JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 6, Number 4, Fall 1994

FULLY-DISCRETE COLLOCATION METHODS FOR AN INTEGRAL EQUATION OF THE FIRST KIND

WILLIAM MCLEAN

ABSTRACT. Using a model boundary integral equation of the first kind, we study some very simple numerical integration schemes for implementing spline collocation methods. The logarithmic singularity in the kernel is handled by combining special correction terms with standard composite integration rules of Gauss or Lobatto type. We prove that the stability and asymptotic convergence properties of the collocation method are maintained despite the quadrature errors. Numerical experiments confirm the error analysis.

1. Introduction. Consider the logarithmic-kernel integral equation of the first kind,

$$\int_\Gamma U(Y) \log \frac{\omega}{|X-Y|}\, ds_Y = F(X) \quad \text{for } X \in \Gamma. \tag{1.1}$$

Here Γ is a smooth, closed curve in the plane, $|X-Y|$ is the Euclidean distance between the points X and Y, and ds_Y is the element of arc length at Y. (The role of the parameter ω is explained below.) A standard numerical technique for solving boundary integral equations such as (1.1) is the collocation method. In this paper we use the theory in [**8**] to design some new and very simple numerical integration techniques for handling the integrals that define the entries of the collocation matrix. The resulting fully-discrete methods exhibit the same rates of convergence as would be achieved using the collocation method with exact integration.

Symm's equation arises in boundary integral reformulations of the Dirichlet problem for the Laplace equation in two dimensions, see [**7**]. Other second-order elliptic partial differential equations lead to similar integral equations of the first kind, with kernels that have the same

Received by the editors on August 12, 1994.
AMS (1991) *Mathematics Subject Classifications.* 65N38, 65D30, 65N12.
The author was supported by the Australian Research Council and, while a visitor at the University of Stuttgart, by the Alexander von Humboldt Foundation. Programming assistance was provided by David Dowsett.

qualitative behavior, i.e., a logarithmic singularity when $X = Y$. This singular behavior makes it difficult to use numerical quadratures to evaluate accurately those entries of the collocation matrix that are on or near the diagonal.

Equation (1.1) has a unique solution U for every F, provided

$$\omega \neq \operatorname{cap}(\Gamma), \tag{1.2}$$

where $\operatorname{cap}(\Gamma)$ denotes the logarithmic capacity (or transfinite diameter) of Γ; see [**6**, Chapter 16] or [**15**]. Generally we are free to choose any value for ω, and it is always the case that $\operatorname{cap}(\Gamma)$ is smaller than the (ordinary Euclidean) diameter of Γ. Thus, if the exact value of $\operatorname{cap}(\Gamma)$ is not known, then a simple way of ensuring that (1.2) holds is to choose ω larger than the diameter of Γ. Henceforth, we shall assume that (1.2) is satisfied.

We introduce a one-periodic parametric representation $\gamma : \mathbf{R} \to \Gamma$ and put

$$u(y) = U[\gamma(y)]|\gamma'(y)| \quad \text{and} \quad f(x) = F[\gamma(x)],$$

so as to recast (1.1) as a one-periodic integral equation on the real line,

$$Lu = f, \tag{1.3}$$

where the integral operator L is defined by

$$Lu(x) = \int_0^1 u(y) \log \frac{\omega}{|\gamma(x) - \gamma(y)|}\, dy \quad \text{for } x \in \mathbf{R}. \tag{1.4}$$

Next we choose a step-size $h = 1/N$ and define a uniform mesh,

$$t_j = jh. \tag{1.5}$$

For a fixed integer $r \geq 1$, let S_h denote the space of one-periodic smoothest splines of order r with breakpoints t_j. In other words, a one-periodic function v belongs to S_h if and only if v is a piecewise polynomial of degree at most $r - 1$ having, if $r \geq 2$, continuous derivatives up to and including order $r - 2$. We choose $\varepsilon \in [0, 1)$ and define the collocation points

$$x_j = t_j + \varepsilon h. \tag{1.6}$$

In the standard collocation method (i.e., with exact integration) the numerical solution $u_h \in S_h$ satisfies

$$L u_h(x_j) = f(x_j) \quad \text{for } 0 \le j \le N-1. \tag{1.7}$$

One way of computing u_h is to take a one-periodic B-spline basis $\{B_0, \dots, B_{N-1}\}$ for S_h, and to substitute an expansion of the form

$$u_h(x) = \sum_{k=0}^{N-1} u_k B_k(x)$$

into (1.7), obtaining an $N \times N$ linear system

$$\sum_{k=0}^{N-1} a_{jk} u_k = f(x_j) \quad \text{for } 0 \le j \le N-1.$$

The coefficients in this linear system are given by

$$a_{jk} = L B_k(x_j) = \int_0^1 B_k(y) \log \frac{\omega}{|\gamma(x_j) - \gamma(y)|}\, dy. \tag{1.8}$$

When the a_{jk} are evaluated using some kind of quadrature formula, the numerical solution u_h no longer satisfies (1.7), but instead

$$L_h u_h(x_j) = f(x_j) \quad \text{for } 0 \le j \le N-1, \tag{1.9}$$

where L_h approximates L. In this paper we shall study fully-discrete methods for which L_h has the form

$$L_h u(x_j) = h \sum_{p=1}^{P} \kappa_p h^{n_p} u^{(n_p)}(x_{jp}) + \sum_{k=1}^{N} h \sum_{q=1}^{Q} w_q u(y_{kq}) \log \frac{\omega}{|X_j - Y_{kq}|}. \tag{1.10}$$

Here

$$X_j = \gamma(x_j), \qquad X_{jp} = \gamma(x_{jp}), \qquad Y_{kq} = \gamma(y_{kq}),$$

where

$$x_{jp} = t_j + \xi_p h \quad \text{and} \quad y_{kq} = t_k + \eta_q h.$$

We will choose w_q and η_q to be the weights and integration points of an appropriate quadrature rule, so that

$$\int_0^1 g(\eta)\,d\eta \approx \sum_{p=1}^{P} w_p g(\eta_p). \tag{1.11}$$

The double sum in (1.10) is then just the result of applying the corresponding composite quadrature rule to the integral in the definition (1.4) of $Lu(x)$, with $x = x_j$. We think of the first sum in (1.10) as a correction term that compensates for the logarithmic singularity in the integrand.

Table 1 lists six different methods of the form described above. The number ρ is the order of accuracy of the method, as defined in Section 3. In practical terms, for any C^ρ function G we shall see that

$$\int_\Gamma G(Y)U(Y)\,ds_Y = \sum_{k=0}^{N-1} h \sum_{q=1}^{Q} w_q G(Y_{kq}) u_h(y_{kq}) + O(h^\rho). \tag{1.12}$$

In particular, by taking $G(Y) = \log(\omega/|X - Y|)$ for $X \notin \Gamma$, we obtain an $O(h^\rho)$ approximation to the single layer potential of U, away from Γ. As an illustrative example, consider the first and simplest method in Table 1, using piecewise-constant, midpoint collocation $(r = 1, \varepsilon = 1/2)$ with the two-point Gauss rule and a one-point correction term $(Q = 2, P = 1)$. In this case the B-spline B_k is just the (periodized) characteristic polynomial of the k-th subinterval, and the entries of the discrete collocation matrix are given by

$$a_{jk} = L_h B_k(x_j) = \left(\kappa_1 \delta_{jk} + \frac{1}{2}\sum_{q=1}^{Q} \log\frac{\omega}{|X_j - Y_{kq}|}\right)h, \tag{1.13}$$

where $\delta_{jk} = 1$ if $j = k$ and 0 otherwise. For the particular value of κ_1 listed in the table, this method has order $\rho = 3$, the same as would be achieved using exact integration as in (1.8). However, if κ_1 takes any other value, in particular, if the correction term is omitted altogether, then the order of the fully-discrete method is only $\rho = 1$. A related result (Lemma 5.2) is that if the diagonal entries a_{jj} are computed *exactly* but the off-diagonal entries a_{jk} for $j \neq k$ are evaluated using

TABLE 1. Discrete collocation methods.

r	ε	ρ	n_p	ξ_p	κ_p	η_p, w_p
1	1/2	3	0	0.5	0.45437 86189 24146	2-point Gauss
2	0	3	0	0.0	0.20894 80843 10412	2-point Gauss
3	1/2	5	0	-0.5	-0.00288 62720 33753	4-point Lobatto
			0	0.5	0.33493 64898 26654	
			0	1.5	-0.00288 62720 33753	
3	1/2	5	0	0.5	0.32916 39457 59148	4-point Lobatto
			2	0.5	-0.00288 62720 33753	
4	0	5	0	-1.0	0.00024 39406 66840	3-point Gauss
			0	0.0	0.10418 00739 24350	
			0	1.0	0.00024 39406 66840	
4	0	5	0	0.0	0.10466 79552 58029	3-point Gauss
			2	0.0	0.00024 39406 66840	

the two-point Gauss rule, then the order is also only $\rho = 1$. Thus, quadrature errors arising from the *near*-diagonal entries degrade the asymptotic rate of convergence as $h \to 0$. (However, Lemma 5.2 also shows that in practice this degradation should not be apparent until h is quite small.) Thus, the higher convergence rate achieved by using (1.13), with only the diagonal entries involving κ_1, is a little surprising.

