For the problem (29) we have

Theorem 3.1. *If $f \in H^{-1/2}(\Gamma)$, then the problem (29) has a unique solution $(u_h, p_h) \in (S_h, M_h)$.*

The proof of Theorem 3.1 is omitted here because it is similar to the proof of Theorem 2.1. Now assume that $f \in L_{5/4}(\Gamma)$. Taking $v_h = u_h$,

$q_h = p_h$ in (29), we obtain

$$a_0\left(\frac{du_h}{ds}, \frac{du_h}{ds}\right) + a_1(u_h, u_h) + a_0(p_h, p_h) = (f, u_h).$$

Using Young's inequality and the results in the Lemmas 2.1–2.2, we get

$$||u_h||^5_{0,5,\Gamma} \le C_4(||f||_{0,5/4,\Gamma})^{5/4}, \tag{30}$$

where C_4 is a constant independent of h. Similarly for the solution (u, p) of (12), we have

$$||u||^5_{0,5,\Gamma} \le C_4(||f||_{0,5/4,\Gamma})^{5/4}. \tag{31}$$

We now analyze the error of the approximation solution (u_h, p_h). Let $e_h = u - u_h$, $\varepsilon_h = p - p_h$. Then we have

(32)
$$a_0\left(\frac{de_h}{ds}, \frac{dv_h}{ds}\right) + a_1(u, v_h) - a_1(u_h, v_h) + b(\varepsilon_h, v_h) = 0, \qquad \forall\, v_h \in S_h,$$
$$-b(q_h, e_h) + a_0(q_h, \varepsilon_h) = 0, \qquad \forall\, q_h \in M_h.$$

By (32), a computation shows

$$\begin{aligned} a_0&\left(\frac{de_h}{ds}, \frac{de_h}{ds}\right) + a_1(u, e_h) - a_1(u_h, e_h) + a_0(\varepsilon_h, \varepsilon_h) \\ &= a_0\left(\frac{de_h}{ds}, \frac{d(u - v_h)}{ds}\right) + a_1(u, u - v_h) - a_1(u_h, u - v_h) \\ &\quad + a_0(p - q_h, \varepsilon_h) + b(\varepsilon_h, u - v_h) - b(p - q_h, \varepsilon_h), \\ &\qquad\qquad \forall\, v_h \in S_h, \quad q_h \in M_h. \end{aligned}$$

On the other hand, from Lemmas 2.1–2.4, we have

$$\begin{aligned} a_0&\left(\frac{de_h}{ds}, \frac{de_h}{ds}\right) + a_1(u, e_h) - a_1(u_h, e_h) + a_0(\varepsilon_h, \varepsilon_h) \\ &\ge \beta_0\left(\left\|\frac{de_h}{ds}\right\|^2_{-1/2,2,\Gamma} + ||\varepsilon_h||^2_{-1/2,2,\Gamma}\right) + \frac{\gamma}{4}\int_\Gamma (|u|^3 + |u_h|^3)e_h^2\, ds \end{aligned}$$

and

$$\left|a_0\left(\frac{de_h}{ds}, \frac{d(u-v_h)}{ds}\right)\right| \le M_0\left\|\frac{de_h}{ds}\right\|_{-1/2,2,\Gamma}\left\|\frac{d(u-v_h)}{ds}\right\|_{-1/2,2,\Gamma}$$

