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PALEY-WIENER THEOREM AND THE FACTORIZATION

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ABSTRACT. In this note we shall generalize the Paley-Wiener theorem to self-adjoint operators similar to $(-id/dx)$. By using a simple factorization result, it is shown that the Paley-Wiener theorem holds if $\Gamma'(\lambda)$ is analytic, where $\Gamma(\lambda)$ is the spectral function.

1. Introduction. Recall that with each self-adjoint operator is associated a transform or a unitary operator by which the self-adjoint operator is equivalent to a multiplication by the independent variable. For instance, $-id/dx$ is self-adjoint in the Hilbert space L^2_{dx} and $\mathcal{F}(f)(\lambda) = \int_{\mathbf{R}} f(x)e^{i\lambda x}\,dx$ defines a unitary operator called the Fourier transform

$$L^2_{dx} \xrightarrow{\mathcal{F}} L^2_{d\lambda/2\pi}.$$

One of the most interesting features of the Fourier transform is the Paley-Wiener theorem: Let $F(\lambda)$ be an entire function

$$\left.\begin{matrix} |F(\lambda)| < Me^{a|\lambda|} \\ F(\lambda) \in L^2_{d\lambda} \end{matrix}\right\} \Longleftrightarrow \left\{\begin{matrix} F(\lambda) = \displaystyle\int_{-a}^{a} f(x)e^{i\lambda x}\,dx \\ f(\lambda) \in L^2_{dx} \end{matrix}\right.$$

$e^{i\lambda x}$ are clearly the eigenfunctionals of the operator $-id/dx$. Our question is: Characterize self-adjoint operators in $L^2_{dM(x)}$ such that if $e^{i\lambda x}$ is replaced by its eigenfunctionals, then does a similar Paley-Wiener theorem hold? For the sake of simplicity, it is sufficient to consider self-adjoint operators with a simple spectrum, σ say. Let L be a self-adjoint operator acting in the separable Hilbert space $L^2_{dM(x)}$, and let $y(x,\lambda)$ be its eigenfunctionals, i.e., $Ly(x,\lambda) = \lambda y(x,\lambda)$ in the weak sense, see [**3**]. This gives rise to the y-transform, F_y

$$\forall\, f \in L^2_{dM(x)} \quad F_y(f)(\lambda) \equiv \int f(x)y(x,\lambda)\,dM(x).$$

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The inverse is simply defined by:

$$f(x) = \int F_y(f)(\lambda)\overline{y(x,\lambda)}\, d\Gamma(\lambda)$$

where $\Gamma(\lambda)$ is the spectral function associated with the operator L. Parseval equality associated with the operator L is given by

$$\int F_y(f)(\lambda)\overline{F_y(\psi)(\lambda)}\, d\Gamma(\lambda) = \int f(x)\overline{\psi(x)}\, dM(x).$$

Recall that we have assumed that the spectrum $\sigma = \operatorname{supp}\Gamma(\lambda) \equiv \mathbf{R}$.

2. Statement of the problem. Let $F(\lambda)$ be an entire function of λ. Under what conditions would

$$|F(\lambda)| < Me^{a|\lambda|} \Longleftrightarrow F_y(f)(\lambda) = \int_{-a}^{a} f(x)y(x,\lambda)\, dM(x).$$

In other words we would like to generalize the Paley-Wiener theorem to different and more general transforms. Throughout this work we shall need the following condition:

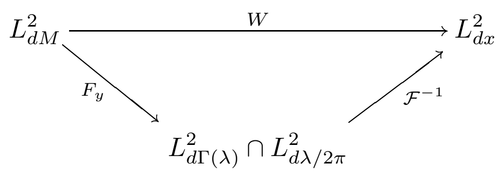
Condition [A]. Let L be a self-adjoint operator in the separable Hilbert space $L^2_{dM(x)}$ having a simple spectrum, $\sigma = R$. The associated unitary operator will be called the y-transform and denoted by $F_y(f)(\lambda) = \int f(x)y(x,\lambda)\, dM(x)$.