The conventional approach to studying the effect of numerical integration in the boundary element method is based on a perturbation analysis of the coefficient matrix, and typically involves estimating the quadrature error for each entry; see, e.g., Wendland [**16**, Theorem 3.4]. Our approach is quite different, being based on the theory developed in [**8**], and is thus related to the *qualocation* method, see Chandler and Sloan [**5**] and the survey paper [**14**], and also to the fully-discrete Galerkin methods in [**10**] and [**11**]. There are many other fully-discrete schemes for the log-kernel equation (1.1), such as the one studied in a recent paper of Bialecki and Yan [**3**]. These schemes, however, have no direct interpretation as full discretizations of the standard spline-collocation method.

The paper is organized as follows. Section 2 gathers together some preliminary matter needed for applying the results in [**8**]. In Section 3 we describe how the stability and order-of-accuracy properties of the collocation method (1.7) and its fully-discrete version (1.9) hinge upon certain conditions involving ε, r and the parameters that define the quadrature formula and the correction term. Next, in Section 4, we derive the methods in Table 1 and show that they are stable and achieve the order of accuracy ρ shown. This result is stated as Theorem 4.4. In Section 5 we look at a related method that uses exact integration on subintervals where the integrand is singular or near-singular, combined with a quadrature formula (1.11) on the remaining subintervals. Finally, Section 6 presents the results of some simple numerical experiments that confirm the theoretical analysis in Sections 3–5.

2. Fourier analysis of the collocation method. We begin this section by showing that the integral operator L defined in (1.4) fits into the framework of the theory in [**8**]. Denote the kernel of L by

$$K(x,y) = \log \frac{\omega}{|\gamma(x) - \gamma(y)|},$$

and write

$$K(x,y) = K_A(x-y) + K_B(x,y), \tag{2.1}$$

where

$$K_A(x-y) = 1 + \log \frac{1}{|2\sin \pi(x-y)|}$$

and

$$K_B(x,y) = \begin{cases} -1 + \log \frac{|2\omega \sin \pi(x-y)|}{|\gamma(x)-\gamma(y)|}, & \text{if } x-y \text{ is not an integer,} \\ -1 + \log \frac{2\omega\pi}{|\gamma'(x)|}, & \text{if } x-y \text{ is an integer.} \end{cases}$$

The splitting (2.1) means that

$$L = A + B, \tag{2.2}$$

where A is the one-periodic, translation-invariant operator

$$Au(x) = \int_0^1 K_A(x-y)u(y)\,dy,$$

and B is the one-periodic smoothing operator

$$Bu(x) = \int_0^1 K_B(x,y)u(y)\,dy.$$

Define a function $\sigma : \mathbf{R} \to \mathbf{R}$ by

$$\sigma(0) = 1 \quad \text{and} \quad \sigma(y) = 1/(2|y|) \quad \text{for } y \neq 0, \tag{2.3}$$

and denote the complex Fourier coefficients of u by

$$\hat{u}(m) = \int_0^1 u(x)\exp(-i2\pi mx)\,dx \quad \text{for } m \in \mathbf{Z}.$$

It follows from the Fourier expansion

$$\log\frac{1}{|2\sin\pi x|} = \sum_{m=1}^{\infty}\frac{1}{m}\cos 2\pi mx = \sum_{m\neq 0}\frac{1}{2|m|}\exp(i2\pi mx) \tag{2.4}$$

that $\sigma(m) = \widehat{K}_A(m)$, and so σ is the global periodic symbol of the operator A, i.e.,

$$Au(x) = \sum_{m=-\infty}^{\infty}\sigma(m)\hat{u}(m)\exp(i2\pi mx).$$

Thus, the symbol of A is even and is positive-homogeneous of degree -1, so $\beta = -1$ in the notation of [**8**], and thus $L : H^s \to H^{s+1}$ is a bounded and invertible linear operator for all $s \in \mathbf{R}$, where H^s denotes the usual one-periodic Sobolev space of order s.

Next, we recall some more facts from [**8**]. Define

$$\Delta_r(\xi,y) = y^r\sum_{m\neq 0}\frac{1}{(m+y)^r}\exp(i2\pi m\xi) \quad \text{for } \xi \in \mathbf{R} \text{ and } |y| \leq 1/2,$$

and

$$\Omega_k(\xi,y) = \frac{y^k}{\sigma(y)}\sum_{m\neq 0}\frac{\sigma(m+y)}{(m+y)^k}\exp(i2\pi m\xi), \tag{2.5}$$

for $\xi \in \mathbf{R}$, $|y| \leq 1/2$ and k is an integer. Let

$$\Lambda_h = \{\mu \in \mathbf{Z} : -N/2 < \mu \leq N/2\},$$

define $\psi_{r,0}(x) = 1$ and

$$\psi_{r,\mu}(x) = \sum_{m \equiv \mu \bmod N} \left(\frac{\mu}{m}\right)^r \exp(i2\pi mx) \quad \text{for } \mu \in \Lambda_h,$$

and put

$$\phi_m(x) = \exp(i2\pi mx) \quad \text{for } m \in \mathbf{Z}.$$

It turns out that $\{\psi_{r,\mu} : \mu \in \Lambda_h\}$ is a basis for S_h, and that we can think of $\psi_{r,\mu}$ as a spline substitute for ϕ_μ. In fact,

$$\psi_{r,\mu}(x) = \phi_\mu(x)[1 + \Delta_r(Nx, \mu h)] \quad \text{for } \mu \in \Lambda_h, \tag{2.6}$$

so $\Delta_r(Nx, \mu h)$ is the relative error in the approximation $\phi_\mu(x) \approx \psi_{r,\mu}(x)$.

Using the collocation points (1.6), we define a discrete inner product

$$\langle f, v\rangle_h = \sum_{j=1}^{N} h f(x_j)\overline{v(x_j)},$$

that approximates the L_2-inner product

$$\langle f, v\rangle = \int_0^1 f(x)\overline{v(x)}\, dx.$$

We then define three discrete sesquilinear forms,

$$l_h(u, v) = \langle Lu, v\rangle_h, \qquad a_h(u, v) = \langle Au, v\rangle_h, \qquad b_h(u, v) = \langle Bu, v\rangle_h. \tag{2.7}$$

In view of (2.2), these satisfy $l_h(u, v) = a_h(u, v) + b_h(u, v)$, and the collocation method (1.7) is equivalent to the discrete Petrov-Galerkin method

$$l_h(u_h, \phi_\mu) = \langle f, \phi_\mu\rangle_h \quad \text{for } \mu \in \Lambda_h. \tag{2.8}$$

It is shown in [**8**, subsection 3.3] that for $\mu, \nu \in \Lambda_h$ and for $k \in \mathbf{Z}$,

$$a_h(\psi_\mu, \phi_\nu) = \begin{cases} d_h, & \text{if } \mu = \nu = 0, \\ D(\mu h)\sigma(\mu), & \text{if } \mu = \nu \neq 0, \\ 0, & \text{if } \mu \neq \nu, \end{cases} \tag{2.9}$$

and

$$\langle \phi_{\mu+kN}, \phi_\nu \rangle_h = \begin{cases} D_k, & \text{if } \mu = \nu, \\ 0, & \text{if } \mu \neq \nu, \end{cases} \tag{2.10}$$

where

$$d_h = 1, \qquad D(y) = 1 + \Omega_r(\varepsilon, y), \qquad D_k = \exp(i2\pi k\varepsilon). \tag{2.11}$$

As we shall explain in Section 3, the behavior of the function D determines the stability and convergence properties of the collocation method.

Likewise, the fully-discrete collocation method (1.9) is equivalent to a discrete Petrov-Galerkin method of the form (2.8), but instead of (2.7),

$$l_h(u,v) = \langle L_h u, v\rangle_h, \qquad a_h(u,v) = \langle A_h u, v\rangle_h, \qquad b_h(u,v) = \langle B_h u, v\rangle_h.$$

Here $L_h = A_h + B_h$ is defined as in (1.10), with

$$A_h u(x_j) = h \sum_{p=1}^{P} \kappa_p h^{n_p} u^{(n_p)}(x_{jp}) + \sum_{k=1}^{N} h \sum_{q=1}^{Q} w_q K_A(x_j - y_{kq}) u(y_{kq})$$

and

$$B_h u(x) = \sum_{k=1}^{N} h \sum_{q=1}^{Q} w_q K_B(x, y_{kq}) u(y_{kq}).$$

The relations (2.9) and (2.10) are again valid, but now

$$\begin{aligned} d_h &= 1 + h\Bigg(\sum_{n_p=0} \kappa_p + \sum_{q=1}^{Q} w_q [K_A(\varepsilon - \eta_q) - 1] \Bigg), \\ D(y) &= 2|y| \sum_{p=1}^{P} \kappa_p T_1(r, \varepsilon, n_p, \xi_p, y) + \sum_{q=1}^{Q} w_q T_2(r, \varepsilon, \eta_q, y), \end{aligned} \tag{2.12}$$

where

$$
\begin{aligned}
T_1(r, \varepsilon, n, \xi, y) &= (i2\pi y)^n [1 + \Delta_{r-n}(\xi, y)] \exp[i2\pi(\xi - \varepsilon)y], \\
T_2(r, \varepsilon, \eta, y) &= [1 + \Delta_r(\eta, y)][1 + \Omega_0(\varepsilon - \eta, y)].
\end{aligned} \tag{2.13}
$$

These formulae follow from (2.6) and [**8**, subsections 3.6 and 3.7].

