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WIENER-HOPF-HANKEL OPERATORS FOR SOME WEDGE DIFFRACTION PROBLEMS WITH MIXED BOUNDARY CONDITIONS

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ABSTRACT. An operator theoretic approach is used to study problems of diffraction of time-harmonic electromagnetic (or acoustic) waves by right angle wedges Ω_w such that one of the faces is perfectly conducting (soft) and the other either nonconducting (hard) or imperfectly conducting (with a finite impedance). The correspondent boundary value problems for the two dimensional Helmholtz equation are shown to be well posed in the energy space $H_1(\mathbf{R}^2\backslash\overline{\Omega}_w)$. These problems are reduced to equivalent integral equations in $L_2^+(\mathbf{R})$ of Wiener-Hopf-Hankel type, which can be explicitly solved by obtaining canonical generalized factorizations of certain nonrational 2×2 matrix-valued symbols.

1. Introduction. In this paper we consider the diffraction problem of an electromagnetic (or acoustic) wave by a rectangular wedge $\{(x, y, z) \in \mathbf{R}^3 : x < 0, y < 0\}$ one of whose faces is perfectly conducting (or soft) and the other face has a prescribed impedance, either finite or infinite. The wedge is supposed to be immersed in a homogeneous and lossy medium, and we assume a time-harmonic incident field with only one component, parallel to the edge $x = y = 0$, $z \in \mathbf{R}$ of the wedge.

Splitting the total field into the incident and diffracted field, the above assumptions, together with Maxwell's equations, lead to the following boundary value problem $\mathcal{P}_\lambda$ for the two dimensional Helmholtz equation in the exterior of $\overline{\Omega}_w = \{(x, y) \in \mathbf{R}^2; x \leq 0, y \leq 0\}$,

$$(1.1)\qquad (\Delta + k_0^2)u(x, y) = 0, \qquad (x, y) \in \Omega = \mathbf{R}^2\backslash\overline{\Omega}_w$$

$$(1.2)\qquad \left(\frac{\partial}{\partial y} - \lambda\right) u(x, 0+) = f(x), \quad x < 0$$

$$(1.3)\qquad u(0+, y) = g(y), \quad y < 0$$

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where Δ denotes the Laplacian, u is the scalar potential associated to the diffracted field, f and g correspond with the incident field on the boundary $\partial\Omega$, k_0 stands for the wave number which, as usual, we assume to fulfill the conditions

$$\operatorname{Re} k_0 > 0, \qquad \operatorname{Im} k_0 > 0, \tag{1.4}$$

and $\lambda = k_0/\eta$, for η denoting the normalized impedance of the face $\{(x, 0, z) \in \mathbf{R}^3 : x < 0\}$ of the wedge. We also have either $\operatorname{Re} \lambda > 0$ and $\operatorname{Im} \lambda > 0$ or $\lambda = 0$ (for the hard/soft case).

For the particular case of an incident plane wave, this problem has been studied by several authors (see [**1, 6, 8, 14, 15, 21**]), who had proposed different function-theoretic methods to obtain its particular solution in analytical form.

Here we shall present a new operator-theoretic approach to the problem, in a Sobolev space setting, following a method already developed in [**19, 20**] for the perfectly conducting wedge. This yields the representation of the resolvent as a continuous operator between the data and resolution spaces and makes it possible to obtain more information about a priori estimates, stability, regularity, asymptotic behavior and well-posed settings for numerical methods.

In Section 2, after giving a rigorous formulation to the boundary value problem (1.1)–(1.3) in the energy norm space $H_1(\Omega)$, for general data in the trace spaces $H_{\pm 1/2}(\mathbf{R}^-)$, we decompose the problem into an auxiliary Sommerfeld half-plane type problem and a particular wedge problem (with a homogeneous Dirichlet boundary condition instead of (1.3)). The former can be explicitly solved by inverting an equivalent scalar Wiener-Hopf operator (see [**16, 19**]). The latter (Problem $\mathcal{P}^0_\lambda$) is seen to be equivalent, in the sense of Theorem 3.3, to a system of pseudodifferential equations of combined Wiener-Hopf and Hankel type. This system will be discussed separately in Sections 4 and 5 for the cases $\lambda = 0$ and $\lambda \neq 0$, respectively. It turns out that for the mixed boundary value problem ($\lambda = 0$) the system can be reduced, by symmetrization and lifting, to a scalar Hankel integral equation in $L_2^+(\mathbf{R})$ which is uniquely solvable, thus yielding the existence and uniqueness of the solution to this boundary value problem, as well as an explicit formula for it.