The main idea is to compare the operator L with $-id/dx$. To apply the comparison theorem we need to use rigged spaces, see [**1**]. However, for the sake of simplicity we shall present a different method that would not use rigged spaces. To this end, we introduce the following

Definition 1. Let condition [A] hold. W is said to be a transition operator if

(1) $$F_y(f)(\lambda) \equiv \mathcal{F}(Wf)(\lambda) \quad \forall\, \lambda \in \mathbf{R}$$

where $\mathcal{F}$ is the Fourier transform



Using equation (1) we easily deduce the following

Proposition 2. *We have the following obvious properties*

i) W *is always densely defined in* L^2_{dM}

ii) $D_W \equiv \{f \in L^2_{dM} / \int_\sigma |F_y(f)|^2 \, d\lambda < \infty\}$

iii) $R_W \equiv \{f \in L^2_{dx} / \int_R |\mathcal{F}(f)|^2 \, d\Gamma(\lambda) < \infty\}$.

iv) W *is bounded if and only if* $L^2_{d\Gamma(\lambda)} \subset L^2_{d\lambda/2\pi}$.

v) W^{-1} *exists* $\Leftrightarrow \int_\sigma |F_y(f)|^2 \, d\lambda = 0 \Rightarrow f = 0$.

It is also clear that the operators W and W^{-1} are always well defined but may be unbounded, depending on the nature of $\Gamma(\lambda)$.

Remark. The operator W^{-1} is defined similarly by

$$F_y(W^{-1}f)(\lambda) \equiv \mathcal{F}(f)(\lambda).$$

Theorem 3. *Let condition* [A] *hold and* $F(\lambda)$ *be an entire function, then*

$$\left\{ \begin{array}{l} |F(\lambda)| < Me^{a|\lambda|} \\ F(\lambda) \in L^2_{d\Gamma(\lambda)} \cap L^2_{d\lambda/2\pi} \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{l} F(\lambda) = \displaystyle\int f(x) y(x,\lambda) \, dM(x) \\ f \in DW \subset L^2_{dM(x)} \text{ and} \\ \operatorname{supp} Wf \subset [-a,a] \end{array} \right\}$$

Proof. The proof is a simple consequence of the definition of the operator W. Assume that the righthand side is true. Then

$F(\lambda) \in L^2_{d\Gamma(\lambda)}$ and, since $Wf \in L^2_{dx}$, then $|\mathcal{F}(Wf)(\lambda)| < e^{a|\lambda|}$ and $\mathcal{F}(Wf)(\lambda) \in L^2_{d\lambda/2\pi}$. By using the definition of the operator W, i.e., $\mathcal{F}(Wf) = F_y(f)$, we deduce $F(\lambda) \in L^2_{d\lambda/2\pi} \cap L^2_{d\Gamma(\lambda)}$ and $|F(\lambda)| < e^{a|\lambda|}$. Conversely, if $F(\lambda)$ satisfies the lefthand side then there exists $f(x) \in D_W$ such that $F(\lambda) \equiv F_y(f)(\lambda) = \mathcal{F}(Wf)(\lambda)$. By the Paley-Wiener theorem we deduce that $\operatorname{supp} Wf \subset [-a, a]$. □

Definition 4. An operator L is said to have the *Paley-Wiener property* if, for any entire function $F(\lambda)$,

$$\left\{ \begin{array}{c} |F(\lambda)| < Me^{a|\lambda|} \\ F(\lambda) \in L^2_{d\Gamma(\lambda)} \cap L^2_{d\lambda/2\pi} \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{l} F(\lambda) = \displaystyle\int_{-a}^{a} f(x) y(x,\lambda)\, dM(x) \\ f \in D_W \subset L^2_{dM(x)} \end{array} \right\}.$$

It is readily seen that for the Paley-Wiener property to hold we only need W and W^{-1} to be support preserving operators, i.e., $\operatorname{supp} Wf \subset [-a, a]$ if and only if $\operatorname{supp} f \subset [-a, a]$.

To proceed further, we shall need to define the concept of support preserving operators.