To conclude this section, we collect together some more notation and cite some results that will be needed later. Following [**5**], let

$$
\begin{aligned}
F_\alpha^+(\xi, y) = G_\alpha^+(\xi, y) + iH_\alpha^+(\xi, y) &= \sum_{m \neq 0} \frac{1}{|m + y|^\alpha} \exp(i2\pi m\xi), \\
F_\alpha^-(\xi, y) = G_\alpha^-(\xi, y) + iH_\alpha^-(\xi, y) &= \sum_{m \neq 0} \frac{\operatorname{sign}(m)}{|m + y|^\alpha} \exp(i2\pi m\xi),
\end{aligned} \tag{2.14}
$$

and note that

$$
\Delta_r(\xi, y) = \begin{cases} y^r F_r^-(\xi, y), & \text{if } r \text{ is odd}, \\ y^r F_r^+(\xi, y), & \text{if } r \text{ is even}, \end{cases} \tag{2.15}
$$

and

$$
\Omega_k(\xi, y) = \begin{cases} \operatorname{sign}(y)|y|^{k+1} F_{k+1}^-(\xi, y), & \text{if } k \text{ is odd}, \\ |y|^{k+1} F_{k+1}^+(\xi, y), & \text{if } k \text{ is even}. \end{cases} \tag{2.16}
$$

We shall also make use of the trigonometric series

$$
G_\alpha(\xi) = 2 \sum_{m=1}^{\infty} \frac{1}{m^\alpha} \cos(2\pi m\xi),
$$

$$
H_\alpha(\xi) = 2 \sum_{m=1}^{\infty} \frac{1}{m^\alpha} \sin(2\pi m\xi),
$$

that arise in the Taylor expansions of $G_\alpha^\pm(\xi, y)$ and $H_\alpha^\pm(\xi, y)$ about $y = 0$. Brown et al. [**4**] discuss some relevant properties of $G_\alpha^\pm$ and $H_\alpha^\pm$, including the following two lemmas (see also the Appendix of [**5**], but note the erratum).

Lemma 2.1. *Let $\alpha > 0$.*

(i) *The functions G_α and H_α are infinitely differentiable on the open interval $(0,1)$, and satisfy $G_\alpha(1-\xi) = G_\alpha(\xi)$ and $H_\alpha(1-\xi) = -H_\alpha(\xi)$ for $0 < \xi < 1$.*

(ii) *The function G_α is strictly decreasing on the interval $(0,1/2)$ where it has a unique zero ξ_α^* that is a monotonically increasing function of α.*

(iii) *The function H_α is strictly positive on $(0,1/2)$.*

Lemma 2.2. *For $\alpha > 0$, $0 \le \xi \le 1$ and $0 \le y \le 1/2$,*

(i) $1 + y^\alpha G_\alpha^+(\xi, y) \ge 0$ *with equality if and only if $\xi = y = 1/2$;*

(ii) $1 + y^\alpha G_\alpha^-(\xi, y) \ge 0$ *with equality if and only if $\xi = 0$ or 1, and $y = 1/2$;*

(iii) $H_\alpha^+(\xi, y) \le 0$ *if $0 \le \xi \le 1/2$ and $H_\alpha^+(\xi, y) \ge 0$ if $1/2 \le \xi \le 1$;*

(iv) $H_\alpha^-(\xi, y) \ge 0$ *if $0 \le \xi \le 1/2$ and $H_\alpha^-(\xi, y) \le 0$ if $1/2 \le \xi \le 1$.*

Notice that, by (2.4),

$$K_A(\xi) = 1 + (1/2)G_1(\xi). \tag{2.17}$$

Also we point out that for $k \ge 1$ the restrictions of H_{2k-1} and G_{2k} to the unit interval are polynomials of degree $2k-1$ and $2k$, respectively. In fact, apart from constant factors they are just Bernoulli polynomials; see Abramowitz and Stegun [**1**, p. 805].

3. Stability and order of accuracy. The error analysis in [**8**] involves three stability conditions and three order-of-accuracy conditions, labelled S1–S3 and O1–O3, respectively. Conditions S1, S2, O1 and O2 relate to the approximations $a_h(u,v) \approx \langle Au, v\rangle$ and $\langle f, v\rangle_h \approx \langle f, v\rangle$, whereas S3 and O3 relate to the approximation $b_h(u,v) \approx \langle Bu, v\rangle$. For the methods considered in this paper we have $D_k = \exp(i2\pi k\varepsilon)$ and consequently condition O2 is satisfied, and O1 implies S1. Thus, of the first group of four conditions, it suffices to verify only two:

S2. There is a positive constant c such that

$$|D(y)| \ge c \quad \text{for } 0 < |y| \le 1/2.$$

O1. There is a number $\rho > 0$ such that

$$d_h = 1 + O(h^\rho) \quad \text{as } h \to 0^+,$$
$$D(y) = 1 + O(|y|^\rho) \quad \text{as } y \to 0.$$

The number ρ appearing in condition O1 is called the *order* of the numerical method because (see below) we cannot do better than $O(h^\rho)$ convergence in any Sobolev norm. (Strictly speaking, the order is the *largest* ρ for which O1 holds.) The second group of conditions, S3 and O3, involve Sobolev indices s and t. Let M denote the order of precision of the quadrature rule (1.11) used to define L_h, i.e., suppose that the rule integrates any polynomial of degree $M - 1$ exactly. In the case of the collocation method (1.7) with exact integration, we formally define $M = \infty$. By [**8**, Theorems 6.3 and 6.5], conditions S3 and O3 hold for

$$r - M - 1 < s < r - 1/2 \quad \text{and} \quad 0 \le t - s \le M.$$

We will say that the numerical method (1.7) or (1.9) is *stable and of order* ρ if conditions S2 and O1 hold, and if

$$M \ge \rho. \tag{3.1}$$

This restriction on M guarantees that conditions S3 and O3 hold for s and t satisfying (3.2) below. Therefore, [**8**, Theorem 5.2] yields the following error estimates where, as usual, $||\cdot||_s$ denotes the norm in the Sobolev space H^s.

Theorem 3.1. *Consider the collocation method* (1.7) *or its fully-discrete variant* (1.9) *applied to the logarithmic-kernel integral equation* (1.3)*, and let s and t be real numbers satisfying*

$$r - \rho \le s \le t \le r, \qquad s < r - 1/2, \qquad -1/2 < t. \tag{3.2}$$

If the numerical method is stable and of order ρ, and if the exact solution u belongs to $H^{t+\max(-1-s,0)}$, then for all h sufficiently small there exists a unique numerical solution u_h, and

$$||u_h - u||_s \le ch^{t-s}||u||_{t+\max(-1-s,0)}.$$

In particular, we can easily recover the results of Arnold and Wendland [**2**] and Schmidt [**13**] for the standard case, in which the entries of the collocation matrix are computed exactly. Recall the definition (1.6) of the collocation points, and remember that we assume $0 \le \varepsilon < 1$.

Theorem 3.2. *The collocation method* (1.7) *applied to the logarithmic-kernel integral equation* (1.3) *is stable and of order* (*at least*) $r+1$ *if*

$$r \text{ is odd and } \varepsilon \neq 0,$$

or if

$$r \text{ is even and } \varepsilon \neq 1/2.$$

Proof. By (2.3), (2.5) and (2.11), we have

$$D(y) = 1 + O(|y|^{r+1}),$$

so O1 holds with $\rho = r+1$. Furthermore, since

$$\Omega_k(\xi, -y) = \overline{\Omega_k(\xi, y)}, \tag{3.3}$$

we have

$$D(-y) = \overline{D(y)},$$

and if $0 \le y \le 1/2$, then

$$\operatorname{Re} D(y) = \begin{cases} 1 + y^{r+1} G^-_{r+1}(\varepsilon, y), & \text{if } r \text{ is odd}, \\ 1 + y^{r+1} G^+_{r+1}(\varepsilon, y), & \text{if } r \text{ is even}. \end{cases}$$

By Lemma 2.2, condition S2 holds except if r is odd and $\varepsilon = 0$, or if r is even and $\varepsilon = 1/2$. $\square$

Saranen [**12**] observed that in some cases the order of accuracy is in fact greater than $r+1$. Indeed, by Taylor expansion,

$$\begin{aligned} D(y) = 1 + |y|^{r+1} & \\ \times & \begin{cases} iH_{r+1}(\varepsilon)\operatorname{sign}(y) - (r+1)G_{r+1}(\varepsilon)y + O(y^2), & \text{if } r \text{ is odd}, \\ G_{r+1}(\varepsilon) - i(r+1)H_{r+1}(\varepsilon)y + O(y^2), & \text{if } r \text{ is even}, \end{cases} \end{aligned}$$

so if

$$r \text{ is odd and } H_{r+1}(\varepsilon) = 0,$$

or if

$$r \text{ is even and } G_{r+1}(\varepsilon) = 0,$$

then $D(y) = 1 + O(|y|^{r+2})$, and thus by Lemma 2.1 the following is true.

Theorem 3.3. *The collocation method applied to the logarithmic kernel integral equation is stable and of order $r+2$ for the special choice*

$$\varepsilon = \begin{cases} 1/2, & \text{if } r \text{ is odd,} \\ \xi_{r+1}^* \text{ or } 1 - \xi_{r+1}^*, & \text{if } r \text{ is even.} \end{cases}$$

Before proceeding to the next section we show that the error estimate (1.12) holds, as claimed in the Introduction.