For the impedance case ($\lambda \neq 0$), the system of pseudodifferential equations is no more decoupled as before. However, by exploiting some particular symmetry properties, we reduce it to a scalar second-order integral equation of Wiener-Hopf type (see (5.11)), where an undetected compatibility condition becomes apparent. Then the incorporation of this condition allows us to deduce a new scalar equation involving only the sum of a Wiener-Hopf and a Hankel operator. The study of the solvability properties of this last equation can be done by associating with it a certain Riemann-Hilbert problem in $[L_2^+(\mathbf{R})]^2$ with a 2×2 symbol of piecewise-continuous functions, which is seen to be of normal type and has zero index [**11**]. But, unfortunately, it is a nonrational symbol that does not fit into the classes for which factorization methods are presently available (see [**7**] and [**13**]). Therefore, the invertibility of the correspondent operator, and consequently the existence and uniqueness of solution to the impedance boundary-value problem, needs a more sophisticated approach and is planned to be treated in a forthcoming publication.

2. Formulation of $\mathcal{P}_\lambda$ and reduction to a simpler boundary value problem. In what follows, we will use the same notations and basic results which were introduced already in [**20**, Section 2] and can be consulted for details.

Let $\mathbf{R}^\pm = \{x \in \mathbf{R} : \pm x > 0\}$ and Ω denote the unbounded Lipschitz domain defined in (1.1), whose boundary we represent as the union of the origin with the half-lines

$$\Gamma_1 = \{(x,0) : x < 0\} \quad \text{and} \quad \Gamma_2 = \{(0,y) : y < 0\}, \tag{2.1}$$

which we will identify with $\mathbf{R}^-$.

Because of energy considerations, the natural space to formulate the boundary value problem (1.1)–(1.3) is the Sobolev space $H_1(\Omega)$. Then, by the trace theorems, we must take the Cauchy data in the trace spaces $H_{-1/2}(\Gamma_1)$ and $H_{1/2}(\Gamma_2)$, which makes clear the following general formulation to problem (1.1)–(1.3) (see [**3, 5, 12**] for background):

Problem $\mathcal{P}_\lambda$. Find a weak solution $u \in H_1(\Omega)$ of the Helmholtz equation

$$(\Delta + k_0^2)u = 0 \quad \text{in } \Omega, \tag{2.2}$$

which satisfies the boundary conditions

$$\left(\frac{\partial}{\partial y} - \lambda\right) u\Big|_{\Gamma_1} = f \tag{2.3}$$

and

$$u_{|_{\Gamma_2}} = g \tag{2.4}$$

where $u_{|\Gamma_i}$ represents the trace of u on Γ_i $(i = 1, 2)$, $\partial u/\partial y_{|\Gamma_1}$ denotes the trace of $\partial u/\partial y$ on Γ_1 and $f \in H_{-1/2}(\mathbf{R}^-)$, $g \in H_{1/2}(\mathbf{R}^-)$ are given distributions.

In order to get a simpler problem which afterwards can be reduced to certain pseudodifferential equations of Wiener-Hopf-Hankel type, it is convenient to consider firstly an auxiliary Sommerfeld "half-plane" type problem (corresponding to the "wedge face" Γ_2). For this, let

$$\mathbf{R}^2_r = \{(x, y) : x > 0\}, \qquad \mathbf{R}^2_l = \mathbf{R}^2 \backslash \overline{R}^2_r$$

denote the right and left half-spaces, and let $\Gamma'_2 = \{(0, y) : y > 0\}$ represent the "complementary half-plane."

Problem $\mathcal{P}_S$. Find a weak solution $w \in L_2(\mathbf{R}^2)$, with $w^{l,r} = w_{|_{\mathbf{R}^2_{l,r}}} \in H_1(\mathbf{R}^2_{l,r})$, which is a solution to the Helmholtz equation

$$(\Delta + k_0^2)w = 0 \quad \text{in } \mathbf{R}^2_{l,r} \tag{2.5}$$

and satisfies the boundary conditions

$$w_0^{l,r} = w^{l,r}(x, \cdot)_{|x=\mp 0} = g \quad \text{in } \Gamma_2 \tag{2.6}$$

and the transmission conditions

$$\begin{matrix} w_0^l - w_0^r = 0 \\ w_1^l - w_1^r = 0 \end{matrix} \quad \text{in } \Gamma'_2 \tag{2.7}$$

where $w_0^{l,r} \in H_{1/2}(\mathbf{R})$ and $w_1^{l,r} \in H_{-1/2}(\mathbf{R})$ are the traces on $x = 0$ of $w^{l,r}$ and $\partial w^{l,r}/\partial x$, respectively, and $g \in H_{1/2}(\mathbf{R}^-)$ is given.