Definition 5.

i) W is said to be Support Preserving (S.P.) if

$$\operatorname{supp} Wf \subset \operatorname{supp} f \quad \text{for all } f \in D_W$$

ii) W is said to be Weak Support Preserving (W.S.P.) if

$$\operatorname{supp} f \subset [-a, a] \Longrightarrow \operatorname{supp} Wf \subset [-a, a] \quad \text{for all } f \in D_W.$$

Examples of S.P. operators. We now use the idea of chains of projections. Let $P_\xi f(x) \equiv 1_{[-|\xi|, |\xi|]}(t) f(t)$ and let X_+ be a bounded operator on $L^2_{(a,b)}$ where $-\infty \le a, b \le \infty$. Recall that an operator X_+ is said to be upper triangular if

$$X_+ P_\xi = P_\xi X_+ P_\xi.$$

It is readily seen that upper triangular operators are W.S.P. operators. Indeed, let f be given such that $\operatorname{supp} f \subset [-c, c]$. Then for all ξ such that $|\xi| > c$, we obviously have $P_\xi f = f$ and $X_+ f = P_\xi X_+ f$. Thus, $\operatorname{supp} X_+ f \subset [-c, c]$.

i) If X_+ is a Volterra operator, of upper triangular type with respect to the chain P_ξ, then $1 + X_+$ is W.S.P. and its inverse $[1 + X_+]^{-1}$ is also W.S.P. since it is of the same type, see [**4**].

ii) Let $Wf \equiv f + \int_{-\infty}^{-|x|} K(x,s) f(s)\, dM(s) + \int_{|x|}^{\infty} K(x,s) f(s)\, dM(s)$.

If $\iint |K(x,t)^2|\, dM(s)\, dx < \infty$, then W^{-1} is W.S.P. since

$$W^{-1} f \equiv f + \int_{-\infty}^{-|x|} H(x,s) f(s)\, dM(s) + \int_{|x|}^{\infty} H(x,s) f(s)\, dM(s),$$

iii) By the local property $\sum_{n \geq 0} a_n(x)(d^n/dx^n)$ is an S.P. operator.

iv) Let $Uf(x) \equiv r(x) f(t(x))$ where $r(x) > 0$ and $t(x) \nearrow$ and odd, then $\operatorname{supp} Uf \subset [-a, a] \Longleftrightarrow \operatorname{supp} f \subset [-t(a), t(a)]$.

The operator U in this last example, is similar to a W.S.P. since the supports are rescaled by the function $t(x)$.

Thus, we have a simple

Corollary 6. *Let condition* [A] *hold. If W and W^{-1} are* W.S.P. *operators, then the Paley-Wiener property holds.*

We now give necessary and sufficient conditions for W and W^{-1} to be S.P. Recall that in [**1**] the following operator was introduced

$$L^2_{dM(t)} \xrightarrow{G} L^2_{dM(t)}$$
$$f \to Gf(x) \equiv \int F_y(f)(\lambda) \overline{y(x,\lambda)}\, d\lambda / 2\pi.$$

We recall that, from the factorization theorem, it follows that

$$G = W'W. \tag{2}$$

We also have a similar factorization if we consider W^{-1},

$$S = [W^{-1}]'[W^{-1}] \tag{3}$$

where

$$Sf(x) \equiv \int \mathcal{F}(f)(\lambda)e^{-i\lambda x}\, d\Gamma(\lambda).$$

Recall that in case $\Gamma'(\lambda)$ is locally summable, then equation (3) reduces to

$$2\pi \frac{d\Gamma}{d\lambda}\left(\frac{-id}{ix}\right) = [W^{-1}]'[W^{-1}]. \tag{4}$$

For details of the above results, see Theorem 4 in [**1**].

Theorem 7. *Let condition* [A] *hold and let*

a) W *be* S.P.,

b) G^{-1} *be* S.P..