Theorem 3.4. *Assume that $r \le \rho \le M$ and $u \in H^{\rho-1}$. If the hypotheses of Theorem 3.1 hold, then*

$$\left| \int_0^1 g(y)u(y)\,dy - \sum_{k=0}^{N-1} h \sum_{q=1}^{Q} w_q g(y_{kq}) u_h(y_{kq}) \right| \le ch^\rho ||g||_\rho ||u||_{\rho-1}$$

for any $g \in H^\rho$.

Proof. Without loss of generality, we can assume that u and u_h are real-valued, then our task is to estimate $\langle g, u\rangle - \langle g, u_h\rangle_h$, where $\langle \cdot, \cdot \rangle_h$ is the discrete inner product based on the composite form of the quadrature rule (1.11). By Theorem 3.1,

$$\begin{aligned} |\langle g, u\rangle - \langle g, u_h\rangle| &= |\langle g, u - u_h\rangle| \\ &\le ||g||_{\rho-r} ||u - u_h||_{r-\rho} \\ &\le ch^\rho ||g||_\rho ||u||_{\rho-1}, \end{aligned}$$

and since $\rho \le M$ we can apply [**8**, Lemma 6.2 ii)] to obtain

$$|\langle g, u_h\rangle - \langle g, u_h\rangle_h| \le ch^\rho ||g||_\rho ||u_h||_0.$$

Taking $s = t = 0$ in the error estimate of Theorem 3.1, we see that $||u_h||_0 \leq c||u||_0$, and the result follows. $\square$

4. Derivation of the fully-discrete methods. Throughout this section we shall assume that the collocation points are given by

$$\varepsilon = \begin{cases} 1/2, & \text{if } r \text{ is odd}, \\ 0, & \text{if } r \text{ is even}. \end{cases} \tag{4.1}$$

This is the case for each of the methods in Table 1, and the next lemma will allow us to simplify their analysis. Recall the definitions of T_1 and T_2 given in (2.13).

Lemma 4.1. *Assume that the collocation points are chosen according to* (4.1).

(i) *If the correction term satisfies*

$$n_p \text{ is even}, \quad \kappa_{P-p+1} = \kappa_p$$

and

$$\xi_{P-p+1} = 2\varepsilon - \xi_p$$

for $1 \leq p \leq P$, *then the sum* $\sum_{p=1}^{P} \kappa_p T_1(r, \varepsilon, n_p, \xi_p, y)$ *is a real-valued, even function of* y.

(ii) *If the quadrature rule satisfies*

$$w_{Q-q+1} = w_q \quad \text{and} \quad \eta_{Q-q+1} = 1 - \eta_q \quad \text{for } 1 \leq q \leq Q,$$

then the sum $\sum_{q=1}^{Q} w_q T_2(r, \varepsilon, \eta_q, y)$ *is a real-valued, even function of* y.

Proof. For $\varepsilon = 0$ or $1/2$,

$$\Delta_r(2\varepsilon - \xi, y) = \overline{\Delta_r(\xi, y)} \quad \text{and} \quad \Omega_k(2\varepsilon - \eta, y) = \overline{\Omega_k(\eta, y)}$$

and thus, provided n is even,

$$T_1(r, \varepsilon, n, 2\varepsilon - \xi, y) = \overline{T_1(r, \varepsilon, n, \xi, y)}$$

and

$$T_2(r, \varepsilon, 1-\eta, y) = \overline{T_2(r, \varepsilon, \eta, y)}.$$

Therefore, both in part (i) and in part (ii) the symmetry condition implies that the sum is real. Also, it follows from the identities (3.3) and $\Delta_r(\xi, -y) = \overline{\Delta_r(\xi, y)}$ that

$$T_1(r, \varepsilon, n, \xi, -y) = \overline{T_1(r, \varepsilon, n, \xi, y)} \tag{4.2}$$

and

$$T_2(r, \varepsilon, \eta, -y) = \overline{T_2(r, \varepsilon, \eta, y)},$$

so in the symmetric case each sum is an even function of y. ◻

The next lemma will help in establishing stability.

Lemma 4.2. *Let* $0 \le \eta \le 1$.

(i) *If* r *is odd, then*

$$\min_{0 \le y \le 1/2} \operatorname{Re} T_2(r, 1/2, \eta, y) > 0 \quad \textit{for } 0 < \eta < 1,$$

but $T_2(r, 1/2, \eta, 1/2) = 0$ *for* $\eta = 0$ *or* 1.

(ii) *If* r *is even, then*

$$\min_{0 \le y \le 1/2} \operatorname{Re} T_2(r, 0, \eta, y) > 0 \quad \textit{for } \eta \ne 1/2,$$

but $\operatorname{Re} T_2(r, 0, 1/2, 1/2) = 0$.

Proof. If r is odd, then by (2.13), (2.15) and (2.16),

$$T_2(r, 1/2, \eta, y) = [1 + y^r F_r^-(\eta, y)][1 + |y| F_1^+(1/2 - \eta, y)],$$

so

$$\begin{aligned} \operatorname{Re} T_2(r, 1/2, \eta, y) = {} & [1 + y^r G_r^-(\eta, y)][1 + |y| G_1^+(1/2 - \eta, y)] \\ & - y^r |y| H_r^-(\eta, y) H_1^+(1/2 - \eta, y). \end{aligned} \tag{4.3}$$

Lemma 2.2 implies that for $0 \le y \le 1/2$,

$$[1 + y^r G_r^-(\eta, y)][1 + yG_1^+(1/2 - \eta, y)] \ge 0,$$

with equality if and only if $\eta = 0$ or 1, and $y = 1/2$. Moreover,

$$H_r^-(\eta, y)H_1^+(1/2 - \eta, y) \le 0,$$

so part (i) follows after noting that $1 + (1/2)^r F_r^-(0, 1/2) = 0$.

If r is even, then

$$T_2(r, 0, \eta, y) = [1 + y^r F_r^+(\eta, y)][1 + |y|F_1^+(-\eta, y)],$$

so, because G_1^+ and H_1^+ are one-periodic in their first arguments,

$$\begin{aligned} \operatorname{Re} T_2(r, 0, \eta, y) = {} & [1 + y^r G_r^+(\eta, y)][1 + |y|G_1^+(1 - \eta, y)] \\ & - y^r |y| H_r^+(\eta, y) H_1^+(1 - \eta, y). \end{aligned} \tag{4.4}$$

Part (ii) follows after using Lemma 2.2 and noting that $1+ (1/2)^r F_r^+(1/2, 1/2) = 0$. $\square$

The following lemma will help in determining the order ρ of the various fully-discrete methods. For a proof, see [**10**, Lemma 4.5] and [**5**, Lemma A.2].

Lemma 4.3. *As $y \to 0$,*

$$\sum_{q=1}^{Q} w_q \Delta(\eta_q, y) = O(y^{\max(r,M)}).$$

and

$$\operatorname{Re} \Delta_r(\eta, y)\Omega_0(\varepsilon - \eta, y) = \begin{cases} O(y^{r+2}), & \text{if } r \text{ is odd,} \\ O(y^{r+1}), & \text{if } r \text{ is even.} \end{cases}$$

We are now ready to prove the main results of the paper.

Theorem 4.4. *Each of the fully-discrete methods defined in Table 1 is stable and of order ρ, where*

$$\rho = \begin{cases} 3, & \text{for the piecewise-constant } (r=1) \\ & \text{and piecewise-linear } (r=2) \text{ cases,} \\ 5, & \text{for the piecewise-quadratic } (r=3) \\ & \text{and piecewise-cubic } (r=4) \text{ cases.} \end{cases}$$

Proof. In every case, the correction term and quadrature rule are symmetric in the sense of Lemma 4.1, so we see from (2.12) and (2.17) that

$$\begin{aligned} d_h &= 1 + h\biggl(\sum_{n_p=0} \kappa_p + \frac{1}{2}\sum_{q=1}^{Q} w_q G_1(\varepsilon - \eta_q)\biggr), \\ D(y) &= 2|y|\sum_{p=1}^{P} \kappa_p \operatorname{Re} T_1(r,\varepsilon,n_p,\xi_p,y) + \sum_{q=1}^{Q} w_q \operatorname{Re} T_2(r,\varepsilon,\eta_q,y). \end{aligned} \tag{4.5}$$

By (2.13) and Lemma 4.3,

$$\sum_{q=1}^{Q} w_q \operatorname{Re} T_2(r,\varepsilon,\eta_q,y) = 1 + \sum_{q=1}^{Q} w_q |y| G_1^+(\varepsilon - \eta_q, y) + O(|y|^\rho) \tag{4.6}$$

and using Taylor's theorem one finds that

$$G_1^+(\varepsilon - \eta, y) = G_1(\varepsilon - \eta) + G_3(\varepsilon - \eta)y^2 + G_5(\varepsilon - \eta)y^4 + \cdots \tag{4.7}$$

for $|y| < 1$. Note also that $D(-y) = D(y)$, so when verifying condition S2 it suffices to show that $D(y) \neq 0$ for $0 \leq y \leq 1/2$.

We shall deal with the six methods in the order in which they are listed in Table 1.

