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PARABOLIC VOLTERRA INTEGRODIFFERENTIAL EQUATIONS OF CONVOLUTION TYPE

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ABSTRACT. Linear abstract parabolic Volterra integrodifferential equations of convolution type with L^1 kernel are considered. Under suitable assumptions it is proved that strict solutions exist and that many of the maximal regularity properties of the solutions of parabolic evolution equations are inherited. The results are then applied to parabolic partial integrodifferential equations.

1. Introduction. Let X be a Banach space, and let $A : D(A) \subset X \to X$ be the infinitesimal generator of an analytic semigroup $S(t)$ on X. Moreover, let $B : D(B) \subset X \to X$ be a linear operator with domain $D(B) \supseteq D(A)$. Given $T > 0$ we shall consider the following initial value problem

$$
(1.1) \qquad \begin{aligned} &u'(t) = Au(t) + \int_0^t k(t-s)Bu(s)\,ds + f(t), \qquad t \in \,]0,T[\\ &u(0) = x \end{aligned}
$$

where f and x are given. In this paper we do not assume that f is continuous.

Problem (1.1) arises as an abstract version of parabolic partial integrodifferential equations of the following kind:

$$
(1.2) \qquad \begin{aligned} &u_t(t,\xi) = Eu(t,\xi) + \int_0^t k(t-s)Lu(s,\xi)\,ds + f(t,\xi), \\ &\qquad\qquad (t,\xi) \in \,]0,T[\, \times \Omega \\ &u(0,\xi) = x(\xi), \qquad \xi \in \Omega \end{aligned}
$$

with suitable boundary conditions. Here E is an elliptic operator in Ω and L is a differential operator of order less than or equal to the order

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of E. Moreover, Ω is a bounded subset of $\mathbf{R}^n$ with regular boundary $\partial\Omega$.

The main object of this paper is the existence of differentiable solutions of problem (1.1) in the sense of L^1. By this we mean a function $u \in W^{1.1}(0,T;X) \cap L^1(0,T;D(A))$ satisfying (1.1). Under the assumption $k \in L^1(]0,T[)$ we give existence results for solutions of (1.1) and prove that they satisfy many of the regularity properties of the solutions of the differential problem (see [**5**] and the Appendix)

$$(1.3) \qquad \begin{aligned} &u'(t) = Au(t) + f(t), \qquad t \in]0,T[\\ &u(0) = x \end{aligned}$$

under the same assumptions on the data.

We use the following notation. $X(\theta) := (D(A),X)_{\theta,1}$, $\theta \in]0,1[$, are the real interpolation spaces between $D(A)$ and $X, X(\theta+1) := \{x \in D(A), Ax \in X(\theta)\}$, $X(0)$ denotes an intermediate space between $D(A)$ and X and $W^{\theta,1}(0,T;X)$ are the Sobolev spaces of fractional order (see Section 2).

In particular in Section 4 we prove the following results.

If f belongs to a Sobolev space we prove that:

(A) *Let $f \in W^{\theta,1}(0,T;X)$ for some $\theta \in]0,1[$. Then if $x \in X(\theta)$ the solution u of* (1.1) *satisfies*

(i) $u \in C(0,T;X(\theta))$,

(ii) u', $Au \in W^{\theta,1}(0,T;X)$,

(iii) $u' \in L^1(0,T;X(\theta))$.

We note that the assumption $x \in X(\theta)$ is also necessary in order to get a solution satisfying (ii). This result proves that $W^{\theta,1}(0,T;X)$ is a space of maximal regularity.

If f takes values in some interpolation space we will show that:

(B) *Let $f \in L^1(0,T;X(\theta))$ for some $\theta \in]0,1[$. Then if $x \in X(\theta)$ the solution u of* (1.1) *satisfies*

(i) $u \in C(0,T;X(\theta))$,

(ii) $Au \in W^{\theta,1}(0,T;X)$

(iii) $u' \in L^1(0,T;X(\theta))$.

If in addition $B : X(\theta+1) \mapsto X(\theta)$ (*for example if* $B = A$) *then*

(iv) $Au \in L^1(0,T;X(\theta))$.

Also in this case the assumption $x \in X(\theta)$ is necessary in order to get (ii). Moreover, in case (iv) this result proves that $L^1(0,T;X(\theta))$ is a space of maximal regularity.

Finally, if k satisfies the additional assumption

$$\int_0^T dt \int_0^T ds \frac{|k(t)-k(s)|}{|t-s|} < +\infty \tag{1.4}$$

then we can prove the following result:

(C) *Let* f *satisfy the property*

$$\int_0^T dt \int_0^T ds \frac{||f(t)-f(s)||}{|t-s|} < +\infty$$

then if $x \in X(0)$ *the solution* u *satisfies*

$$u', Au \in L^1(0,T;X). \tag{$*$}$$

Again we note that the assumption $x \in X(0)$ is also necessary for $(*)$.

For $k = 0$ assertion (C) gives an existence result for differentiable solutions in L^1 to problem (1.3) (see Theorem 7.6) which seems to be new. This result is analogous to the one proved by Webb [**16**] for the case $u(0) = 0$ and f of bounded variation.

Next we study in the case where $B = A$ the following weak form of problem (1.1):

$$\begin{aligned} u(t) = S(t)x + A\int_0^t S(t-s)\,ds \int_0^s k(s-\sigma)u(\sigma)\,d\sigma \\ + \int_0^t S(t-s)f(s)\,ds. \end{aligned} \tag{1.5}$$

When $f \in L^1(0,T;X)$ and $x \in X$ we prove that the solutions u of (1.5) satisfy $u \in W^{\theta,1}(0,T;X) \cap L^1(0,T;X(\theta))$, for each $\theta \in]0,1[$. If, in addition, (1.4) holds, then $u \in C(0,T;X)$.

Finally, in Section 6 we apply the above results to the partial integrodifferential problem (1.2).

Problem (1.1) with noncontinuous f has been studied by various authors under different assumptions on k, B and X.

Concerning maximal regularity results we refer to Clément and Da Prato [**3**] who study (1.1) on the whole line, with $B = A$ and a kernel of monotone type, in the spaces $W^{\theta,p}(0,T;X)$ and $L^p(0,T;D(\theta,p))$; Di Blasio [**6**] considers (1.1) in the nonautonomous case in the spaces $L^p(0,T;D(\theta,p))$, $p > 1$, under the assumptions $D(B^2) = D(A^2)$ and $k \in L^{p'}(]0,T[)$; in Lorenzi and Paparoni [**9**] (1.1) is studied in the spaces $W^{\theta,p}(0,T;X)$ with $k \in L^1(]0,T[)$, $p, q > 1$; and, finally, Prüss and Sohr [**13**] study (1.1) in the spaces $L^p(0,T;X)$ in the case where $p > 1$ and X is ζ-convex.

Regularity results are obtained in [**11**] by Prüss who considers (1.1) with the integral term $\int_0^t k(t-s)Bu(s)\,ds$ replaced by $\int_0^t dB(t-s)u(s)$, where $B(t)$ is a family of linear operators of bounded variation satisfying $D(A) \subseteq D(B(t))$. Given f in a suitable subspace of $L^1(0,T;X)$, denoted by $B^\alpha(0,T;X)$, Prüss gives sufficient conditions to $u(0)$ for the existence of solutions u such that u', $Au \in B^\alpha(0,T;X)$. In [**11**] a result similar to (C) is also proved in the case where $u(0) = 0$ and f is of bounded variation.

Finally, we recall the recent monograph of Prüss [**12**] where maximal regularity results for integral equations of parabolic type are obtained. From these results it is possible to deduce assertion (ii) of (A) under additional assumptions on f.

We refer to [**12**] and the references therein also for results for problem (1.1) in the case where f is continuous which has not been considered in this paper.