Then W^{-1} *is* S.P. *and the Paley-Wiener property holds.*

Proof. We would like to see when W^{-1} is S.P. To this end it is sufficient to show that if $\operatorname{supp} f = [a,b]$, then $\operatorname{supp} W^{-1}f \subset \operatorname{supp} f$. Thus, we first need to show that

$$\int W^{-1}f(x)\overline{\psi(x)}\, dM(x) = 0 \quad \forall\, \psi \in D_{W'^{-1}} \in L^2_{dM}$$
$$\text{such that } \operatorname{supp}\psi \cap \operatorname{supp} f = \varnothing.$$

From equation (2),

$$\int W^{-1}f(x)\overline{\psi(x)}\, dM(x) = \int f(x)\overline{W^{-1'}\psi}\, dx$$
$$= \int f(x)\overline{WG^{-1}\psi(x)}\, dx.$$

Recall that WG^{-1} is support preserving and therefore $\operatorname{supp} f \cap \operatorname{supp} WG^{-1}\psi \subset \operatorname{supp} f \cap \operatorname{supp}\psi = \varnothing$, thus

$$\int W^{-1}f(x)\overline{\psi(x)}\, dM(x) = 0. \tag{5}$$

To end the proof we need to see that $D_{W^{-1'}}$ is dense in L^2_{dM}. From the previous remark W and W^{-1} are densely defined. Therefore $W^{-1'} = W'^{-1}$, thus

$$\overline{D_{W^{-1'}}} = \overline{D_{W'^{-1}}} = \overline{R_{W'}} = \{\operatorname{Ker} W\}^{\perp} = 0^{\perp} = L^2_{dM}.$$

Hence we have that $W^{-1'}$ is a densely defined operator. Denote by

$$L_K \equiv \{\psi(x) \in L^2_{dM(t)}/\operatorname{supp}\psi(x) \subset K\}.$$

Then clearly $L_K \subset L^2_{dM(t)}$; we have $D_{W^{-1'}} \cap L_K$ dense in L_K, and so equation (5) means $\operatorname{supp} W^{-1} f = 0$ if $\operatorname{supp} f \cap K = \varnothing$. Therefore, W^{-1} is S.P. $\square$

We can also use equation (4) to obtain a more practical result, which is the main result in this section.

Theorem 8. *Assume that condition* [A] *holds and let W be* S.P. *If $(d\Gamma/d\lambda)(\lambda)$ is analytic, then W^{-1} is* S.P. *and the Paley-Wiener property holds.*

Proof. It is sufficient to observe that equation (4) holds and

$$W^{-1} \equiv 2\pi W' \frac{d\Gamma}{d\lambda}\left(\frac{-id}{ix}\right).$$

Therefore

$$\begin{aligned}\int W^{-1} f(x)\overline{\psi(x)}\, dM(x) &= 2\pi \int W' \frac{d\Gamma}{d\lambda}\left(\frac{-id}{ix}\right) f(x)\bar{\psi}(x)\, dM(x)\\ &= 2\pi \int \frac{d\Gamma}{d\lambda}\left(\frac{-id}{ix}\right) f(x)\overline{W\psi(x)}\, dx\\ &= 0.\end{aligned}$$

Since $(d\Gamma/d\lambda)(-id/dx)$ and W are S.P. operators and $\operatorname{supp} f \cap \operatorname{supp}\psi = \varnothing$. To end the proof use the fact that $W^{-1'}$ is densely defined.

Proposition 9. *Let $W' \equiv 1 + X_+^*$ where X_+^* is a lower triangular Volterra operator with respect to the chain $P_\xi \equiv 1_{[-|\xi|,|\xi|]}(t)$, then the Paley-Wiener property holds.*

Proof. This is a simple consequence from the fact that the inverse of a Volterra operator of the second kind is a Volterra operator of the second kind, see [**4**]. □

What remains is to obtain simple conditions such that W is S.P.

If the solution

$$y(x,\lambda) = \sum_{n\geq 0} a_n(x)\lambda^n e^{i\lambda x}$$

$$y(x,\lambda) = \sum_{n\geq 0} a_n(x)\left(\frac{-id}{ix}\right)^n e^{i\lambda x}$$

Then formally

$$Wf(x) \equiv \sum_{n\geq 0} \left(\frac{-id}{ix}\right)^n [a_n(x)f(x)]$$

and so W is S.P..