Method 1. In this case $r = 1$, $\varepsilon = 1/2$ and (1.11) is the two-point, Gauss-Legendre rule for the unit interval, i.e., $Q = 2$ and

$$\eta_1 = \frac{1}{2}\biggl(1 - \frac{1}{\sqrt{3}}\biggr), \qquad \eta_2 = \frac{1}{2}\biggl(1 + \frac{1}{\sqrt{3}}\biggr), \qquad w_1 = w_2 = \frac{1}{2}. \tag{4.8}$$

The order of precision is $M = 4$. Regarding the correction term we assume for the moment simply that $P = 1$ and $\xi_1 = 1/2$. Using Taylor expansions, we find that

$$\operatorname{Re} T_1(1, 1/2, 0, 1/2, y) = 1 + yG_1^-(1/2, y) = 1 + O(y^2), \tag{4.9}$$

so (4.5), (4.6) and (4.7) imply that

$$d_h = 1 + (\kappa_1 + e_1)h \quad \text{and} \quad D(y) = 1 + 2(\kappa_1 + e_1)|y| + O(y^3),$$

where

$$e_1 = \frac{1}{2}\sum_{q=1}^{2} w_q G_1\left(\frac{1}{2} - \eta_q\right) = \frac{1}{2}G_1\left(\frac{1}{2\sqrt{3}}\right) = -\log\left[2\sin\left(\frac{\pi}{2\sqrt{3}}\right)\right].$$

In Table 1 we have chosen $\kappa_1 = -e_1$ so that O1 holds with $\rho = 3$. By (4.9) and Lemma 2.2 we see that $\operatorname{Re} T_1(1, 1/2, 0, 1/2, y) \geq 0$ for $0 \leq y \leq 1/2$, so it follows from Lemma 4.2 that condition S2 holds. (Note that κ_1, w_1 and w_2 are all positive.) Since $M = 4$ and $\rho = 3$ satisfy (3.1), we have proved that the method is stable and of order 3.

Method 2. Now $r = 2$ and $\varepsilon = 0$, but we continue using the two-point Gauss rule (4.8). Assuming for the correction term that $P = 1$ and $\xi_1 = 0$, we find since $G_2(0) = \pi^2/3$ that

$$\operatorname{Re} T_1(2, 0, 0, 0, y) = 1 + y^2 G_2^+(0, y) = 1 + O(y^2), \tag{4.10}$$

so

$$d_h = 1 + (\kappa_1 + e_3)h$$

and

$$D(y) = 1 + 2(\kappa_1 + e_2)|y| + O(y^3),$$

where

$$e_2 = \frac{1}{2}\sum_{q=1}^{2} w_q G_1(-\eta_q) = \frac{1}{2}G_1\left(\frac{1}{2} - \frac{1}{2\sqrt{3}}\right) = -\log\left[2\cos\left(\frac{\pi}{2\sqrt{3}}\right)\right].$$

Putting $\kappa_1 = -e_2$ as in Table 1, we see that condition O1 holds with $\rho = 3$. By (4.10) and Lemma 2.2, we have $\operatorname{Re} T_1(2,0,0,0,y) \geq 0$ for $0 \leq y \leq 1/2$, so it follows from Lemma 4.2 that condition S2 holds. Since $M = 4$ and $\rho = 3$ satisfy (3.1), the method is stable and of order 3.

Method 3. We use $r = 3$, $\varepsilon = 1/2$ and the four-point Lobatto rule for the unit interval: $Q = 4$ and

$$\begin{gathered}\eta_1 = 0, \qquad \eta_2 = \frac{1}{2}\left(1 - \frac{1}{\sqrt{5}}\right), \qquad \eta_3 = \frac{1}{2}\left(1 + \frac{1}{\sqrt{5}}\right), \qquad \eta_4 = 1, \\ w_1 = w_4 = \frac{1}{12}, \qquad w_2 = w_3 = \frac{5}{12}.\end{gathered} \tag{4.11}$$

The order of precision is $M = 6$ and, in the correction term, $P = 3$ and $n_1 = n_2 = n_3 = 0$. We find that

$$\begin{aligned}\operatorname{Re} T_1(3,1/2,0,\xi,y) &= [1 + y^3 G_3^-(\xi,y)] \cos 2\pi(\xi - 1/2)y \\ &\qquad - y^3 H_3^-(\xi,y) \sin 2\pi(\xi - 1/2)y \\ &= 1 - 2\pi^2(\xi - 1/2)^2 y^2 + O(y^4)\end{aligned} \tag{4.12}$$

so

$$d_h = 1 + \left(e_3 + \sum_{p=1}^{3} \kappa_p\right) h$$

and

$$D(y) = 1+2\left(e_3+\sum_{p=1}^{3}\kappa_p\right)|y|+4\pi^2\left(e_4-\sum_{p=1}^{3}\kappa_p(\xi_p-1/2)^2\right)|y|^3+O(y^5),$$

where

$$\begin{aligned}e_3 &= \frac{1}{2}\sum_{q=1}^{4} w_q G_1\left(\frac{1}{2} - \eta_q\right) = \frac{1}{12}\left[G_1\left(\frac{1}{2}\right) + 5G_1\left(\frac{1}{2\sqrt{5}}\right)\right] \\ &= -0.329163945759148\end{aligned}$$

and, evaluating G_3 as in [**9**],

$$\begin{aligned}e_4 &= \frac{1}{4\pi^2}\sum_{q=1}^{4} w_q G_3\left(\frac{1}{2} - \eta_q\right) = \frac{1}{24\pi^2}\left[G_3\left(\frac{1}{2}\right) + 5G_3\left(\frac{1}{2\sqrt{5}}\right)\right] \\ &= -0.005772544067506.\end{aligned}$$

The correction term has the form

$$\xi_2 = 1/2, \qquad \xi_3 = 1 - \xi_1, \qquad \kappa_3 = \kappa_1, \tag{4.13}$$

so to satisfy condition O1 with $\rho = 5$ it suffices to choose ξ_1, κ_1 and κ_2 such that

$$2\kappa_1 + \kappa_2 = -e_3 \quad \text{and} \quad 2\kappa_1(\xi_1 - 1/2)^2 = e_4. \tag{4.14}$$

Here there are three unknowns but only two equations, so we have one degree of freedom. The method given in Table 1 arises by taking

$$\xi_1 = -1/2, \qquad \kappa_1 = e_4/2, \qquad \kappa_2 = -e_3 - e_4,$$

and in this case we find using (4.12) that

$$\sum_{p=1}^{3} \kappa_p \mathrm{Re}\, T_1(3, 1/2, 0, \xi_p, y) = [\kappa_2 + 2\kappa_1 \cos 2\pi y][1 + y^3 G_3^-(1/2, y)].$$

Since $\kappa_2 > 2|\kappa_1|$, we conclude with the help of Lemmas 2.2 and 4.2 that condition S2 holds. It follows that the method is stable, because $M = 6$ and $\rho = 5$ satisfy the condition (3.1).

Method 4. As in Method 3, $r = 3$, $\varepsilon = 1/2$ and (1.11) is the four-point Lobatto rule (4.11). However, this time, in the correction term we have $P = 2$, $\xi_1 = \xi_2 = 1/2$, $n_1 = 0$ and $n_2 = 2$. Since

$$\mathrm{Re}\, T_1(3, 1/2, 2, 1/2, y) = -4\pi^2 y^2 [1 + y G_1^-(1/2, y)] = -4\pi^2 y^2 + O(y^4),$$

we find that

$$d_h = 1 + (e_3 + \kappa_1)h$$

and

$$D(y) = 1 + 2(e_3 + \kappa_1)|y| + 4\pi^2(e_4 - 2\kappa_2)|y|^3 + O(y^5).$$

The method shown in Table 1 has

$$\kappa_1 = -e_3 \quad \text{and} \quad \kappa_2 = e_4/2,$$

so condition O1 holds with $\rho = 5$. Moreover, because $\kappa_1 > 0$ and $\kappa_2 < 0$, it follows by part (ii) of Lemma 2.2 that

$$\begin{aligned}\sum_{p=1}^{2} \kappa_p \mathrm{Re}\, T_1(3, 1/2, n_p, 1/2, y) = \kappa_1[1 + y^3 G_3^-(1/2, y)] \\ - 4\pi^2 \kappa_2 y^2 [1 + y G_1^-(1/2, y)] \geq 0\end{aligned}$$

for $0 \leq y \leq 1/2$. Thus, condition S2 holds and we conclude that the method is stable.