2. Notation. Let X be a Banach space with norm $||\cdot||$ and let $A : D(A) \subseteq X \to X$ be the infinitesimal generator of an analytic semigroup $S(t)$. Without loss of generality we assume that 0 belongs to the resolvent set of A. This can always be achieved by replacing A

by $A - \alpha I$, for suitable α. From this we have that there exist M and $\omega > 0$ such that for $t > 0$ and $x \in X$

$$(2.1) \qquad ||S(t)x|| \leq Me^{-\omega t}||x||, \qquad t||AS(t)x|| \leq Me^{-\omega t}||x||.$$

Moreover, $D(A)$ is a Banach space under the norm

$$||x||_{D(A)} = ||Ax||.$$

For $\theta \in [0,1[$ we denote by $X(\theta)$ the intermediate space between $D(A)$ and X defined as (see [**2**] for the case $\theta \in]0,1[$ and [**8**] for the case $\theta = 0$).

$$(2.2) \quad X(\theta) = \left\{ x \in X : H_\theta(x) := \int_0^{+\infty} \sigma^{-\theta} ||AS(\sigma)x||\, d\sigma < +\infty \right\}$$

with norm

$$||x||_\theta = ||x|| + H_\theta(x).$$

It is known that for $\theta \in]0,1[$ the spaces $X(\theta)$ are real interpolation spaces between X and $D(A)$. We refer to [**2**] and [**8**] for a detailed description of the properties of these spaces and to [**14**] for the case in which A is the realization of the Laplace operator. Here we recall the following inclusions which will be used throughout

$$D(A) \subseteq X(\theta') \subseteq X(\theta) \subseteq X, \qquad 0 \leq \theta \leq \theta' < 1.$$

We denote by $X(\theta + 1)$ the Banach space

$$X(\theta + 1) = \{x \in D(A) : Ax \in X(\theta)\}$$

with norm

$$||x||_{\theta+1} = ||x|| + ||Ax||_\theta.$$

In what follows we shall be concerned with the following spaces of functions u from an interval $[0, b]$ into a Banach space Y.

i) $L^1(0, b; Y)$ is the Banach space of integrable functions u on $]0, b[$ with norm

$$||u||_{L^1(0,b;Y)} = \int_0^b ||u(t)||_Y\, dt$$

ii) $C(0,b;Y)$ is the Banach space of continuous functions on $[0,b]$ with norm

$$||u||_{C(0,b;Y)} = \sup_{0\le t\le b} ||u(t)||_Y$$

iii) $W^{\alpha,1}(0,b;Y)$, for $\alpha \in [0,1[$, is the Banach space of functions u in $L^1(0,b;Y)$ such that

$$N_1^\alpha(0,b;u) := \int_0^b dt \int_0^b \frac{||u(t)-u(s)||_Y}{|t-s|^{\alpha+1}}\, ds < +\infty$$

with norm

$$||u||_{W^{\alpha,1}(0,b;Y)} = ||u||_{L^1(0,b;Y)} + N_1^\alpha(0,b;u).$$

For $\alpha \in]0,1[$ it can be proved (for a proof see, e.g., [**5**]) that if $u \in W^{\alpha,1}(0,b;Y)$ then $u(t)t^{-\alpha} \in L^1(0,b;Y)$ and we have

$$N_2^\alpha(0,b;u) := \int_0^b t^{-\alpha}||u(t)||_Y\, dt \le c(b)||u||_{W^{\alpha,1}(0,b;Y)}.$$

Therefore if we denote by $\widehat{W}^{\alpha,1}(0,b;Y)$ the space of functions $u \in W^{\alpha,1}(0,b;Y)$ with norm

$$||u||_{\widehat{W}^{\alpha,1}(0,b;Y)} = ||u||_{W^{\alpha,1}(0,b;Y)} + N_2^\alpha(0,b;u)$$

we have that $\widehat{W}^{\alpha,1}(0,b;Y) \simeq W^{\alpha,1}(0,b;Y)$ $(\alpha \in]0,1[)$.

iv) $W^{1,1}(0,b;Y)$ is the Banach space of functions u in $L^1(0,b;Y)$ with their first distributional derivative u' in $L^1(0,b;Y)$ with norm

$$||u||_{W^{1,1}(0,b;Y)} = ||u||_{L^1(0,b;Y)} + ||u'||_{L^1(0,b;Y)}.$$

3. Properties of the convolution operator. Let $k \in L^1(]0,T[)$ for given $T > 0$. In what follows we denote by $K : L^1(0,T;X) \to L^1(0,T;X)$ the operator defined as

$$(Ku)(t) = (k * u)(t) := \int_0^t k(t-s)u(s)\, ds.$$

The following lemmas collect some properties of the operator K.

Lemma 3.1. *Let $u \in L^1(0,b;X(\theta))$, with $b \le T$. Then $Ku \in L^1(0,b;X(\theta))$ and we have*

$$||Ku||_{L^1(0,b;X(\theta))} \le |k|_b ||u||_{L^1(0,b;X(\theta))} \tag{3.1}$$

where we have set

$$|k|_b = \int_0^b |k(s)|\, ds.$$

Proof. Interchanging the order of integration, we have

$$||Ku||_{L^1(0,b;X)} \le \int_0^b dt \int_0^t ||k(t-s)u(s)||\, ds \le |k|_b ||u||_{L^1(0,b;X)}.$$

Moreover

$$\begin{aligned} ||H_\theta(Ku)||_{L^1(0,b;X)} &= \int_0^b dt \int_0^{+\infty} \left\| AS(\sigma) \int_0^t k(t-s)u(s)\, ds \right\| \frac{d\sigma}{\sigma_\theta} \\ &\le |k|_b ||u||_{L^1(0,b;X(\theta))} \end{aligned}$$

and the result follows. □

Lemma 3.2. *Let $u \in W^{\alpha,1}(0,b;X)$, with $\alpha \in]0,1[$ and $b \le T$. Then $Ku \in W^{\alpha,1}(0,b;X)$ and we have*

$$||Ku||_{W^{\alpha,1}(0,b;X)} \le 2|k|_b [N_1^\alpha(0,b;u) + \alpha^{-1} N_2^\alpha(0,b;u)] \tag{3.2}$$

and

$$N_2^\alpha(0,b;Ku) \le |k|_b N_2^\alpha(0,b;u). \tag{3.3}$$

Proof. Interchanging the order of integration we have

$$\begin{aligned} ||Ku||_{L^1(0,b;X)} &\le \int_0^b dt \int_0^t ||k(t-s)u(s)||\, ds \\ &\le |k|_b ||u||_{L^1(0,b;X)}. \end{aligned}$$

Moreover,

$$\begin{aligned} N_1^\alpha(0,b;Ku) =&2\int_0^b dt \\ &\times \int_0^t \frac{ds}{(t-s)^{\alpha+1}} \bigg\| \int_0^t k(\sigma)u(t-\sigma)\,d\sigma - \int_0^s k(\sigma)u(s-\sigma)\,d\sigma \bigg\| \\ \le& 2\int_0^b dt \int_0^t \frac{ds}{(t-s)^{\alpha+1}} \bigg\| \int_0^s k(\sigma)[u(t-\sigma)-u(s-\sigma)]\,d\sigma \bigg\| \\ &+ 2\int_0^b dt \int_0^t \frac{ds}{(t-s)^{\alpha+1}} \bigg\| \int_s^t k(\sigma)u(t-\sigma)\,d\sigma \bigg\| =: I_1 + I_2. \end{aligned}$$

Now

$$I_1 \le |k|_b N_1^\alpha(0,b;u)$$

and

$$I_2 \le 2\alpha^{-1}|k|_b \int_0^b t^{-\alpha}||u(t)||\,dt$$

and (3.2) follows.

Finally,

$$\begin{aligned} N_2^\alpha(0,b;Ku) &= \bigg\| \int_0^b k(s)\,ds \int_s^t u(t-s)t^{-\alpha}\,dt \\ &\le |k|_b N_2^\alpha(0,b;u), \end{aligned}$$

and the proof is complete. □

4. Solutions of the integrodifferential equation. We now investigate sufficient conditions for the existence of solutions of the problem

$$\begin{aligned} &u'(t) = Au(t) + \int_0^t k(t-s)Bu(s)\,ds + f(t), \qquad t \in\,]0,T[\\ &u(0) = x \end{aligned} \tag{4.1}$$

where A satisfies the assumptions of Section 2, $k \in L^1(]0,T[)$ and $B : D(B) \subseteq X \to X$ is a linear operator with $D(B) \supseteq D(A)$ satisfying, for some $\beta > 0$,

$$||Bx|| \le \beta||Ax||, \quad \forall\, x \in D(A). \tag{4.2}$$

Given $f \in L^1(0,T;X)$ we say that u is a *strict solution* of problem (4.1) if Au, $u' \in L^1(0,T;X)$ and (4.1) is satisfied. It is easy to see that if u is a strict solution of (4.1), then u verifies the integral equation

$$
\begin{aligned}
u(t) = S(t)x + \int_0^t S(t-s)(k * Bu)(s)\,ds \\
+ \int_0^t S(t-s)f(s)\,ds.
\end{aligned}
\tag{4.3}
$$

Conversely, if $u \in L^1(0,T;D(A)) \cap W^{1,1}(0,T;X)$ and satisfies (4.3), then u is a strict solution of (4.1). Therefore we study equation (4.3). To this end we introduce the operator

$$
(\Gamma u)(t) = \int_0^t S(t-s)(k * Bu)(s)\,ds. \tag{4.4}
$$

The following lemmas describe the properties of Γ.