$$
\begin{aligned}
&\le \frac{\beta_0}{4}\left\|\frac{de_h}{ds}\right\|^2_{-1/2,2,\Gamma} \\
&\quad + \frac{2M_0^2}{\beta_0}\left\|\frac{d(u-v_h)}{ds}\right\|^2_{-1/2,2,\Gamma}, \\
|a_0(p-q,\varepsilon_h)| &\le \frac{\beta_0}{4}||\varepsilon_h||^2_{-1/2,2,\Gamma} \\
&\quad + \frac{2M_0^2}{\beta_0}||p-q_h||^2_{-1/2,2,\Gamma}, \\
|a_1(u,u-v_h) - a_1(u_h,u-v_h)| &\le 4\gamma\int_\Gamma(|u|^3+|u_h|^3)e_h||u-v_h|\,ds \\
&\le \frac{\gamma}{8}\int_\Gamma(|u|^3+|u_h|^3)e_h^2\,ds \\
&\quad + 32\gamma\int_\Gamma(|u|^3+|u_h|^3)|u-v_h|^2\,ds, \\
|b(\varepsilon_h,u-v_h)| &\le M_2||\varepsilon_h||_{-1/2,2,\Gamma}\left\|\frac{d(u-v_h)}{ds}\right\|_{-1/2,2,\Gamma} \\
&\le \frac{\beta_0}{4}||\varepsilon_h||^2_{-1/2,2,\Gamma} \\
&\quad + \frac{2M_2^2}{\beta_0}\left\|\frac{d(u-v_h)}{ds}\right\|^2_{-1/2,2,\Gamma}, \\
|b(p-q_h,e_h)| &\le M_2||p-q_h||_{-1/2,2,\Gamma}\left\|\frac{d(e_h)}{ds}\right\|_{-1/2,2,\Gamma} \\
&\le \frac{\beta_0}{4}\left\|\frac{de_h}{ds}\right\|^2_{-1/2,2,\Gamma} \\
&\quad + \frac{2M_2^2}{\beta_0}||p-q_h||^2_{-1/2,2,\Gamma}.
\end{aligned}
$$

Thus, we obtain

$$\tag{33} \frac{\beta_0}{2}\left(\left\|\frac{de_h}{ds}\right\|^2_{-1/2,2,\Gamma} + ||\varepsilon_h||^2_{-1/2,2,\Gamma}\right) + \frac{\gamma}{8}\int_\Gamma (|u|^3 + |u_h|^3)e_h^2\, ds \\ \leq 2\left(\frac{M_0^2 + M_2^2}{\beta_0}\right)\left(\left\|\frac{d(u-v_h)}{ds}\right\|^2_{-1/2,2,\Gamma} + ||p-q_h||^2_{-1/2,2,\Gamma}\right) \\ + 32\gamma\int_\Gamma (|u|^3 + |u_h|^3)|u-v_h|^2\, ds.$$

Combining the inequalities (33), (30) and (31), we have the following error estimate.

Theorem 3.2. *Suppose $f \in L_{5/4}(\Gamma)$, let (u,p) and (u_h, p_h) be the solutions of* (12) *and* (29)*, respectively, and let $e_h = u - u_h$, $\varepsilon_h = p - p_h$. Then*

$$\tag{34} \left\|\frac{de_h}{ds}\right\|^2_{-1/2,2,\Gamma} + ||e_h||^2_{-1/2,2,\Gamma} + \int_\Gamma (|u|^3 + |u_h|^3)e_h^2\, ds \\ \leq C\left\{\inf_{v_h\in S_h}\left[\left\|\frac{d(u-v_h)}{ds}\right\|^2_{-1/2,2,\Gamma} + (||f||_{0,5/4,\Gamma})^{3/4}||u-v_h||^2_{0,5,\Gamma}\right] \right. \\ \left. + \inf_{q_h\in M_h}||p-q_h||^2_{-1/2,2,\Gamma}\right\},$$

where C is a constant independent of h.

In applications the subspaces S_h and M_h could be constructed as follows. Assume that the boundary Γ of Ω is represented as

$$x_1 = x_1(s), \qquad x_2 = x_2(s), \qquad 0 \leq s \leq L,$$

and

$$x_j(0) = x_j(L), \qquad j = 1, 2.$$

Then Γ is divided into segments $\{T\}$ by the points $(x_1(s_i), x_2(s_i))$, $i = 1, 2, \ldots, N$ with $s_1 = 0$, $s_{N+1} = L$. Let

$$h = \max_{1\leq i\leq N}|s_{i+1} - s_i|.$$

This partition of Γ is denoted by J_h. Let

$$S_h = \{v_h \in C^0(\Gamma), v_h|_T \text{ is a linear function of parameter } s,\ \forall T \in J_h\}, \tag{35}$$

$$M_h = \{\mu_h|_T \text{ is a constant}, \forall T \in J_h \text{ and } \int_\Gamma \mu_h \, ds = 0\}. \tag{36}$$