Method 5. We use $r = 4$, $\varepsilon = 0$ and the three-point Gauss-Legendre rule for the unit interval: $Q = 3$ and

$$\eta_1 = \frac{1}{2}\left(1 - \sqrt{\frac{3}{5}}\right), \qquad \eta_2 = \frac{1}{2}, \qquad \eta_3 = \frac{1}{2}\left(1 + \sqrt{\frac{3}{5}}\right),$$
$$w_1 = w_3 = \frac{5}{18}, \qquad w_2 = \frac{8}{18}.$$

The order of precision is $M = 6$, and in the correction term $P = 3$ and $n_1 = n_2 = n_3 = 0$. We find that

(4.15)
$$\begin{aligned}\mathrm{Re}\, T_1(4, 0, 0, \xi, y) &= [1 + y^4 G_4^+(\xi, y)] \cos 2\pi\xi y - y^4 H_4^+(\xi, y) \sin 2\pi\xi y \\ &= 1 - 2\pi^2 \xi^2 y^2 + O(y^4),\end{aligned}$$

so

$$d_h = 1 + \left(e_5 + \sum_{p=1}^{3} \kappa_p\right) h$$

and

$$D(y) = 1 + 2\left(e_5 + \sum_{p=1}^{3} \kappa_p\right)|y| + 4\pi^2\left(e_6 - \sum_{p=1}^{3} \kappa_p \xi_p^2\right)|y|^3 + O(y^5),$$

where

$$e_5 = \frac{1}{2}\sum_{q=1}^{3} w_q G_1(\eta_q) = -0.104667955258029$$

and

$$e_6 = \frac{1}{4\pi^2}\sum_{q=1}^{3} w_q G_3(\eta_q) = 0.000487881333679.$$

Now we specify that $\xi_2 = 0$, $\xi_3 = -\xi_1$ and $\kappa_3 = \kappa_1$. To satisfy condition O1 for $\rho = 5$, we have only to choose ξ_1, κ_1 and κ_2 satisfying

$$2\kappa_1 + \kappa_2 = -e_5 \quad \text{and} \quad 2\kappa_1\xi_1^2 = e_6,$$

giving us one degree of freedom. The method shown in Table 1 arises from the solution

$$\xi_1 = -1, \qquad \kappa_1 = e_6/2, \qquad \kappa_2 = -e_5 - e_6,$$

and in this case we see from (4.15) and Lemma 2.2 that

$$\sum_{p=1}^{3} \kappa_p \operatorname{Re} T_1(4, 0, 0, \xi_p, y) = [\kappa_2 + 2\kappa_1 \cos 2\pi y][1 + y^4 G_4^+(0, y)] \geq 0,$$

for $0 \leq y \leq 1/2$. Condition S2 then follows from Lemma 4.2, and we note that $M = 6$ and $\rho = 5$ satisfy (3.1). Hence, the method is stable.

Method 6. This differs from Method 5 only in its correction term. We have $P = 2$, $\xi_1 = \xi_2 = 0$, $n_1 = 0$ and $n_2 = 2$, and since

$$\operatorname{Re} T_1(4, 0, 2, 0, y) = -4\pi^2 y^2[1 + y^2 G_2^+(\xi, y)] = -4\pi^2 y^2 + O(y^4),$$

we find with the help of (4.15) that

$$d_h = 1 + (e_5 + \kappa_1)h$$

and

$$D(y) = 1 + 2(e_5 + \kappa_1)|y| + 4\pi^2(e_6 - 2\kappa_2)|y|^3 + O(y^5).$$

The method shown in Table 1 has

$$\kappa_1 = -e_5 \quad \text{and} \quad \kappa_2 = e_6/2,$$

so condition O1 holds with $\rho = 5$. To verify condition S2, it suffices to show that the quantity

$$\sum_{p=0}^{2} \kappa_p \mathrm{Re}\, T_1(4, 0, n_p, 0, y) = \kappa_1[1+y^4 G_4^+(0, y)] - 4\pi^2\kappa_2 y^2[1+y^2 G_2^+(0, y)] \tag{4.16}$$

is nonnegative for $0 \le y \le 1/2$. Given any $\alpha > 1$, the function $y \mapsto 1 + y^\alpha G_\alpha^+(0, y)$ is nonnegative and monotonically increasing for $y \in [0, 1/2]$, having at $y = 1/2$ the maximum value

$$1 + \left(\frac{1}{2}\right)^\alpha G_\alpha^+\left(0, \frac{1}{2}\right) = 2\sum_{m=0}^{\infty} \frac{1}{(2m+1)^\alpha} = 2(1 - 2^{-\alpha})\zeta(\alpha),$$

where ζ is the Riemann zeta function. Since $\zeta(2) = \pi^2/6$, the righthand side of (4.16) is bounded from below by $\kappa_1 - \pi^4\kappa_2/4 > 0$. $\square$

Remark. In the third method, the choice $\xi_1 = -1/2$ makes verification of the stability condition S2 particularly easy, and also has the practical advantage that $x_{i,1} = x_{i-1,3}$ and $x_{i,3} = x_{i+1,1}$. However, other choices of ξ_1 also lead to reasonable methods, assuming of course that the other ξ_p and the κ_p are selected according to (4.13) and (4.14). For instance, it would be natural to take

$$\xi_1 = 0, \qquad \kappa_1 = 2e_4, \qquad \kappa_2 = -e_3 - 4e_4,$$

in which case $x_{i1} = y_{i1}$ and $x_{i3} = y_{i4}$. Numerical investigations of the behavior of $D(y)$ did not detect any values of ξ_1 for which condition S2 failed, but as ξ_1 approaches $1/2$ the values of κ_1 and κ_2 tend to $-\infty$ and ∞, respectively.

5. A modified method. In this section we consider collocation methods in which L is replaced by the partially-discrete linear operator

$$(\widetilde{L}_h u)(x_j) = \sum_{k \in S_j} \int_{t_k}^{t_{k+1}} u(y) \log \frac{\omega}{|X_j - \gamma(y)|}\, dy + \sum_{k \in S_j'} h \sum_{q=1}^{Q} w_q u(y_{kq}) \log \frac{\omega}{|X_j - Y_{kq}|},$$

where, for some fixed, small nonnegative integers n_1 and n_2,

$$S_j = \{k \in \mathbf{Z} : j - n_1 \le k \le j + n_2\},$$
$$S'_j = \{k \in \mathbf{Z} : j + n_2 + 1 \le k \le N - n_1 - 1\}.$$

Thus, the integration is exact on subintervals where the integrand is singular or near-singular, but elsewhere the quadrature formula (1.11) is used. This modified method approximates many standard implementations of the collocation method, in which special techniques are used to evaluate the singular or near-singular integrals, combined with conventional quadratures for the smooth case (cf. the second part of Section 6). The method will also shed light on the fully-discrete methods of Section 4.

Defining $\tilde{l}_h = \tilde{a}_h + \tilde{b}_h$ in the obvious way, we have the following analogue to formula (2.9).

Lemma 5.1. *For $\mu, \nu \in \Lambda_h$,*

$$\tilde{a}_h(\psi_\mu, \phi_\nu) = \begin{cases} \tilde{d}_h, & \text{if } \mu = \nu = 0, \\ \sigma(\mu)\widetilde{D}_h(\mu h), & \text{if } \mu = \nu \neq 0, \\ 0, & \text{if } \mu \neq \nu, \end{cases}$$

where

$$\tilde{d}_h = 1 - h \sum_{k=-n_1}^{n_2} \left\{ \int_0^1 \log|\varepsilon - k - \eta|\, d\eta - \sum_{q=1}^{Q} w_q \log|\varepsilon - k - \eta_q| \right\} + \frac{h}{2} \sum_{q=1}^{Q} w_q G_1(\varepsilon - \eta_q) + O(h^M)$$

and

$$\widetilde{D}_h(y) = 2|y| \sum_{k=-n_1}^{n_2} \exp(i2\pi k y) \left\{ \int_0^1 T_1(r, \varepsilon, 0, \eta, y) K_A[(\varepsilon - k - \eta)h]\, d\eta - \sum_{q=1}^{Q} w_q T_1(r, \varepsilon, 0, \eta_q, y) K_A[(\varepsilon - k - \eta_q)h] \right\} + \sum_{q=1}^{Q} w_q T_2(r, \varepsilon, \eta_q, y).$$

Proof. Since $\tilde{a}_h$ is invariant under translation by h, we can apply [**8**, Lemma 3.1] and deduce that $\tilde{a}_h(\psi_\mu, \phi_\nu) = 0$ if $\mu \neq \nu$. Next

$$\begin{aligned}(\tilde{A}_h u)(x_j) = \sum_{k\in S_j} \bigg\{ \int_{t_k}^{t_{k+1}} & u(y) K_A(x_j - y)\, dy \\ & - h \sum_{q=1}^{Q} w_q u(y_{kq}) K_A(x_j - y_{kq}) \bigg\} \\ + \sum_{k=0}^{N-1} h \sum_{q=1}^{Q} & w_q u(y_{kq}) K_A(x_j - y_{kq}),\end{aligned}$$

so

$$\tilde{a}_h(\psi_\mu, \phi_\mu) = \langle \tilde{A}_h \psi_\mu, \phi_\mu \rangle_h = I_\mu - II_\mu + III_\mu,$$

where

$$\begin{aligned}I_\mu &= \sum_{j=0}^{N-1} h \sum_{k\in S_j} \int_{t_k}^{t_{k+1}} \psi_\mu(y) K_A(x_j - y)\, dy\, \overline{\phi_\mu(x_j)}, \\ II_\mu &= \sum_{j=0}^{N-1} h \sum_{k\in S_j} h \sum_{q=1}^{Q} w_q \psi_\mu(y_{kq}) K_A(x_j - y_{kq}) \overline{\phi_\mu(x_j)},\end{aligned}$$

and, recalling (2.12) and (2.13),

$$III_\mu = \begin{cases} 1 + h \sum_{q=1}^{Q} w_q [K_A(\varepsilon - \eta_q) - 1], & \text{if } \mu = 0, \\ \sum_{q=1}^{Q} w_q T_2(r, \varepsilon, \eta_q, y), & \text{if } \mu \neq 0. \end{cases}$$

With the help of (2.6), we see that the inner sums of I_μ and II_μ are independent of j, and that

$$\begin{aligned}I_\mu = \sum_{k=-n_1}^{n_2} h \int_0^1 & \exp[i2\pi\mu(k + \eta - \varepsilon)h] \\ & \cdot [1 + \Delta_r(\eta, \mu h)] K_A[(\varepsilon - k - \eta)h]\, d\eta, \\ II_\mu = \sum_{k=-n_1}^{n_2} h \sum_{q=1}^{Q} & w_q \exp[i2\pi\mu(k + \eta_q - \varepsilon)h] \\ & \cdot [1 + \Delta_r(\eta_q, \mu h)] K_A[(\varepsilon - k - \eta_q)h].\end{aligned}$$

The formula for $\widetilde{D}_h(y)$ follows after noting that $h = 2|\mu h|\sigma(\mu)$ for $\mu \neq 0$.