We also have

Proposition 10. *Let*

$$y(x,\lambda) = P(\lambda)e^{i\lambda x} + \int_{-|x|}^{|x|} \sum_{n\geq 0} a_n(x,t)\frac{d^n}{dt^n}e^{i\lambda t}\,dt$$

where $P(\lambda)$ is entire. Then W is W.S.P.

Proof. This defines the shift operator explicitly, see [**1**].

$$W'f \equiv P\left(\frac{-id}{ix}\right)f + \int_{-|x|}^{|x|} \sum_{n\geq 0} a_n(x,t)\frac{d^n}{dt^n}f(t)\,dt.$$

Therefore

$$Wf \equiv \overline{P}\left(\frac{-id}{ix}\right)f(x) + \sum_{n\geq 0} \frac{d^n}{dx^n}\int_{-\infty}^{-|x|} a_n(t,x)f(t)\,dt$$
$$+ \sum_{n\geq 0} \frac{d^n}{dx^n}\int_{|x|}^{-\infty} a_n(t,x)f(t)\,dt. \qquad \square$$

Corollary 11. *Let $y(x,\lambda)$ be an entire function of λ of order one and type $|a(x)|$. If $|a(x)|$ is increasing, then W is* W.S.P.

As a consequence of the special factorization, see [**4**], one obviously obtains a necessary condition

Proposition 12. *Let $W' \equiv 1 + X_+^*$ where X_+^* is a Volterra operator. Then $2\pi(d\Gamma/d\lambda)(-id/dx) - 1 \in \sigma_\infty$, i.e., is a compact operator.*

3. Examples.

A) Consider the following self-adjoint operator in L^2_{dx}

$$L(y) \equiv \frac{idy}{dx} + q(x)y, \quad x \in (-\infty,\infty)$$

where $q(x) \in L^{1,\text{loc}}_{dx}$. The eigenfunctionals are solutions of

$$\begin{cases} iy'(x,\lambda) + q(x)y(x,\lambda) = \lambda y(x,\lambda) \\ y(0,\lambda) = 1. \end{cases}$$

Thus

$$y(x,\lambda) = e^{-i\lambda x} e^{i\int_0^x q(t)\,dt},$$

i.e.,

$$Wf(x) = f(x)e^{-i\int_0^x q(t)\,dt}.$$

In this case W and W^{-1} are both S.P. since $|Wf| = |f|$.

B) Consider the operator defined by

$$L(f) = \frac{-i}{w(x)}\frac{df}{dx} \quad x \in (-\infty, \infty)$$

where $w(x) \geq 0$ and $w(x) \in L^{1,\mathrm{loc}}_{dx}$. Hence L is self adjoint in L^2_{wdx}. The eigenfunctionals are given by

$$y(x, \lambda) = e^{+i\lambda \int_0^x w(s)\,ds}.$$

Using the definition of the operator W

$$F_y(f)(x) = \int e^{+i\lambda \int_0^x w(s)\,ds} f(x)\,dx.$$

Therefore

$$Wf(x) = f(a(x))a'(x)$$

where $a(x)$ is the inverse of $\int_0^x w(s)\,ds$.

C) Second order differential operators. It is well known that

$$\begin{cases} Lf \equiv \dfrac{-d^2}{dx^2} f(x) + q(x)f(x), \quad x \geq 0 \\ nf(0) - f'(0) = 0, \end{cases}$$

defines a self-adjoint operator in $L^2_{dx}[0, \infty)$.

The eigenfunctionals are solutions of

$$\begin{cases} -y''(x, \lambda) + q(x)y(x, \lambda) = \lambda y(x, \lambda) \\ y(0, \lambda) = 1 \text{ and } y'(0, \lambda) = n. \end{cases}$$

Gelfand and Levitan have shown the existence of two functions $H(x, t)$ and $K(x, t)$ such that

$$\begin{cases} y(x, \lambda) = \cos(x\sqrt{\lambda}) + \int_0^x K(x, t)\cos(t\sqrt{\lambda})\,dt \\ \cos(x\sqrt{\lambda}) = y(x, \lambda) + \int_0^x H(x, t)y(t, \lambda)\,dx. \end{cases}$$

In this case the operator W is given by

$$W f(x) = f(x) + \int_x^\infty K(t,x) f(t)\,dt$$
$$W^{-1} f(x) = f(x) + \int_x^\infty H(t,x) f(t)\,dt.$$

Clearly, W and W^{-1} are W.S.P. in $L^2_{(0,\infty)}$.