For this problem, there is a unique solution w, which is easily obtained explicitly by inverting an equivalent scalar Wiener-Hopf operator (see, e.g., [**10**, Section 2], [**16** or **19**]). Furthermore, it was shown in [**20**] that this function w is such that $w \in H_1(\mathbf{R}^2)$ (which is a consequence of imposing the same boundary data g on both sides of the screen) and it is also a weak solution of the Helmholtz equation in $\mathbf{R}^2\backslash\overline{\Gamma}_2$ (due to the transmission conditions (2.7)). Therefore, $w_{|\Omega}$ is a function in $H_1(\Omega)$ which satisfies the Helmholtz equation in Ω, whose trace on the boundary $\partial\Omega$ belongs to $H_{1/2}(\partial\Omega)$. This trace is given by

$$w\Big|_{\partial\Omega} = \begin{cases} w\Big|_{\Gamma_1} & \text{in } \Gamma_1 \\ g & \text{in } \Gamma_2. \end{cases} \tag{2.8}$$

Then, by superposition and the use of the substitution

$$v = u - w\Big|_{\Omega} \tag{2.9}$$

we reduce Problem $\mathcal{P}_\lambda$ to the following one:

Problem $\mathcal{P}_\lambda^0$. Find a solution $v \in H_1(\Omega)$ to the Helmholtz equation

$$(\Delta + k_0^2)v = 0 \quad \text{in } \Omega \tag{2.10}$$

such that

$$\left(\frac{\partial}{\partial y} - \lambda\right) v\Big|_{\Gamma_1} = f' \tag{2.11}$$

and

$$v\Big|_{\Gamma_2} = 0 \tag{2.12}$$

where $f' \in H_{-1/2}(\mathbf{R}^-)$ is a given distribution.

Indeed, just by linearity, we have

Proposition 2.1. *Problems* $\mathcal{P}_\lambda$ *and* $\mathcal{P}_\lambda^0$ *are equivalent in the following sense*:

(i) *Given* $f \in H_{-1/2}(\mathbf{R}^-)$ *and* $g \in H_{1/2}(\mathbf{R}^-)$ *and letting* w *be a solution to Problem* $\mathcal{P}_S$, *then* u *is a solution to Problem* $\mathcal{P}_\lambda$ *if and only if the function* v *defined in* (2.9) *is a solution to Problem* $\mathcal{P}_\lambda^0$ *with* $f' = f - (\partial w/\partial y - \lambda w)_{|\Gamma_1}$.

(ii) *Problem* $\mathcal{P}_\lambda$ *is uniquely solvable if and only if Problem* $\mathcal{P}_\lambda^0$ *is uniquely solvable.*

The above proposition reduces our original Problem $\mathcal{P}_\lambda$ to Problem $\mathcal{P}_\lambda^0$, on which we will concentrate our attention from now on.

The relevance of this reduction lies in the fact that in Problem $\mathcal{P}_\lambda^0$ we have a homogeneous Dirichlet boundary condition on Γ_2. As it was shown in [**19**, **20**], this means that we can give a further equivalent formulation to Problem $\mathcal{P}_\lambda$ as a mixed boundary-transmission problem of the type of Sommerfelds' half-plane problem, then allowing its reduction to an equivalent Wiener-Hopf-Hankel system of equations (see the next section).

Indeed, if there exists a solution $v \in H_1(\Omega)$ of Problem $\mathcal{P}_\lambda^0$, its restriction v^- to the third quadrant Q_3 of $\mathbf{R}^2$ is a function in $H_1(Q_3)$ with zero trace on Γ_2. Then, its odd extension to $\mathbf{R}^2_- = \{(x,y) \in \mathbf{R}^2 : y < 0\}$, given by

$$v_{\mathfrak{o}}^-(x,y) = \begin{cases} v^-(x,y), & (x,y) \in Q_3 \\ -v^-(-x,y), & (x,y) \in Q_4 \end{cases}$$

is a function in $H_1(\mathbf{R}^2_-)$ which furthermore satisfies the Helmholtz equation in $\mathbf{R}^2_-$ (see [**20**, Section 3]). Therefore, it is straightforward to verify that Problem $\mathcal{P}_\lambda^0$ has the following equivalent formulation, where $\mathbf{R}^2_+$ stands for the upper half-space and $\Gamma_1' = \{(x,0) : x > 0\}$.