In the case $\mu = 0$, we have

$$I_0 - II_0 = h \sum_{k=-n_1}^{n_2} \left\{ \int_0^1 K_A[(\varepsilon - k - \eta)h]\, d\eta - \sum_{q=1}^{Q} w_q K_A[(\varepsilon - k - \eta_q)h] \right\}.$$

Write

$$K_A(\xi h) = R(\xi h) - \log 2\pi h - \log|\xi|$$

where

$$R(\xi) = 1 + \log\left|\frac{\pi\xi}{\sin \pi\xi}\right|,$$

and observe that R is C^∞ on the open interval $(-1, 1)$. Since the quadrature formula (1.11) has order of precision M,

$$\int_0^1 R[(\varepsilon - k - \eta)h]\, d\eta - \sum_{q=1}^{Q} w_q R[(\varepsilon - k - \eta_q)h] = O(h^M), \tag{5.1}$$

and the formula for $\tilde{d}_h$ follows. □

For simplicity, we shall study in detail only the piecewise-constant case.

Lemma 5.2. *If $r = 1$, $\varepsilon = 1/2$ and $n_1 = n_2 = n$, and if we use the two-point Gauss-Legendre rule* (4.8), *then for any fixed $n \geq 0$,*

$$\tilde{d}_h = 1 + \tilde{e}_n h + O(h^4)$$

and

$$\widetilde{D}_h(y) = 1 + [2\tilde{e}_n + O(h^4)]|y| + O(|y|^3(1 + |\log h|)),$$

where the coefficient $\tilde{e}_n$ is given by (5.3) *below, and satisfies*

$$\tilde{e}_n = -\frac{1}{1080(n+1)^3} - \frac{1}{720(n+1)^4} + O(n^{-5}) \quad \text{as } n \to \infty.$$

Proof. By Lemma 5.1 the formula for $\tilde{d}_h$ holds with

$$\begin{aligned}\tilde{e}_n = &-\sum_{k=-n}^{n}\left\{\int_0^1 \log\left|\frac{1}{2}-k-\eta\right| d\eta - \sum_{q=1}^{Q} w_q \log\left|\frac{1}{2}-k-\eta_q\right|\right\}\\ &+\frac{1}{2}\sum_{q=1}^{Q} w_q G_1\left(\frac{1}{2}-\eta_q\right).\end{aligned} \tag{5.2}$$

Using the closed form $H_1(\eta) = -2\pi(\eta - 1/2)$ we see that

$$\begin{aligned}T_1(1, 1/2, 0, \eta, y) &= [1 + yG_1^-(\eta, y) + iyH_i^-(n, y)]\exp[i2\pi(\eta - 1/2)y]\\ &= [1 + iH_1(\eta)y + O(y^2)][1 + i2\pi(\eta - 1/2)y + O(y^2)]\\ &= 1 + O(y^2),\end{aligned}$$

and it follows from the symmetry of the Gauss rule that $\widetilde{D}_h(y)$ is real. Thus, by (4.6), (4.7) and Lemma 5.1,

$$\begin{aligned}\widetilde{D}_h(y) = 2|y| \sum_{k=-n}^{n} &\left\{\int_0^1 K_A\left[\left(\frac{1}{2}-k-\eta\right)h\right] d\eta\right.\\ &\left.\quad - \sum_{q=1}^{Q} w_q K_A\left[\left(\frac{1}{2}-k-\eta_q\right)h\right]\right\}\\ &+ 1 + |y|\sum_{q=1}^{Q} w_q G_1\left(\frac{1}{2}-\eta_q\right) + O(|y|^3(1 + |\log h|)),\end{aligned}$$

and, after using (5.1) to simplify the expression in braces, we arrive at the formula for $\widetilde{D}_h(y)$.

To complete the proof, we consider $\tilde{e}_n$ in more detail. Evaluating the integral in (5.2), and inserting the values of the weights and integration points given in (4.8), we obtain

$$\begin{aligned}\tilde{e}_n = &(2n+1) - (2n+1)\log\left(n + \frac{1}{2}\right)\\ &+ \log\left(\frac{1}{2\sqrt{3}}\right) + \sum_{k=1}^{n}\log\left(k^2 - \frac{1}{12}\right) - \log\left[2\sin\left(\frac{\pi}{2\sqrt{3}}\right)\right].\end{aligned} \tag{5.3}$$

The standard infinite product representation of the sine function [**1**, 4.3.89] implies that

$$\log\left(\frac{1}{2\sqrt{3}}\right)+\sum_{k=1}^{n}\log\left(k^2-\frac{1}{12}\right)-\log\left[2\sin\left(\frac{\pi}{2\sqrt{3}}\right)\right]$$
$$=2\log(n!)-\log(2\pi)-\sum_{k=n+1}^{\infty}\log\left(1-\frac{1}{12k^2}\right),$$

and so, using an asymptotic expansion of the logarithm of the Gamma function [**1**, 6.1.40], we find that

$$\tilde{e}_n=I_n+II_n+III_n \quad \text{as } n\to\infty, \tag{5.4}$$

where

$$I_n=(2n+1)\log\left(\frac{n+1}{n+1/2}\right)-1, \qquad II_n=-\sum_{k=n+1}^{\infty}\log\left(1-\frac{1}{12k^2}\right),$$

and, with B_j denoting the j-th Bernoulli number,

$$III_n\sim\sum_{m=1}^{\infty}\frac{B_{2m}}{m(2m-1)(n+1)^{2m-1}}=\frac{1}{6(n+1)}-\frac{1}{180(n+1)^3}+O(n^{-5}).$$

The Taylor expansion of $\log(1-x)$ gives

$$\begin{aligned}I_n&=-\sum_{m=1}^{\infty}\frac{(n+1)^{-m}}{m(m+1)2^m}\\&=-\frac{1}{4(n+1)}-\frac{1}{24(n+1)^2}-\frac{1}{96(n+1)^3}-\frac{1}{320(n+1)^4}+O(n^{-5})\end{aligned}$$

and

$$II_n=\sum_{m=1}^{\infty}\frac{1}{12^m m}\sum_{k=n+1}^{\infty}\frac{1}{k^{2m}}.$$

By the Euler-Maclaurin summation formula,

$$\begin{aligned}\sum_{k=n+1}^{\infty}\frac{1}{k^{2m}}&\sim\frac{1}{(2m-1)(n+1)^{2m-1}}+\frac{1}{2(n+1)^{2m}}\\&\quad+\sum_{k=1}^{\infty}\binom{2m+2k-1}{2k}\frac{B_{2k}}{(2m+2k-1)(n+1)^{2m+2k-1}},\end{aligned}$$

so

$$II_n = \frac{1}{12(n+1)} + \frac{1}{24(n+1)^2} + \frac{13}{864(n+1)^3} + \frac{1}{576(n+1)^4} + O(n^{-5}).$$

Substituting the expansions for I_n, II_n and III_n into (5.4), we obtain the asymptotic behavior of $\tilde{e}_n$. $\square$

The expansions of $\tilde{d}_h$ and $\widetilde{D}_h(y)$ given in Lemma 5.2 show that for any fixed n the method is only first-order accurate (and not third-order accurate, as would be the case if *all* integrations were performed exactly; see Theorem 3.3). Unfortunately, the expansions do not appear to be uniform in n, so it is difficult to make precise statements about what would happen if n were allowed to depend on h. The results do show, however, that the correction terms in the fully-discrete methods of Section 4 do not work just by making the integration sufficiently accurate over the intervals containing the x_{jp}.

In practice, the lower asymptotic rate of convergence of the modified method of Lemma 5.2 will not be apparent unless the mesh is quite fine, due to the small size of the coefficient $\tilde{e}_n$:

n	$\tilde{e}_n$
0	-3.685×10^{-3}
1	-2.386×10^{-4}
2	-5.603×10^{-5}
3	-2.096×10^{-5}
4	-9.977×10^{-6}

For instance, $|\tilde{e}_1| \approx h^2$ if we use $N = 64$ subintervals.

6. Numerical experiments. Each of the six methods from Table 1 was tested on the following simple example. The curve Γ was the ellipse

$$X_1^2/4^2 + X_2^2 = 1,$$

and the righthand side of the integral equation (1.1) was the harmonic function

$$\begin{aligned} F(X) &= \operatorname{Re} \sin[(X_1 - 0.33) + i(X_2 - 0.22)] \\ &= \sin(X_1 - 0.33)\cosh(X_2 - 0.22). \end{aligned}$$

We used the obvious parametric representation $\gamma(x) = (4\cos 2\pi x, \sin 2\pi x)$, and took $\omega = 3$. (Since the logarithmic capacity of an ellipse is equal to the arithmetic mean of its major and minor semi-axes, $\operatorname{cap}(\Gamma) = 2{\cdot}5$ and so the condition (1.2) for unique solvability was satisfied.)