D) Generalized second order differential operators. Consider the following self-adjoint operator acting in the Hilbert space $L^2_{w(x)\,dx}$ and defined by

$$\begin{cases} Lf \equiv \dfrac{-1}{w(x)} \dfrac{d^2}{dx^2} f, & x \geq 0 \\ f'(0) - n f(0) = 0 \end{cases}$$

where $w(x) \geq 0$, $w(x) \asymp x^\alpha$ as $x \to 0$, $w(x) \in L^{1,\mathrm{loc}}_{dx}$ and $\alpha + 1 > 0$. It is known that the eigenfunctionals $\varphi(x,\lambda)$ are solutions of

$$\begin{cases} \dfrac{-1}{w(x)} \dfrac{d^2}{dx^2} \varphi(x,\lambda) = \lambda \varphi(x,\lambda) \\ \varphi(0,\lambda) = 1, \qquad \varphi'(0,\lambda) = n. \end{cases}$$

Clearly, $\varphi(x,\lambda)$ is entire and satisfies $|\varphi(x,\lambda)| \leq e^{\sqrt{|\lambda|}t(x)}$ where $t(x) = \sqrt{2x \int_0^x w(s)\,ds}$. As $\lambda \to \infty$ we have the following asymptotics derived from the WKB method,

$$\varphi(x,\lambda) \asymp \sqrt{\frac{\xi(x)}{p(x)}} \Big\{ c_1 \lambda^{-1/(2(\alpha+2))} \mathcal{J}_{1/(\alpha+2))}(\xi(x)) + c_2^{1/(2(\alpha+2))} \mathcal{J}_{-1/(\alpha+2)}(\xi(x)) \Big\}$$

where $\alpha + 2 > 1$, $p(x) = \sqrt{\lambda w(x)}$, $\xi(x) = \int_0^x p(t)\,dt$, and c_1 and c_2 are just constants. For fixed x, the above estimates show that $\varphi(x,\lambda) = O(\lambda^{1/2})$. Hence, for all $x > 0$,

$$\frac{\varphi(x,\lambda) - \cos(x\sqrt{\lambda}) - n\frac{\sin(x\sqrt{\lambda})}{\sqrt{\lambda}}}{\lambda} \in L^2_{(0,\infty)}$$

is entire of order 1 and type $b(x) \equiv \max(x, t(x))$. In this case, by the Paley-Wiener theorem, there exists a function $K(x,t)$ such that

$$\varphi(x,\lambda) - \cos(x\sqrt{\lambda}) - n\frac{\sin(x\sqrt{\lambda})}{\sqrt{\lambda}} = \lambda \int_0^{b(x)} K(x,t)\cos(t\sqrt{\lambda})\,dt,$$

which can be rewritten as

(6)

$$\varphi(x,\lambda) = \cos(x\sqrt{\lambda}) - n\int_0^x \cos(t\sqrt{\lambda})\,dt + \lambda\int_0^{b(x)} K(x,t)\cos(t\sqrt{\lambda})\,dt.$$

The operator W can be obtained very easily by using the definition

$$Wf(x) = f(x) - n\int_x^\infty f(t)\,dt + \frac{-d^2}{dx^2}\int_{a(x)}^\infty K(t,x)f(t)\,dt$$

where $a(b(x)) = x$, i.e., the inverse of the function $b(x)$. Therefore, W is W.S.P. For the inverse we need to write equation (6) as a Volterra type equation. The inverse would be

$$\cos(x\sqrt{\lambda}) = \varphi(x,\lambda) + \int_0^{b(x)} R(x,t,\lambda)\varphi(t,\lambda)\,dt,$$

where $R(x,t,\lambda) \equiv \sum_{n\geq 0} a_n(x,t)\lambda^n$, i.e., is an entire function of λ. Thus W^{-1} can be written as

$$W^{-1}f(x) = f(x) + \int_{a(x)}^\infty \sum_{n\geq 0} a_n(x,t)L^n f(t)\,dt$$

and since $Lf = (-1/w(x))(d^2/dx^2)f$ is S.P., we deduce that W^{-1} is W.S.P. in the following way.