Reformulation of $\mathcal{P}_\lambda^0$. Given $f' \in H_{-1/2}(\mathbf{R}^-)$, look for a function $v \in L_2(\mathbf{R}^2)$, such that $v^+ = v_{|\mathbf{R}^2_+} \in H_1(\mathbf{R}^2_+)$ satisfies the Helmholtz equation

$$(\Delta + k_0^2)v^+ = 0 \quad \text{in } \mathbf{R}^2_+ \tag{2.14}$$

$v_{\mathfrak{o}}^- = v_{|\mathbf{R}^2_-} \in H_1(\mathbf{R}^2_-)$ is an odd function (with respect to x), which is a solution of the Helmholtz equation

$$(\Delta + k_0^2)v_0^- = 0 \quad \text{in } \mathbf{R}^2_-; \tag{2.15}$$

v^+ satisfies the boundary condition

$$v_1^+ - \lambda v_0^+ = f' \quad \text{in } \Gamma_1 \tag{2.16}$$

where $v_1^+ = \partial v/\partial y(\cdot,y)_{|y=+0} \in H_{-1/2}(\mathbf{R})$ and $v_0^+ = v(\cdot,y)_{|y=+0} \in H_{1/2}(\mathbf{R})$. Moreover, the following transmission conditions must be verified:

$$v_0^+ - v_{\mathfrak{o}0}^- = 0 \tag{2.17}$$

$$\text{in } \Gamma_1'$$

$$v_1^+ - v_{\mathfrak{o}1}^- = 0 \tag{2.18}$$

where $v_{\mathfrak{o}0}^- = v_{\mathfrak{o}}^-(\cdot,y)_{|y=-0} \in H_{1/2}(\mathbf{R})$ and $v_{\mathfrak{o}1}^- = \partial v_{\mathfrak{o}}^-/\partial y(\cdot,y)_{|y=-0} \in H_{-1/2}(\mathbf{R})$.

3. Equivalence to a system of integral equations.

The last formulation to Problem $\mathcal{P}_\lambda^0$ makes it possible to use the so-called representation formula for the solution of Sommerfeld half-plane type problems in terms of the Cauchy data on $y=0$. From [**10**, Theorem 2.1] we readily obtain

Theorem 3.1. *A function $v \in L_2(\mathbf{R}^2)$ with $v_{|\mathbf{R}^2_\pm} \in H_1(\mathbf{R}^2_\pm)$ is a solution to* (2.14)–(2.18) *if and only if it is represented by*

$$v(x,y) = \mathcal{F}^{-1}_{\xi\to x}\{\hat{v}_0^+(\xi)e^{-t(\xi)y}h(y) + \hat{v}_{\mathfrak{o}0}^-(\xi)e^{t(\xi)y}h(-y)\} \tag{3.1}$$

for almost all $(x,y) \in \mathbf{R}^2$, where

$$\hat{v}_0^+(\xi) = \mathcal{F}_{x\to\xi}v_0^+(x) = \int_{\mathbf{R}} e^{ix\xi}v_0^+(x)\,dx, \qquad \hat{v}_{\mathfrak{o}0}^-(\xi) = \mathcal{F}_{x\to\xi}v_{\mathfrak{o}0}^-(x) \tag{3.2}$$

are the Fourier transforms of the data $v_0^+, v_{\mathfrak{o}0}^-$ in the trace space $H_{1/2}(\mathbf{R})$, h represents the Heaviside unit-step function and

$$t(\xi) = (\xi^2 - k_0^2)^{1/2}, \quad \xi \in \mathbf{R} \tag{3.3}$$

for the branch of the square root that tends to $+\infty$ as $\xi \to \pm\infty$, with branch cuts $\Gamma^\pm = \{z \in \mathbf{C} : z = \pm k_0 \pm i\tau,\ \tau > 0\}$.