Lemma 4.1. *For each $\theta \in]0,1[$ the operator Γ maps $W^{\theta,1}(0,T;D(A))$ into itself and there exists $c_1 = c_1(\theta,T)$ such that for each $\tau \le T$*

(i) $||\Gamma u||_{W^{\theta,1}(0,\tau;D(A))} \le c_1 |k|_\tau [N_1^\theta(0,\tau;Au) + N_2^\theta(0,\tau;Au)]$

and

(ii) $N_2^\theta(0,\tau;A\Gamma u) \le c_1 |k|_\tau [N_1^\theta(0,\tau;Au) + N_2^\theta(0,\tau;Au)]$.

Proof. Let $u \in W^{\theta,1}(0,\tau;D(A))$. From (4.2) we have that $Bu \in W^{\theta,1}(0,\tau;X)$, so that by Lemma 3.2 we get $k * Bu \in W^{\theta,1}(0,T;X)$. Hence, using Lemma 7.5 (ii) we get $A\Gamma u \in W^{\theta,1}(0,T;X)$ and assertion (i). Let us prove (ii). We have

$$
\begin{aligned}
N_2^\theta(0,\tau;A\Gamma u) \le \int_0^\tau dt\, t^{-\theta} \left\| \int_0^t AS(t-s)[(k*Bu)(s) - (k*Bu)(t)]\,ds \right\| \\
+ \int_0^\tau dt\, t^{-\theta} \left\| \int_0^t AS(t-s)(k*Bu)(t)\,ds \right\| =: J_1 + J_2.
\end{aligned}
$$

Using (2.1), we get

$$
\begin{aligned}
J_1 &\le M \int_0^\tau dt\, t^{-\theta} \int_0^t ||(k*Bu)(s) - (k*Bu)(t)||(t-s)^{-1}\,ds \\
&\le M N_1^\theta(0,\tau;(k*Bu))
\end{aligned}
$$

and

$$J_2 = \int_0^\tau t^{-\theta}\, dt[S(t)(k * Bu)(t) - (k * Bu)(t)]$$
$$\le (1 + M) N_2^\theta(0, \tau; (k * Bu));$$

therefore, assertion (ii) follows from Lemma 3.2 and (4.2). □

Lemma 4.2. *Let* $B \in \mathcal{L}(X(\theta+1), X(\theta))$. *Then the operator* Γ *maps* $L^1(0, T; X(\theta + 1))$ *into itself and there exists* $c_2 = c_2(\theta, T)$ *such that for each* $\tau \le T$

$$||\Gamma u||_{L^1(0,\tau;X(\theta+1))} \le c_2 |k|_\tau ||u||_{L^1(0,\tau;X(\theta+1))}.$$

Proof. Let $u \in L^1(0, T; X(\theta+1))$. By assumption $Bu \in L^1(0, T; X(\theta))$ so that by Lemma 3.1 we get $k * Bu \in L^1(0, T; X(\theta))$. Therefore, the result follows from Lemma A.4 (ii). □

Remark 4.3. The assumption $B \in \mathcal{L}(X(\theta + 1), X(\theta))$ is obviously verified in the case in which $B = A$.

We now investigate the properties of the operator Γ if the kernel function k satisfies the additional assumption

$$(4.5) \qquad ||k||_T := \int_0^T dt \int_0^T ds \frac{|k(t) - k(s)|}{|t - s|} < +\infty.$$

For example, assumption (4.5) is satisfied if k is of bounded variation on $[0, T]$. Another example is the function $k(t) = t^{-\gamma}$ for $\gamma \in]0, 1[$.

Then we have the following result.

Lemma 4.4. *Let* k *also satisfy* (4.5). *Then if* $u \in L^1(0, T; D(A))$ *we have* $\Gamma u \in L^1(0, T; D(A))$ *and for each* $\tau \le T$,

$$||\Gamma u||_{L^1(0,\tau;D(A))} \le (1 + M)\beta[|k|_\tau + ||k||_\tau]||u||_{L^1(0,\tau;D(A))}$$

where β *is given by* (4.2).

Proof. As

$$A\Gamma u(t) = A\int_0^t S(t-s)\,ds\int_0^s k(s-\sigma)Bu(\sigma)\,d\sigma$$

we get

$$\begin{aligned}\int_0^\tau ||A\Gamma u(t)||\,dt \le& \int_0^\tau dt\left\|\int_0^t ds\int_0^s AS(t-s)k(t-\sigma)Bu(\sigma)\,d\sigma\right\|\\ &+\int_0^\tau dt\int_0^t ds\int_0^s d\sigma\\ &\times ||AS(t-s)[k(t-\sigma)-k(s-\sigma)]Bu(\sigma)||\\ =:& I_1+I_2.\end{aligned}$$

Interchanging the order of integration and using (2.1)

$$\begin{aligned}I_1 &= \int_0^\tau dt\left\|\int_0^t d\sigma\int_\sigma^t ds AS(t-s)k(t-\sigma)Bu(\sigma)\right\|\\ &= \int_0^\tau dt\left\|\int_0^t [S(t-\sigma)-I]k(t-\sigma)Bu(\sigma)\,d\sigma\right\|\\ &\le (1+M)|k|_\tau||Bu||_{L^1(0,\tau;X)}\end{aligned}$$

and

$$\begin{aligned}I_2 &\le M\int_0^\tau dt\int_0^t ds\int_0^s d\sigma\frac{|k(t-\sigma)-k(s-\sigma)|}{t-s}||Bu(\sigma)||\\ &= M\int_0^\tau d\sigma||Bu(\sigma)||\int_\sigma^\tau dt\int_\sigma^t ds\frac{|k(t-\sigma)-k(s-\sigma)|}{t-s}\\ &\le M||k||_\tau||Bu||_{L^1(0,\tau;X)}.\end{aligned}$$

Therefore

$$||\Gamma Au||_{L^1(0,\tau;X)} \le (1+M)[|k|_\tau + ||k||_\tau]||Bu||_{L^1(0,\tau;X)}$$

and hence the conclusion follows from (4.2). □

We now turn to the problem of finding solutions of (4.1).

The following result concerns the case in which f takes values in some interpolation space.

Theorem 4.5. *Let* $f \in L^1(0,T;X(\theta))$ *for some* $\theta \in]0,1[$. *Then if* $x \in X(\theta)$ *there exists a unique strict solution* $u \in W^{\theta,1}(0,T;D(A))$ *of problem* (4.1) *and we have*:

(i) $u \in C(0,T;X(\theta))$

(ii) $u' \in L^1(0,T;X(\theta))$,

(iii) $Au \in W^{\theta,1}(0,T;X)$.

Moreover, there exists $c_3 = c_3(\theta,T)$ *verifying*

(iv) $||u||_{C(0,T;X(\theta))} \leq c_3[||x||_\theta + ||f||_{L^1(0,T;X(\theta))}]$

(v) $||u'||_{L^1(0,T;X(\theta))} \leq c_3[||x||_\theta + ||f||_{L^1(0,T;X(\theta))}]$

(vi) $||Au||_{W^{\theta,1}(0,T;X)} \leq c_3[||x||_\theta + ||f||_{L^1(0,T;X(\theta))}]$

Proof. Set

$$u_0(t) = S(t)x$$

and

$$u_1(t) = \int_0^t S(t-s)f(s)\,ds.$$

If $x \in X(\theta)$ and $f \in L^1(0,T;X(\theta))$ by Lemmas A.2 and A.4 (ii) of the Appendix we have that Au_0, $Au_1 \in W^{\theta,1}(0,T;X)$. Moreover, there exist $a_2 = a_2(\theta)$ and $b_2 = b_2(\theta,T)$ verifying

$$||u_0||_{W^{\theta,1}(0,\tau;D(A))} \leq a_2(\theta)||x||_\theta$$

and

$$||u_1||_{W^{\theta,1}(0,\tau;D(A))} \leq b_2(\theta,T)||f||_{L^1(0,\tau;X(\theta))}.$$

Therefore using Lemma 4.1 we obtain that $u_0 + u_1 + \Gamma$ maps $W^{\theta,1}(0,\tau;D(A))$ into itself, for each $\tau \leq T$. Let us now prove that there exists $\tau \leq T$ such that $u_0 + u_1 + \Gamma$ is a contraction with respect to the norm

$$||\cdot||_{\widehat{W}^{\theta,1}(0,\tau;D(A))} = ||\cdot||_{W^{\theta,1}(0,\tau;D(A))} + N_2^\theta(0,\tau;\cdot).$$