Here

$$\operatorname{supp} f \subset [0,\gamma] \Longleftrightarrow \operatorname{supp} Wf \subset [0, b(\gamma)].$$

E) The following operator was studied in [**2**].

$$Lf \equiv \frac{-1}{A(x)}\frac{d}{dx}\left(A(x)\frac{d}{dx}f\right) \quad x \geq 0$$

where $A(x) \geq 0$ and $1/A(x) \in L^{1,\text{loc}}$. This defines a symmetric operator in $L^2_{A(x)\,dx}[0,\infty)$. Let the eigenfunctions $\varphi(x,\lambda)$ be solutions of

$$\begin{cases} \dfrac{-1}{A(x)}\dfrac{d}{dx}\left(A(x)\,\dfrac{d}{dx}\,[\varphi(\lambda)]\right) = \lambda\varphi(\lambda) \\ \varphi(0,\lambda) = 0 \qquad \lim_{x\to 0} A(x)\varphi'(x,\lambda) = 1. \end{cases}$$

Clearly, if we set

$$t(x) = \int_0^x \frac{1}{A(s)}\,ds \quad \text{and therefore} \quad A(x)\frac{d}{dx} = \frac{d}{dt},$$

$$\begin{aligned} y(t,\lambda) &\equiv \varphi(x(t),\lambda) \\ w(t) &\equiv [A(x(t))]^2 \end{aligned}$$

then

$$\frac{-1}{w(t)}\frac{d^2}{dt^2}y(t,\lambda) \equiv \lambda y(t,\lambda).$$

Then from $y(t,\lambda) \equiv \varphi(x(t),\lambda)$ and the previous example, see equation (6), we deduce that W and W^{-1} are W.S.P. with rescaled support.

F) Let us consider the generalized second order differential operator,

$$Lf \equiv \frac{-1}{p(x)}\frac{d^2}{dx^2}f(x) + \frac{q(x)}{p(x)}f(x), \quad x \geq 0,$$

where $p(x) \geq 0$ and $q(x) \in L^{1,\text{loc}}$. Clearly L is symmetric in $L^2_{p(x)\,dx}$. If the spectrum is bounded below, then there exists a $\lambda_0 < 0$ such that

$$Ly(x,\lambda_0) = \lambda_0 y(x,\lambda_0)$$

and $y(x,\lambda_0) > 0$. Let us set $u(x) \equiv y(x,\lambda_0)$ and clearly $y(x,\lambda)/u(x)$ is a solution of

$$u^2(x)\frac{d}{dx}\left[u^2(x)\frac{d}{dx}\left(\frac{y(x,\lambda)}{u(x)}\right)\right] + (\lambda-\lambda_0)p(x)u^4(x)\left(\frac{y(x,\lambda)}{u(x)}\right).$$

In this case the change of variable is obvious

$$\varphi(\eta,\lambda) \equiv \frac{y(x(\eta),\lambda)}{u(x(\eta))} \quad \text{and} \quad \eta(x) \equiv \int_0^x \frac{1}{u^2(s)}\,ds.$$

Hence $\varphi(\eta,\lambda)$ is a solution of

$$\frac{-1}{u^4p(x(\eta))}\frac{d^2}{d\eta^2}\varphi(\eta,\lambda)+(\lambda-\lambda_0)\varphi(\eta,\lambda)=0.$$

The transition operator in this case is defined by

$$\varphi(\eta,\lambda)\equiv\frac{y(x(\eta),\lambda)}{u(x(\eta))}.$$

By using equation (6), we have that W and W^{-1} are W.S.P.

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