Denote the single layer potential of the exact solution U by

$$V(X) = \int_\Gamma U(Y) \log \frac{\omega}{|X-Y|}\, ds_Y = \int_0^1 u(y) \log \frac{\omega}{|X-\gamma(y)|}\, dy,$$

and define a discrete approximation

$$V_h(X) = \sum_{k=0}^{N-1} h \sum_{q=1}^{Q} w_q u_h(y_{kq}) \log \frac{\omega}{|X - Y_{kq}|},$$

where the quadrature rule (1.11) is the same as in the collocation method used to compute u_h. Let Ω denote the open set enclosed by Γ, then V is the unique solution to the Dirichlet problem

$$\begin{aligned} \nabla^2 V &= F \quad \text{on } \Omega, \\ V &= F \quad \text{on } \Gamma, \end{aligned}$$

and since F itself is harmonic on Ω, we have $V = F$ on Ω. Thus, it follows from Theorem 3.4 that

$$V_h(X) = F(X) + O(h^\rho) \quad \text{for } X \in \Omega. \tag{6.1}$$

TABLE 2. Errors in the discrete single layer potential for Method 1 (piecewise constants, $r=1$) and Method 2 (piecewise linears, $r=2$).

	Method 1		Method 2	
N	error	$h^{-3} \times$ error	error	$h^{-3} \times$ error
8	−4.28e-02	−21.9	1.81e-01	92.8
16	9.77e-03	40.0	1.94e-02	79.3
32	2.07e-03	67.7	9.34e-04	30.6
64	2.51e-04	65.8	1.10e-04	28.9
128	3.10e-05	65.0	1.32e-05	27.7
256	3.86e-06	64.8	1.64e-06	27.5
512	4.83e-07	64.8	2.04e-07	27.4
1024	6.03e-08	64.7	2.55e-08	27.4

TABLE 3. Errors in the discrete single layer potential for Methods 3 and 4 (piecewise quadratics, $r = 3$).

	Method 3		Method 4	
N	error	$h^{-5} \times$ error	error	$h^{-5} \times$ error
8	1.54e-01	5,032	1.52e-01	4,990
16	1.04e-02	1,092	1.09e-02	11,447
32	1.20e-04	4,033	1.40e-04	4,681
64	3.93e-06	4,215	4.54e-06	4,940
128	1.19e-07	4,090	1.38e-07	4,758
256	3.70e-09	4,065	4.31e-09	4,735
512	1.16e-10	4,059	1.34e-10	4,729
1024	3.60e-12	4,058	4.20e-12	4,728

TABLE 4. Errors in the discrete single layer potential for Methods 5 and 6 (piecewise cubics, $r = 4$).

	Method 5		Method 6	
N	error	$h^{-5} \times$ error	error	$h^{-5} \times$ error
8	1.17e-01	3,847	1.18e-01	3,861
16	3.03e-04	318	2.17e-04	228
32	6.17e-05	2,071	5.95e-05	1,995
64	1.33e-06	1,432	1.26e-06	1,357
128	4.02e-08	1,382	3.80e-08	1,307
256	1.24e-09	1,364	1.17e-09	1,289
512	3.86e-11	1,360	3.65e-11	1,284
1024	1.21e-12	1,359	1.14e-12	1,284

Tables 2–4 show the error $V_h(X) - F(X)$ at the point $X = (3.1, -0.2)$ for each of the six methods defined in Table 1, using $N = 2^l$ subintervals for $3 \leq l \leq 10$. The behavior of the ratio $h^{-\rho}[V_h(X) - F(X)]$ confirms that (6.1) holds for the predicted values of ρ.

TABLE 5. Errors and experimental convergence rates for the modified method of Lemma 5.2.

N	$n=0$		$n=1$		$n=2$	
8	−3.15e-02	—	−3.10e-02	—	−3.08e-02	—
16	8.66e-03	1.86	8.11e-03	1.94	8.10e-03	1.93
32	2.21e-03	1.97	2.03e-03	2.00	2.01e-03	2.01
64	3.10e-04	2.83	2.44e-04	3.06	2.39e-04	3.07
128	6.19e-05	2.32	3.18e-05	2.94	3.00e-05	2.99
256	1.94e-05	1.67	4.72e-06	2.75	3.92e-06	2.93
512	8.27e-06	1.23	9.69e-07	2.29	5.80e-07	2.76
1024	3.96e-06	1.06	3.10e-07	1.64	1.17e-07	2.31

TABLE 6. Errors and experimental convergence rates for midpoint collocation ($\varepsilon = 1/2$) with piecewise constants ($r = 1$); comparison between Method 1 (fully-discrete) and the use of exact integration.

N	Method 1		Exact Integration	
8	−4.28e-02	—	−3.62e-02	—
16	9.77e-03	2.13	8.12e-03	2.01
32	2.07e-03	2.24	2.01e-03	2.01
64	2.51e-04	3.04	2.37e-04	3.08
128	3.10e-05	3.02	2.93e-05	3.02
256	3.86e-06	3.00	3.65e-06	3.00
512	4.83e-07	3.00	4.56e-07	3.00
1024	6.03e-08	3.00	5.70e-08	3.00

The modified method from Lemma 5.2 was also tested using the choices of Γ and F above. The exact entries of the collocation matrix,

$$a_{jk} = \int_{t_k}^{t_{k+1}} \log \frac{\omega}{|X_j - \gamma(y)|}\, dy \quad \text{for } k \in S_j,$$

were evaluated by splitting the kernel as follows,

$$\log \frac{\omega}{|X_j - \gamma(y)|} = \log R_{jk}(y) \log \frac{1}{|x_j - y_k + m_{jk}|} \quad \text{for } t_k < y < t_{k+1},$$

where

$$R_{jk}(y) = \begin{cases} \dfrac{\omega|x_j - y + m_{jk}|}{|X_j - \gamma(y)|}, & \text{if } X_j \neq \gamma(y), \\ \dfrac{\omega}{|\gamma'(x_j)|}, & \text{if } X_j = \gamma(y), \end{cases}$$

and

$$m_{jk} = \begin{cases} 0, & \text{if } |x_j - t_k| \leq 1/2, \\ 1, & \text{if } -1 \leq x_j - t_k \leq -1/2, \\ -1, & \text{if } 1/2 < x_j - t_k \leq 1. \end{cases}$$

The smooth term $\log R_{jk}$ was integrated using a six-point Gauss-Legendre rule, whereas the singular term was integrated analytically. We also used the six-point Gauss rule to evaluate the single layer potential, and once again computed the error at $X = (3.1, -0.2)$. The results shown in Table 5 are consistent with our remarks at the end of Section 5.

Finally, Table 6 gives a comparison between our fully-discrete Method 1 and the collocation method using exact integration. (The latter is just the modified method with $n > N/2$.) Both methods use piecewise constants and achieve $O(h^3)$ accuracy, but the error is always slightly larger in the fully-discrete case.

REFERENCES

1. M. Abramowitz and I.A. Stegun, *Handbook of mathematical functions*, Ninth Printing, Dover, New York, 1970.

2. D.N. Arnold and W.L. Wendland, *The convergence of spline collocation for strongly elliptic equations on curves*, Numer. Math. **47** (1985), 317–341.

3. B. Bialecki and Y. Yan, *A rectangular quadrature method for logarithmically singular integral equations of the first kind*, J. Integral Equations Appl. **4** (1992), 337–369.

4. G. Brown, G.A. Chandler, I.H. Sloan and D.C. Wilson, *Properties of certain trigonometric series arising in numerical analysis*, J. Math. Anal. Appl. **162** (1991), 371–380.

5. G.A. Chandler and I.H. Sloan, *Spline qualocation methods for boundary integral equations*, Numer. Math. **58** (1990), 537–567; Erratum *Numer. Math.* **62** (1992), 295.

6. E. Hille, *Analytic function theory*, Volume II, Ginn and Company, Boston, 1962.

7. M.A. Jaswon and G.T. Symm, *Integral equations in potential theory and elastostatics*, Academic Press, London, 1977.

8. W. McLean, *Discrete Petrov-Galerkin methods for periodic pseudodifferential equations*, preprint, 1993.

9. ———, *Numerical evaluation of some trigonometric series*, Math. Comp. **63** (1994), 271–275.

10. W. McLean and I.H. Sloan, *A fully-discrete and symmetric boundary element method*, IMA J. Numer. Anal. **14** (1994), 311–345.

11. W. McLean and T. von Petersdorff, in preparation.

12. J. Saranen, *The convergence of even degree spline collocation solution for potential problems in smooth domains of the plane*, Numer. Math. **53** (1988), 499–512.

13. G. Schmidt, *The convergence of Galerkin and collocation methods with splines for pseudodifferential equations on closed curves*, Z. Anal. Anwendungen **3** (1984), 371–384.

14. I.H. Sloan, *Error analysis of boundary integral methods*, Acta Numerica **1** (1992), 287–339.

15. I.H. Sloan and A. Spence, *The Galerkin method for integral equations of the first kind with logarithmic kernel: Theory*, IMA J. Numer. Anal. **8** (1988), 105–122.

16. W.L. Wendland, *Asymptotic accuracy and convergence for point collocation methods*, in *Topics in boundary element research*, Vol. 2: *Time-dependent and vibration problems* (C.A. Brebbia, ed.), Springer-Verlag, Berlin, 1985, 230–257.

University of New South Wales, School of Mathematics, Sydney 2033, Australia.
E-mail address: W.McLean@unsw.edu.au