As it has been recalled before, we have

$$||\cdot||_{W^{\theta,1}(0,\tau;D(A))} \simeq ||\cdot||_{\widehat{W}^{\theta,1}(0,\tau;D(A))}$$

but the use of this latter space is more convenient in this context. From Lemma 4.1 we have that if τ satisfies

$$\gamma_1 := c_1(\theta,T)|k|_\tau(1+\theta^{-1}) \le 1/2$$

then $u_0+u_1+\Gamma$ is a contraction on $\widehat{W}^{\theta,1}(0,\tau;D(A))$. Consequently there exists $u\in\widehat{W}^{\theta,1}(0,\tau;D(A))$ satisfying

$$u = u_0+u_1+\Gamma u;$$

moreover, we have

$$||u||_{\widehat{W}^{\theta,1}(0,\tau;D(A))} \le \frac{1}{1-\gamma_1}[a_2||x||_\theta + b_2||f||_{L^1(0,\tau;X(\theta))}]. \tag{4.6}$$

Now from Lemma A.5 (i) and (iii) we get $\Gamma u\in C(0,T;X(\theta))$ and $(\Gamma u)'\in L^1(0,\tau;X(\theta))$. Moreover, from Lemmas A.2 and A.4 (i) and (ii) we have that $u_0,u_1\in C(0,T;X(\theta))$ and $u_0',u_1'\in L^1(0,\tau;X(\theta))$. Hence u is a strict solution on $[0,\tau]$ of problem (4.1) and satisfies assertions (i), (ii) and (iii) on $[0,\tau]$. If $\tau<T$ we use standard continuation method and consider the following equation for the function $v(t):=u(t+\tau)$

$$\begin{aligned} v(t) = S(t+\tau)x &+ \int_0^t S(t-s)(KBv)(s)\,ds \\ &+ \int_0^{t+\tau} S(t+\tau-s)(KB\tilde{u})(s)\,ds \\ &+ \int_0^{t+\tau} S(t+\tau-s)f(s)\,ds \end{aligned} \tag{4.7}$$

where $\tilde{u}(t)=u(t)$ if $t\le\tau$ and $\tilde{u}(t)=0$ if $t>\tau$. It is easy to see that $\tilde{u}\in W^{\theta,1}(0,T;D(A))$. Moreover, it can be seen that if a function w belongs to $W^{\theta,1}(0,T;D(A))$ then the function $t\mapsto w(t+\tau)$ belongs to $W^{\theta,1}(0,\tau\wedge(T-\tau);D(A))$ and moreover

$$||w(\cdot+\tau)||_{W^{\theta,1}(0,\tau\wedge(T-\tau);D(A))} = ||w||_{W^{\theta,1}(\tau,\tau+\tau\wedge(T-\tau);D(A))}.$$

Therefore the functions $t \mapsto u_0(t+\tau)$, $t \mapsto u_1(t+\tau)$ and

$$t \mapsto u_2(t+\tau) := \int_0^{t+\tau} S(t+\tau-s)(KB\tilde{u})(s)\,ds$$

belong to $W^{\theta,1}(0,\tau\wedge(T-\tau);D(A))$, and we have

(4.8)
$$||u_0(\cdot+\tau)||_{W^{\theta,1}(0,\tau\wedge(T-\tau);D(A))} = ||u_0||_{W^{\theta,1}(\tau,\tau+\tau\wedge(T-\tau);D(A))}$$
(4.9)
$$||u_1(\cdot+\tau)||_{W^{\theta,1}(0,\tau\wedge(T-\tau);D(A))} = ||u_1||_{W^{\theta,1}(\tau,\tau+\tau\wedge(T-\tau);D(A))}$$

and

$$(4.10)\quad \begin{aligned} &||u_2(\cdot+\tau)||_{W^{\theta,1}(0,\tau\wedge(T-\tau);D(A))} \\ &\quad = ||u_2||_{W^{\theta,1}(\tau,\tau+\tau\wedge(T-\tau);D(A))} \\ &\quad \le ||u_2||_{W^{\theta,1}(0,\tau+\tau\wedge(T-\tau);D(A))} \\ &\quad \le c_1|k|_{\tau+\tau\wedge(T-\tau)}||u||_{W^{\theta,1}(0,\tau;D(A))} \\ &\quad \le c_1|k|_{\tau+\tau\wedge(T-\tau)}\frac{1}{1-\gamma_1}[a_2||x||_\theta + b_2||f||_{L^1(0,\tau;X(\theta))}] \end{aligned}$$

where we used Lemma 4.1 and (4.6).

Hence, by an argument similar to the one used above we can prove that $u_0(\cdot+\tau)+u_1(\cdot+\tau)+u_2+\Gamma$ is a contraction on $\widehat{W}^{\theta,1}(0,\tau\wedge(T-\tau);D(A))$. Consequently, we find that there exists $v \in \widehat{W}^{\theta,1}(0,\tau\wedge(T-\tau);D(A))$ which satisfies (4.7). Therefore, the function $U(t)=u(t)$ if $t\in[0,\tau]$, and $U(t)=v(t-\tau)$, if $t\in[\tau,\tau+\tau\wedge(T-\tau)]$ satisfies (4.1) on $[0,\tau+\tau\wedge(T-\tau)]$. Moreover, using (4.6), (4.8), (4.9) and (4.10) we get that U satisfies an estimate similar to (4.6) on $[0,\tau+\tau\wedge(T-\tau)]$. Iterating this procedure we then obtain the existence of a unique solution $u\in W^{\theta,1}(0,T;D(A))$ of (4.1); moreover, u satisfies (i), (ii) and (iii) and

$$||u||_{W^{\theta,1}(0,T;D(A))} \le c_3[||x||_\theta + ||f||_{L^1(0,T;X(\theta))}].$$

Finally, assertion (iv) can be accomplished by using Lemma A.4. □

If B preserves spatial regularity, then Theorem 4.5 can be improved by the following.

Theorem 4.6. *Let $f \in L^1(0,T;X(\theta))$, for some $\theta \in]0,1[$. Assume in addition that $B \in \mathcal{L}(X(\theta+1), X(\theta))$. Then if $x \in X(\theta)$ there exists a unique strict solution $u \in W^{\theta,1}(0,T;D(A))$ of problem* (4.1) *and we have*

(i) $u \in C(0,T;X(\theta))$,

(ii) u', $Au \in L^1(0,T;X(\theta))$,

(iii) $Au \in W^{\theta,1}(0,T;X)$.

Moreover, there exists $c_4 = c_4(\theta,T)$ satisfying

(iv) $||u||_{C(0,T;X(\theta))} \le c_4[||x||_\theta + ||f||_{L^1(0,T;X(\theta))}]$

(v) $||u'||_{L^1(0,T;X(\theta))}$, $||Au||_{L^1(0,T;X(\theta))} \le c_4[||x||_\theta + ||f||_{L^1(0,T;X(\theta))}]$

(vi) $||Au||_{W^{\theta,1}(0,T;X)} \le c_4[||x||_\theta + ||f||_{L^1(0,T;X(\theta))}]$.

Proof. Let u_0 and u_1 be the functions introduced in the proof of Theorem 4.5. By Lemmas A.2 and A.4 (ii) we get Au_0, $Au_1 \in L^1(0,T;X(\theta))$. Hence by Lemma 4.2 we find that $u_0 + u_1 + \Gamma$ maps $L^1(0,T;X(\theta+1))$ into itself. Moreover, using Lemma 4.2 and a fixed point argument we get that if τ is sufficiently small then there exists $u \in L^1(0,\tau;X(\theta+1))$ satisfying (4.3) on $[0,\tau]$. Furthermore, using Lemma 5.1 and Lemmas A.2 and A.4 (iii), we get $A\Gamma u$, Au_0, $Au_1 \in W^{\theta,1}(0,\tau;X)$ and hence $u \in W^{\theta,1}(0,\tau;D(A))$. If $\tau \le T$ the result can be extended on $[0.T]$ by using the usual continuation procedure. Finally the remaining assertions can be proved by using Theorem 4.5. □

The following result concerns the case where f belongs to a Sobolev space.

Theorem 4.7. *Let $f \in W^{\theta,1}(0,T;X)$ for some $\theta \in]0,1[$. Then if $x \in X(\theta)$ there exists a unique strict solution $u \in W^{\theta,1}(0,T;D(A))$ of problem* (4.1) *and we have*

(i) $u \in C(0,T;X(\theta))$,

(ii) u', $Au \in W^{\theta,1}(0,T;X)$,

(iii) $u' \in L^1(0,T;X(\theta))$.