The subspaces S_h and M_h are two regular finite element spaces in the sense of Babuska and Aziz [**2**] that satisfy the following approximation property:

$$\begin{aligned} \inf_{v_h \in S_h} ||u - v_h||^2_{1/2,2,\Gamma} &\le Ch^2 ||u||^2_{3/2,2,\Gamma}, \\ \inf_{q_h \in M_h} ||p - q_h||^2_{-1/2,2,\Gamma} &\le Ch^2 ||p||^2_{1/2,2,\Gamma}. \end{aligned} \tag{37}$$

From the imbedding theorem, we have

$$||v||^2_{0,5,\Gamma} \le C||v||^2_{1/2,2,\Gamma}, \qquad \forall\, v \in H^{1/2}(\Gamma). \tag{38}$$

Furthermore, we obtain

$$\inf_{v_h \in S_h} ||u - v_h||^2_{0,5,\Gamma} \le Ch^2 ||u||^2_{3/2,2,\Gamma}. \tag{39}$$

Combining (34) and (37)–(39), we have

Theorem 3.3. *Suppose that the solution* (u, p) *of* (12) *satisfies* $u \in H^{3/2}(\Gamma)$, $p \in H_*^{-1/2}(\Gamma) \cap H^{1/2}(\Gamma)$, *and the subspaces* S_h *and* M_h *are given by* (35) *and* (36). *Then the following error estimate holds*

$$\begin{aligned} \left\| \frac{d(u - u_h)}{ds} \right\|^2_{-1/2,2,\Gamma} &+ ||p - p_h||^2_{-1/2,2,\Gamma} + \int_\Gamma (|u|^3 + |u_h|^3)(u - u_h)^2 \, ds \\ &\le C\{[1 + (||f||_{0,5/4,\Gamma})^{3/4}]||u||^2_{3/2,2,\Gamma} + ||p||^2_{1/2,2,\Gamma}\}h^2. \end{aligned}$$

4. Numerical examples. Suppose that the domain $\Omega = \{|x| < 1\}$ with boundary Γ, which is the unit circle. Let (r, θ) denote the polar coordinates in the plane, namely $x_1 = r\cos\theta$, $x_2 = r\sin\theta$. Then the boundary Γ is given by

$$x_1 = \cos\theta, \qquad x_2 = \sin\theta, \qquad 0 \le \theta \le 2\pi.$$

Let $h = 2\pi/N$ and $\theta_i = (i-1)h$, $i = 1, 2, \dots, N$, where N is an integer. In each of the following three examples, we take $N = 4, 8, 16$, then get the numerical solutions u_4, u_8, u_{16} of the problem (29), correspondingly, by Newton's method. In every example we select zero initial data for computing u_4 and we select $\{u_4(\theta_i), i = 1, 2, \dots, 8\}$ with $\theta_i = (i-1)(\pi/8)$ as the initial data for computing u_8. Similarly, we select initial data from u_8 for computing u_{16}.

Example 1. In this example we select $\gamma = 1$ and $f(x) = 2$. Then we know that the exact solution of (1)–(2) is

$$u(x) = 2^{1/4} = 1.189207\dots.$$

Example 2. We take $\gamma = 1$, $f(x)|_\Gamma = (1 + |\cos\theta|^3)\cos\theta$. In this case it is straightforward to check that the exact solution of the problem (1)–(2) is

$$u(x) = x_1, \qquad u(x)|_\Gamma = \cos\theta.$$

Example 3. We take $\gamma = 1$, $f(x)|_\Gamma = e^{\cos\theta}\cos(\theta + \sin\theta) + [e^{\cos\theta}\cos(\sin\theta)]^4$. Then the exact solution of problem (1)–(2) is

$$u(x) = e^{x_1}\cos(x_2).$$

The numerical results are shown in Tables 1, 2 and 3, in which k denotes the number of iterates computed in Newton's method, and $E_N = \max_{1\le i\le N} |u|_\Gamma(\theta_i) - u_N(\theta_i)|$ for $N = 4$, 8 and 16.

TABLE 1. Example 1.

N	4	8	16
k	58	1	1
E_N	0.00000	0.00000	0.00000

TABLE 2. Example 2.

N	4	8	16
k	10	4	3
E_N	0.06900	0.02848	0.00671

TABLE 3. Example 3.

N	4	8	16
k	68	6	4
E_N	1.24939	0.19676	0.03617

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