Moreover, there exists $c_5 = c_5(\theta,T)$ verifying

(iv) $||u||_{C(0,T;X(\theta))} \le c_5[||x||_\theta + ||f||_{\widehat{W}^{\theta,1}(0,T;X)}]$

(v) $||u'||_{W^{\theta,1}(0,T;X)}, ||Au||_{W^{\theta,1}(0,T;X)} \le c_5[||x||_\theta + ||f||_{\widehat{W}^{\theta,1}(0,T;X)}]$

(vi) $||u'||_{L^1(0,T;X(\theta))} \le c_5[||x||_\theta + ||f||_{\widehat{W}^{\theta,1}(0,T;X)}]$.

Proof. Let u_0 and u_1 be the functions introduced in the proof of Theorem 4.5. Then by a computation similar to the one used in Theorem 4.5 we get the existence of a unique $u \in W^{\theta,1}(0,T;D(A))$ satisfying $u = u_0 + u_1 + \Gamma u$ and

$$||u||_{W^{\theta,1}(0,T;D(A))} \le c_5[||x||_\theta + ||f||_{\widehat{W}^{\theta,1}(0,T;X)}].$$

Furthermore, from Lemmas A.2 and A.5(i) we have that $u_0, u_1, \Gamma u \in C(0,T;X(\theta))$ and that $u_0', Au_0, u_1', Au_1, (\Gamma u)' \in W^{\theta,1}(0,T;X)$. Finally, from Lemmas A.2 and A.5 (iii) we get $u_0', Au_0, u_1', Au_1, (\Gamma u)' \in L^1(0,T;X(\theta))$. Therefore, u satisfies (i)–(iv). $\square$

Remark 4.8. If $f \in L^1(0,T;X(\theta))$ it follows from (4.3), Lemmas 3.2 and 7.4 that $Au \in W^{\theta,1}(0,T;X)$ only if $x \in X(\theta)$. Therefore, the assumption $x \in X(\theta)$ is also necessary in Theorem 4.5. Similarly, if $f \in W^{\theta,1}(0,T;X)$.

From Theorems 4.5, 4.6 and 4.7, we get the existence of a solution $u \in W^{\theta,1}(0,T;D(A))$. The following result concerns existence of strict solutions.

Theorem 4.9. *Let k satisfy* (4.5)*. Let $x \in X(0)$, and let $f \in W^{0,1}(0,T;X)$. Then there exists a unique strict solution of* (4.1)*. Moreover, there exists c_6 satisfying*

(i) $||u||_{L^1(0,T;D(A))} \le c_6[||x||_0 + ||f||_{W^{0,1}(0,T,X)}]$,

(ii) $||u||_{W^{1,1}(0,T,X)} \le c_6[||x||_0 + ||f||_{W^{0,1}(0,T,X)}]$.

Proof. Let u be a strict solution of (4.1). Using Lemmas 4.4 and A.6, we find that if τ satisfies

$$\gamma_2 := (1+M)\beta[|k|_\tau + ||k||_\tau] \le 1/2 \tag{4.11}$$

then

$$||Au||_{L^1(0,\tau;X)} \le \frac{3M}{1-\gamma_2}[||x||_0 + ||f||_{W^{0,1}(0,\tau;X)}].$$

Therefore, if $x = 0$ and $f = 0$ we have $u = 0$, which implies uniqueness.

To prove existence, let $\{x_n\} \subset X(\theta)$ and $\{f_n\} \subset W^{\theta,1}(0,T,X)$ verifying $x_n \to x$ in $X(0)$ and $f_n \to f$ in $W^{0,1}(0,T,X)$. From Theorem 4.7 there exists $v_n \in W^{\theta,1}(0,T,D(A))$ satisfying

$$v_n'(t) = Av_n(t) + \int_0^t k(t-s)Bv_n(s)\,ds + f_n(t), \quad t \in]0,T[$$
$$v_n(0) = x_n.$$

Moreover, using Lemma 4.4 and Theorem A.6, we find that if τ satisfies (4.11), then

$$||Av_n||_{L^1(0,\tau;X)} \le \frac{3M}{1-\gamma_2}[||x_n||_0 + ||f_n||_{W^{0,1}(0,\tau;X)}]$$

and

$$||Av_n - Av_m||_{L^1(0,\tau;X)} \le \frac{3M}{1-\gamma_2}[||x_n - x_m||_0 + ||f_n - f_m||_{W^{0,1}(0,\tau;X)}].$$

Therefore we find that v_n is Cauchy in $L^1(0,\tau;D(A))$. Since A is closed we have that there exists $u \in L^1(0,\tau;D(A))$ such that

$$v_n \to u, \quad \text{in } L^1(0,\tau;D(A)).$$

Moreover,

$$||Au||_{L^1(0,\tau;X)} \le \frac{3M}{1-\gamma_2}[||x||_0 + ||f||_{W^{0,1}(0,\tau;X)}].$$

This in turn implies that $u \in W^{1,1}(0,\tau;X)$ and $v_n' \to u'$ in $L^1(0,\tau;X)$. Therefore u is a strict solution of (4.1) on $[0,\tau]$ that satisfies (i).

Finally, from (4.1) we obtain

$$\begin{aligned}||u'||_{L^1(0,\tau;X)} &\le ||Au||_{L^1(0,\tau;X)}(1-\gamma_2) + ||f||_{L^1(0,T;X)}\\ &\le c_6[||x||_0 + ||f||_{W^{0,1}(0,T,X)}],\end{aligned}$$

so that u satisfies (ii) on $[0,\tau]$.

If $\tau < T$ we can iterate this procedure and obtain the existence of a function $u \in L^1(0,T;D(A)) \cap W^{1,1}(0,T;X)$ satisfying (4.1) and assertions (i) and (ii). $\square$

Remark 4.10. If $f \in W^{0,1}(0,T;X)$ it follows from Theorem A.6 and Lemma 4.4 that if $u \in L^1(0,T;D(A))$ then $x \in X(0)$. Therefore the assumption $x \in X(0)$ in Theorem 4.9 is also necessary.

5. Solutions of the integrodifferential equation in the case $B = A$. We now study the existence of solutions of the following problem

$$\begin{aligned} &u'(t) = Au(t) + \int_0^t k(t-s)Au(s)\,ds + f(t), \qquad 0 < t < T \\ &u(0) = x. \end{aligned} \tag{5.1}$$

As before, a function $u \in L^1(0,T;D(A)) \cap W^{1,1}(0,T;X)$ which satisfies (5.1) is called a strict solution of (5.1). It is easy to see that if u is a strict solution of (5.1), then we have

$$u(t) = S(t)x + A\int_0^t S(t-s)(Ku)(s)\,ds + \int_0^t S(t-s)f(s)\,ds. \tag{5.2}$$

Equation (5.2) will be called the *mild* form of (5.1). A function $u \in L^1(0,T;X)$ such that

$$t \mapsto \int_0^t S(t-s)(Ku)(s)\,ds$$

belongs to $L^1(0,T;D(A))$ and that (5.2) is satisfied will be called a *mild* solution of (5.1).

Existence results for strict solutions of (5.1) have been obtained in Section 4; we now study the existence of mild solutions. To this end, we introduce the operator

$$(\overline{\Gamma}u)(t) = A\int_0^t S(t-s)(Ku)(s)\,ds. \tag{5.3}$$

The following results concern the properties of the operator $\overline{\Gamma}$.

Lemma 5.1. *For each $\theta \in]0,1[$, the operator $\overline{\Gamma}$ maps $L^1(0,T;X(\theta))$ into itself, and for each $\tau \le T$ we have*

$$||\overline{\Gamma}u||_{L^1(0,\tau;X(\theta))} \le b_2|k|_\tau ||u||_{L^1(0,\tau;X(\theta))}$$

where $b_2 = b_2(\theta,T)$ is given by Lemma A.4.

Proof. Let $u \in L^1(0,T;X(\theta))$. Then by Lemma 3.1 we have that $k * u \in L^1(0,T;X(\theta))$. Moreover, by Lemma A.4 (ii) there exists $b_2 = b_2(\theta,T)$ verifying

$$||\overline{\Gamma}u||_{L^1(0,\tau;X(\theta))} \le b_2||k * u||_{L^1(0,\tau;X(\theta))}.$$

Hence the conclusion follows from (3.1). □

We now study the property of the operator $\overline{\Gamma}$ in the case where k satisfies the additional assumption:

$$||k||_T := \int_0^T dt \int_0^T ds \frac{|k(t)-k(s)|}{|t-s|} < +\infty. \tag{5.4}$$

We have

Lemma 5.2. *Let k also satisfy assumption* (5.4). *Then the operator $\overline{\Gamma}$ maps $C(0,T;X)$ into itself and we have for each $\tau \le T$*

$$||\overline{\Gamma}u||_{C(0,\tau;X)} \le (1+M)[|k|_\tau + ||k||_\tau]||u||_{C(0,\tau;X)}.$$

Proof. We have

$$\begin{aligned} ||\overline{\Gamma}u(t)|| \le & \left\| \int_0^t ds \int_0^s AS(t-s)k(t-\sigma)u(\sigma)\,d\sigma \right\| \\ & + \int_0^t ds \int_0^s d\sigma ||AS(t-s)[k(t-\sigma)-k(s-\sigma)]u(\sigma)|| \\ =: & \, I_1 + I_2. \end{aligned}$$

Now

$$\begin{aligned} I_1 &= \left\| \int_0^t d\sigma \int_\sigma^t ds AS(t-s)k(t-\sigma)u(\sigma) \right\| \\ &= \left\| \int_0^t [S(t-\sigma)-I]k(t-\sigma)u(\sigma)\,d\sigma \right\| \\ &\le (1+M)|k|_\tau ||u||_{C(0,\tau;X)} \end{aligned}$$

and

$$\begin{aligned} I_2 &\le M\int_0^t ds \int_0^s d\sigma \frac{|k(t-\sigma)-k(s-\sigma)|}{t-s}||u(\sigma)|| \\ &= ||u||_{C(0,\tau;X)} M \int_0^t ds \int_0^s d\sigma \frac{|k(t-\sigma)-k(s-\sigma)|}{t-s} \\ &\le M||k||_\tau ||u||_{C(0,\tau;X)}. \end{aligned}$$

Therefore

$$||\overline{\Gamma}u|| \le (1+M)[|k|_\tau + ||k||_\tau]||u||_{C(0,\tau;X)}$$

and the result is proved. □

We are now able to prove the following existence result.

Theorem 5.3. *For each $x \in X$ and $f \in L^1(0,T;X)$ there exists a unique mild solution $u \in L^1(0,T;X(\alpha))$ for some $\alpha \in]0,1[$, of problem* (5.1). *In addition, $u \in L^1(0,T;X(\theta)) \cap W^{\theta,1}(0,T;X)$ for each $\theta \in]0,1[$, and there exists $d_1 = d_1(\theta,T)$ satisfying*

(i) $||u||_{L^1(0,T;X(\theta))} \le d_1[||x|| + ||f||_{L^1(0,T;X)}]$,

(ii) $||u||_{W^{\theta,1}(0,T;X)} \le d_1[||x|| + ||f||_{L^1(0,T;X)}]$.

Proof. Let $u \in L^1(0,T;X(\alpha))$ be a mild solution of (5.1). Using Lemma 5.1 and Lemmas A.1 and A.3(i) we get that if τ satisfies

$$b_2|k|_\tau \le 1/2, \tag{5.5}$$

then

$$||u||_{L^1(0,\tau;X(\alpha))} \le \frac{a_1+b_1}{1-b_2|k|_\tau}[||x|| + ||f||_{L^1(0,\tau;X)}].$$

Hence, if $x = 0$ and $f = 0$ we get $u = 0$ on $[0, \tau]$, which implies uniqueness.

To prove existence, let $\{x_n\} \subset D(A)$, $\{f_n\} \subset L^1(0, T, D(A))$, $x_n \to x$ in X and $f_n \to f$ in $L^1(0, T, X)$. By virtue of Theorem 4.5 there exists $v_n \in W^{\theta,1}(0, T, D(A))$ satisfying

$$(5.6)\quad v_n(t) = S(t)x_n + A\int_0^t S(t-s)(Kv_n)(s)\,ds + \int_0^t S(t-s)f_n(s)\,ds.$$

Moreover, using Lemma 5.1 and Lemmas A.1 and A.3(i) we get that if τ satisfies (5.5) then

$$||v_n||_{L^1(0,\tau;X(\theta))} \le \frac{a_1 + b_1}{1 - b_2|k|_\tau}[||x_n|| + ||f_n||_{L^1(0,\tau;X)}]$$

and

$$||v_n - v_m||_{L^1(0,\tau;X(\theta))} \le \frac{a_1 + b_1}{1 - b_2|k|_\tau}[||x_n - x_m|| + ||f_n - f_m||_{L^1(0,\tau;X)}].$$

Therefore, $\{v_n\}$ is Cauchy in $L^1(0, \tau; X(\theta))$ for each $\theta \in]0, 1[$. Consequently, there exists u such that

$$v_n \to u \quad \text{in } L^1(0, \tau; X(\theta))$$

and

$$||u||_{L^1(0,\tau;X(\theta))} \le \frac{a_1 + b_1}{1 - b_2|k|_\tau}[||x|| + ||f||_{L^1(0,\tau;X)}].$$

Furthermore, passing to the limit as $n \to \infty$ in (5.6) and using the fact that A is closed, we obtain that u is a solution of (5.2) on $[0, \tau]$.

If $\tau < T$ we can iterate this procedure and obtain a solution $u \in L^1(0, T; X(\theta))$ satisfying (i).

Finally, using Lemmas A.1 and A.3(ii) and Lemma A.4(iii) we find that $u \in W^{\theta,1}(0, T; X)$ for each $\theta \in]0, 1[$. Using (5.2) and (i), we get

$$\begin{aligned}||u||_{W^{\theta,1}(0;X)} &\le a_1||x|| + b_1||f||_{L^1(0,T;X)} + b_2|k|_\tau ||u||_{L^1(0,T;X(\theta))}\\ &\le \text{const}\,[||x|| + ||f||_{L^1(0,T;X)}]\end{aligned}$$

so that u satisfies (ii). □

From Theorem 5.3 we have the following uniqueness result for strict solutions.

Corollary 5.4. *The strict solution of* (5.1) *is unique.*

If k is assumed to be more regular we can prove the following.

Theorem 5.5. *Let k also satisfy property* (5.4). *Then there exists a unique mild solution $u \in C(0,T;X)$ of* (5.1). *Moreover, there exists d_2 satisfying*

$$||u||_{C(0,T;X)} \leq d_2[||x|| + ||f||_{L^1(0,T;X)}].$$

Proof. Let $u \in C(0,T;X)$ be a mild solution of (5.1). Using (2.1) and Lemma 5.2 we find that if τ satisfies

$$\gamma_3 := (1+M)[|k|_\tau + ||k||_\tau] \leq 1/2 \tag{5.7}$$

then

$$||u||_{C(0,\tau;X)} \leq \frac{M}{1-\gamma_3}[||x|| + ||f||_{L^1(0,\tau;X)}].$$

Therefore if $x = 0$ and $f = 0$ we have $u = 0$ on $[0,\tau]$, which implies uniqueness. To prove existence we use a density argument similar to the one used in the proof of Theorem 5.3. □

6. Parabolic partial integrodifferential equations. Let Ω be a bounded subset of $\mathbf{R}^n$ with C^2 boundary $\partial\Omega$, and let E be an elliptic operator in Ω which, for simplicity in notation, we take of order 2

$$Eu = \sum_{i,j=1}^{n} (a_{ij}(\xi)u_{\xi_i})_{\xi_j} + \sum_{i=1}^{n} b_i(\xi)u_{\xi_i} + c(\xi)u$$

where a_{ij}, b_i and c are given functions satisfying the properties

$$a_{ij} \in C^1(\overline{\Omega}); \qquad b_i \in C(\overline{\Omega}).$$

Let L be a linear differential operator on Ω with suitable regular coefficients, and let $k \in L^1(]0,T[)$.

We want to study the following problem

$$
\begin{aligned}
&u_t(t,\xi) = Eu(t,\xi) + \int_0^t k(t-s)Lu(s,\xi)\,ds + f(t,\xi),\\
&\qquad\qquad\qquad (t,\xi) \in \,]0,T[\, \times \Omega\\
&u(t,\xi) = 0, \qquad (t,\xi) \in \,]0,T[\, \times \partial\Omega\\
&u(0,\xi) = x(\xi), \qquad \xi \in \Omega.
\end{aligned}
\tag{6.1}
$$

It is well known that the realization of E with homogeneous Dirichlet boundary conditions in the spaces $C(\overline{\Omega})$ or $L^p(\Omega)$, $1 \le p < \infty$, generates an analytic semigroup. Hence, using the results of the preceding sections we can study (6.1) in these spaces. As an example, we choose $X = L^1(\Omega)$ since this case seems to be less studied in the literature.

We introduce some notation. Let A be the operator defined by

$$
\begin{aligned}
D(A) &= \{u \in H_0^{1,1}(\Omega) : Eu \in L^1(\Omega)\}\\
Au &= Eu
\end{aligned}
$$

where Eu is understood in the sense of distributions. Then it is known, see [**1, 10, 15**] that A generates an analytic semigroup on $L^1(\Omega)$. Moreover, we have the following characterization of the interpolation spaces $X(\theta)$, $\theta \in \,]0,1[$, defined by (2.2) (see [**7**])

$$
X(\theta) = \begin{cases} H^{2\theta,1}(\Omega), & \text{if } 2\theta < 1\\ B^{1,1}(\Omega), & \text{if } 2\theta = 1\\ H_0^{2\theta,1}(\Omega), & \text{if } 2\theta > 1 \end{cases}
\tag{6.2}
$$

where $H^{\theta,1}(\Omega)$ are the Sobolev space of fractional order and $B^{1,1}(\Omega)$ is the Besov space.

If $\theta = 0$ then we do not have a simple concrete characterization of such spaces. We refer to [**14**] for the case $E = \Delta$.

Finally, if the order of L is < 2, we denote by B the following operator

$$
\begin{aligned}
D(B) &= \{u \in L^1(\Omega) : Lu \in L^1(\Omega)\}\\
Bu &= Lu,
\end{aligned}
$$

whereas if L has order 2 we define

$$\begin{aligned} D(B) &= \{u \in H_0^{1,1}(\Omega) : Lu \in L^1(\Omega)\} \\ Bu &= Lu. \end{aligned}$$

In what follows, we assume that

$$D(B) \supseteq D(A). \tag{6.3}$$

For example, assumption (4.2) is satisfied if the order of L is < 2 or if L and E have the same principal part.

Using the results of Section 4 we get the following:

Theorem 6.1. *Let $f \in W^{\theta,1}(0,T;L^1(\Omega))$, for some $\theta \in]0,1[$, and let $x \in X(\theta)$, where $X(\theta)$ is defined by* (6.2). *Then the solution of* (6.1) *satisfies*

(i) $u \in C(0,T : X(\theta))$,

(ii) $u_t, E_u \in W^{\theta,1}(0,T;L^1(\Omega))$,

(iii) $u_t \in L^1(0,T;X(\theta))$.

Proof. The results follow from Theorem 4.7. □

If f is regular with respect to ξ, we have

Theorem 6.2. *Let $f \in L^1(0,T;X(\theta))$ and $x \in X(\theta)$ for some $\theta \in]0,1[$. Then the solution of* (6.1) *satisfies*

(i) $u \in C(0,T : X(\theta))$,

(ii) $u_t \in L^1(0,T;X(\theta))$,

(iii) $Eu \in W^{\theta,1}(0,T;L^1(\Omega)))$.

If, in addition, $L = \alpha(\xi)E$, with α regular and $\alpha \geq \alpha_0 > 0$, then

(iv) $Eu \in L^1(0,T;X(\theta))$.

Proof. Assertions (i), (ii) and (iii) follow from Theorem 4.5. Assertion (iv) follows from Theorem 4.6 and from the fact that if $u \in x(\theta+1)$, then $Bu = \alpha Eu \in X(\theta)$ if α is regular. □

We now consider the case where k also satisfies

$$\int_0^T dt \int_0^t |k(t) - k(s)|(t-s)^{-1}\, ds < +\infty. \tag{6.4}$$

We have:

Theorem 6.3. *Let k also satisfy* (6.4), *let $x \in X(0)$ and*

$$\int_0^T dt \int_0^t (t-s)^{-1}\, ds \int_\Omega |f(t,\xi) - f(s,\xi)|\, d\xi < +\infty.$$

Then the solution of (6.1) *satisfies $u_t, Eu \in L_1(]0,T[\times \Omega)$.*

Proof. The result follows from Theorem 4.9. □

Finally we consider the case $L = E$.

Theorem 6.4. *Let $f \in L^1(]0,T[\times\Omega)$, and let $x \in L^1(\Omega)$. Then there exists a function $u \in L^1(0,T;X(\theta)) \cap W^{\theta,1}(0,T;L^1(\Omega))$, for each $\theta \in]0,1[$, satisfying in the mild sense* (6.1) *with $L = E$. If, in addition, k satisfies* (6.4), *then $u \in C(0,T;L^1(\Omega))$.*

Proof. The results follow from Theorems 5.3 and 5.5. □

Appendix

In this section we recall the results for abstract parabolic equations which have been used before.

Let $A : D(A) \subseteq X \to X$ be the infinitesimal generator of an analytic semigroup $S(t)$ in a Banach space X satisfying the assumptions of Section 2. Given $f \in L^1(0,T;X)$ and $x \in X$, we consider the following problem

$$\begin{aligned} u'(t) &= Au(t) + f(t), \qquad 0 < t < T \\ u(0) &= x. \end{aligned} \tag{A.1}$$

A function u is called a *strict solution* of problem (A.1) if $u \in L^1(0,T;D(A)) \cap W^{1,1}(0,T;X)$ and we have $u'(t) = Au(t) + f(t)$ for almost every $t \in]0,T[$ and $u(0)=x$. It is known that if u is a strict solution of (A.1), then for each $t \in]0,T[$, we have

$$u(t) = S(t)x + \int_0^t S(t-s)f(s)\,ds. \tag{A.2}$$

Conversely, if the function u given by (A.2) satisfies $u \in L^1(0,T;D(A)) \cap W^{1,1}(0,T;X)$ then u is a strict solution of (A.1). As usual, we call the function u given by (A.2) the *mild solution* of (A.1). The regularity properties of the mild solutions of (A.1) are widely investigated in [**4**] and [**5**]. In the following we summarize the results that are used in this paper. To this end, it is useful to introduce the functions

$$u_0(t) = S(t)x$$

and

$$u_1(t) = \int_0^t S(t-s)f(s)\,ds.$$

We have:

Lemma A.1. *For each $x \in X$ we have that $u_0 \in C(0,T;X)$ and that $u_0 \in L^1(0,T;X(\theta)) \cap W^{\theta,1}(0,T;X)$ for each $\theta \in]0,1[$. Moreover, there exists $a_1 = a_1(\theta,T)$ verifying, for each $\tau \le T$,*

(i) $||u_0||_{L^1(0,\tau;X(\theta))} \le a_1(\theta,T)||x||$

(ii) $||u_0||_{W^{\theta,1}(0,\tau;X)} \le a_1(\theta,T)||x||$.

Proof. For the proof we refer to Theorems 2 and 7 of [**5**]. □

Lemma A.2. *Let $x \in X(\theta)$ for some $\theta \in]0,1[$. Then we have that $u_0 \in C(0,T;X(\theta))$ and $u_0', Au_0 \in L^1(0,T;X(\theta)) \cap W^{\theta,1}(0,T;X)$. Moreover, there exists $a_2 = a_2(\theta)$ verifying, for each $\tau \le T$,*

(i) $||u_0||_{C(0,T;X(\theta))} \le a_2(\theta)||x||_\theta$,

(ii) $||u_0'||_{L^1(0,\tau;X(\theta))}, ||Au_0||_{L^1(0,\tau;X(\theta))} \le a_2(\theta)||x||_\theta$,

(iii) $||u_0'||_{W^{\theta,1}(0,\tau;X)}, ||Au_0||_{W^{\theta,1}(0,\tau;X)} \le a_2(\theta)||x||_\theta$.

Proof. See Theorems 5 and 15 of [**5**]. □

Lemma A.3. *For each $f \in L^1(0,T;X)$ we have that $u_1 \in C(0,T;X)$ and that $u_1 \in L^1(0,T;X(\theta)) \cap W^{\theta,1}(0,T;X)$ for each $\theta \in]0,1[$. Moreover, there exists $b_1 = b_1(\theta,T)$ verifying, for each $\tau \le T$,*

(i) $||u_1||_{L^1(0,\tau;X(\theta))} \le b_1(\theta,T)||f||_{L^1(0,\tau;X)}$

(ii) $||u_1||_{W^{\theta,1}(0,\tau;X)} \le b_1(\theta,T)||f||_{L^1(0,\tau;X)}$.

Proof. See Theorems 17 and 18 of [**5**]. □

Lemma A.4. *Let $f \in L^1(0,T;X(\theta))$ for some $\theta \in]0,1[$. Then $u_1 \in C(0,T;X(\theta))$, $u_1', Au_1 \in L^1(0,T;X(\theta))$ and $Au_1 \in W^{\theta,1}(0,T;X)$. Moreover, there exists $b_2 = b_2(\theta,T)$ verifying, for each $\tau \le T$,*

(i) $||u_1||_{C(0,\tau;X(\theta))} \le b_2(\theta,T)||x||_\theta$,

(ii) $||u_1'||_{L^1(0,\tau;X(\theta))}, ||Au_1||_{L^1(0,\tau;X(\theta))} \le b_2(\theta,T)||f||_{L^1(0,\tau;X(\theta))}$,

(iii) $||Au_1||_{W^{\theta,1}(0,\tau;X)} \le b_2(\theta,T)||f||_{L^1(0,\tau;X(\theta))}$.

Proof. For the proof we refer to Theorems 20, 21, 22 and 23 of [**5**]. □

Lemma A.5. *Let $f \in W^{\theta,1}(0,T;X)$ for some $\theta \in]0,1[$. Then $u_1 \in C(0,T;X(\theta))$, $u_1', Au_1 \in W^{\theta,1}(0,T;X)$ and $u_1' \in L^1(0,T;X(\theta))$. Moreover, there exists $b_3 = b_3(\theta,T)$ verifying, for each $\tau \le T$,*

(i) $||u_1||_{C(0,\tau;X(\theta))} \le b_3(\theta,T)[||f||_{W^{\theta,1}(0,\tau;X)} + N_2^\theta(0,\tau;f)]$,

(ii) $||u_1'||_{W^{\theta,1}(0,\tau;X)}, ||Au_1||_{W^{\theta,1}(0,\tau;X)} \le b_3(\theta,T)[||f||_{W^{\theta,1}(0,\tau;X)} + N_2^\theta(0,\tau;f)]$,

(iii) $||u_1'||_{L^1(0,\tau;X(\theta))} \le b_3(\theta,T)[||f||_{W^{\theta,1}(0,\tau;X)} + N_2^\theta(0,\tau;f)]$.

Proof. See Lemma 2 and Theorems 25 and 27 of [**5**]. □

Finally we prove the following existence result (which seems to be new) for L^1-strict solutions of problem (A.1).

Theorem A.6. *Let $x \in X(0)$ and $f \in W^{0,1}(0,T;X)$. Then we have $u_0, u_1 \in C(0,T;X(0))$ and $u_0, u_1 \in L^1(0,T;D(A)) \cap W1,1(0,T;X)$.*

Moreover,

(i) $||u_0||_{C(0,T;X(0))} \leq M||x||_0$

(ii) $||u_0||_{L^1(0,T;D(A))\cap W^{1,1}(0,T;X)} \leq ||x||_0$,

(iii) $||u_1||_{C(0,T;X(0))} \leq [2M + M(T\omega)^{-1}]||f||_{W^{0,1}(0,T;X)}$,

(iv) $||u_1||_{L^1(0,T;D(A))\cap W^{1,1}(0,T;X)} \leq 3M||f||_{W^{0,1}(0,T;X)}$.

Proof. We have

$$(\text{A.3})\qquad ||Au_0||_{L^1(0,T;X)} = ||u_0'||_{L^1(0,T;X)} = \int_0^T ||AS(t)x||\,dt \leq H_0(x)$$

and hence $u_0 \in L^1(0,T;D(A)) \cap W^{1,1}(0,T,X)$, and (ii) is verified.

Moreover,

$$||u_0(t)||_0 = \int_0^{+\infty} ||AS(t+\sigma)x||\,d\sigma \leq MH_0(x)$$

and (i) follows.

To prove $u_1 \in L^1(0,T;D(A)) \cap W^{1,1}(0,T,X)$, we use a density argument. Let $\{f_n\} \subset W^{1,1}(0,T,X)$ verifying $f_n \to f$ in $W^{0,1}(0,T,X)$, and set

$$v_n = \int_0^t S(t-s)f_n(s)\,ds.$$

We have $v_n \in L^1(0,T;D(A)) \cap W^{1,1}(0,T,X)$ and $v_n \to u_1$ in $L^1(0,T;X)$. Moreover, using (2.1), we get

$$\begin{aligned}
||Av_n||_{L^1(0,T;X)} &\leq \int_0^T dt\left\|A\int_0^t S(t-s)[f_n(s)-f_n(t)]\,ds\right\| \\
&\quad + \int_0^T dt\left\|A\int_0^t S(t-s)f_n(t)\,ds\right\| \\
&\leq M\int_0^T dt\int_0^t ||f_n(t)-f_n(s)||(t-s)^{-1}\,ds \\
&\quad + \int_0^T dt||S(t)-I)f_n(t)|| \\
&\leq 2M||f_n||_{W^{0,1}(0,T,X)}
\end{aligned}$$

and hence

$$||Av_n - Av_m||_{L^1(0,T;X)} \leq 2M||f_n - f_m||_{W^{0,1}(0,T,X)}.$$

Therefore Av_n is Cauchy in $L^1(0,T;X)$ so that $u_1 \in L^1(0,T;D(A))$ and $Av_n \to Au_1$ in $L^1(0,T;X)$ since A is closed.

Finally, from the equality $v_n' = Av_n + f_n$, we get that $u_1 \in W^{1,1}(0,T;X)$ and that $u_1' = Au_1 + f$. Therefore, u_1 satisfies (iv). Let us prove (iii). We have

$$\begin{aligned} H_0(u_1(t)) &= \int_0^T \left\| \int_0^t AS(\sigma+t-s)f(s)\,ds \right\| d\sigma \\ &\quad + \int_T^{+\infty} \left\| \int_0^T AS(\sigma+t-s)f(s)\,ds \right\| d\sigma \\ &=: I_1 + I_2. \end{aligned}$$

By a computation similar to the one used above we find

$$I_1 \leq 2M||f||_{W^{0,1}(0,T;X)}.$$

Moreover,

$$I_2 \leq MT^{-1} \int_T^{+\infty} e^{-\omega\sigma}\,d\sigma ||f||_{L^1(0,T;X)},$$

and (iii) follows. □

Remark A.7. It follows from (A.3) that the assumption $x \in X(0)$ is also necessary for property (ii).

REFERENCES

1. H. Amann, *Dual semigroups and second order linear boundary value problems*, Israel J. Math. **45** (1983), 225–254.

2. P.L. Butzer and H. Berens, *Semigroups of operators and approximation*, Springer-Verlag, New York, 1967.

3. P. Clément and G. Da Prato, *Existence and regularity results for an integral equation with infinite delay in a Banach space*, Integral Equations Operator Theory **11** (1988), 480–500.

4. G. Da Prato and P. Grisvard, *Sommes d' opérateurs nonlineaires et equations différentielles opérationelles*, J. Math. Pures Appl. **54** (1975), 305–387.

5. G. Di Blasio, *Linear parabolic equations in L^p-spaces*, Ann. Mat. Pura Appl. **138** (1984), 55–104.

6. ———, *Nonautonomous integrodifferential equations in L^p spaces*, J. Integral Equations **10** (1985), 111–121.

7. ———, *Analytic semigroups generated by elliptic operators in L^1 and parabolic equations*, Osaka J. Math. **28** (1991), 367–384.

8. ———, *Holomorphic semigroups in interpolation and extrapolation spaces*, Semigroup Forum **47** (1993), 105–114.

9. A. Lorenzi and E. Paparoni, *Direct and inverse problems in the theory of materials with memory*, Rend. Sem. Mat. Padova **87** (1992), 105–138.

10. A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, New York, 1983.

11. J. Prüss, *On linear Volterra equations of parabolic type in Banach spaces*, Trans. Amer. Math. Soc. **301** (1987), 691–721.

12. ———, *Evolutionary integral equations and applications*, Birkhäuser, Basel, New York, 1993.

13. J. Prüss and H. Sohr, *On operators with bounded imaginary powers in Banach spaces*, Math. Z. **203** (1990), 429–452.

14. M.H. Taibleson, *On the theory of Lipschitz spaces of distributions on Euclidean n-spaces*, J. Math. Mech. **13** (1964), 407–479.

15. H. Tanabe, *On semilinear equations of elliptic and parabolic type,* in *Functional analysis and numerical analysis* (H. Fujita, ed.), Japan-France Seminar, Tokyo and Kyoto, 1976, Japan Society for the Promotion of Sciences, 1978, 455–463.

16. G. Webb, *Regularity of solutions to an abstract inhomogeneous linear differential equation*, Proc. Amer. Math. Soc. **62** (1977), 271–277.

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