It is convenient to introduce as a new *ansatz* the jump vector (see (2.17), (2.18)):

$$\begin{pmatrix} \varphi_0^- \\ \varphi_1^- \end{pmatrix} = \begin{pmatrix} v_0^+ - v_{\mathfrak{o}0}^- \\ v_1^+ - v_{\mathfrak{o}1}^- \end{pmatrix} \in H^-_{1/2}(\mathbf{R}) \times H^-_{-1/2}(\mathbf{R}) \tag{3.4}$$

where $H^-_{\pm 1/2}(\mathbf{R})$ denote the closed subspaces of $H_{\pm 1/2}(\mathbf{R})$ formed by all the functions (distributions) whose support is contained in $\overline{\mathbf{R}}^-$ [**4**]. As an easy consequence of Theorem 3.1, we have

Corollary 3.2. *The Dirichlet data* (*on* $y = 0$) *of a solution* v *to* (2.14)–(2.18) *are related to the jump vector* $(\varphi_0^-, \varphi_1^-)^T$ *by*

$$\begin{pmatrix} v_0^+ \\ v_{\mathfrak{o}0}^- \end{pmatrix} = B \begin{pmatrix} \varphi_0^- \\ \varphi_1^- \end{pmatrix} \tag{3.5}$$

where $B = \mathcal{F}^{-1}\sigma_B\mathcal{F} : H_{1/2}(\mathbf{R}) \times H_{-1/2}(\mathbf{R}) \to [H_{1/2}(\mathbf{R})]^2$ *is the invertible convolution* (*or pseudodifferential*) *operator with symbol*:

$$\sigma_B = -\frac{1}{2}\begin{bmatrix} -1 & \frac{1}{t} \\ 1 & \frac{1}{t} \end{bmatrix}. \tag{3.6}$$

Proof. The result is obtained by direct computation, noting that (3.1) yields

$$v_0^+ = -Av_1^+ \quad \text{and} \quad v_{\mathfrak{o}0}^- = Av_{\mathfrak{o}1}^- \tag{3.7}$$

where A is the invertible pseudodifferential operator (of order -1)

$$A = \mathcal{F}^{-1}\,\frac{1}{t}\,\mathcal{F} : H_{-1/2}(\mathbf{R}) \to H_{1/2}(\mathbf{R}). \qquad \square \tag{3.8}$$

From relations (3.7) we also have

$$\begin{pmatrix} v_1^+ - \lambda v_0^+ \\ v_{\mathfrak{o}1}^- \end{pmatrix} = \mathcal{F}^{-1}\ \mathrm{diag}\,[-(t+\lambda), t]\mathcal{F}\begin{pmatrix} v_0^+ \\ v_{\mathfrak{o}0}^- \end{pmatrix} \tag{3.9}$$

and, therefore, by the use of (3.5), (3.6), it holds

$$C\begin{pmatrix} \varphi_0^- \\ \varphi_1^- \end{pmatrix} = \begin{pmatrix} v_1^+ - \lambda v_0^+ \\ v_{\mathfrak{o}1}^- \end{pmatrix} \tag{3.10}$$

where $C = \mathcal{F}^{-1}\sigma_c\mathcal{F} : H^-_{1/2}(\mathbf{R}) \times H^-_{-1/2}(\mathbf{R}) \to [H_{-1/2}(\mathbf{R})]^2$ is the pseudodifferential operator with symbol σ_c given by

$$\sigma_c = -\frac{1}{2}\begin{bmatrix} t+\lambda & -\frac{t+\lambda}{t} \\ t & 1 \end{bmatrix}. \tag{3.11}$$

In the usual Sommerfeld half-plane problem, the restrictions to $\mathbf{R}^-$ of the trace functions $v_1^+ - \lambda v_0^+$ and $v_{\mathfrak{o}1}^-$ are known. However, in our problem, only the boundary data function f' is given (see (2.16)), and the boundary condition in the lower bank of Γ_1 is replaced by the property of $v_{\mathfrak{o}1}^-$ to be odd. Let us now make use of this condition. For, let $r_\pm$ denote the restriction operators to $\mathbf{R}^\pm$, respectively, and let $l^{\mathfrak{o}} : H_{-1/2}(\mathbf{R}^\pm) \to H_{-1/2}(\mathbf{R})$ stand for the operators of odd extension which are continuous (see [**9**]). Obviously, we have $l^{\mathfrak{o}} r_- v_{\mathfrak{o}1}^- = l^{\mathfrak{o}} r_+ v_{\mathfrak{o}1}^- = l^{\mathfrak{o}} r_+ v_1^+$ where in the last equality we used the transmission condition (2.18). Consequently, $l^{\mathfrak{o}} r_- v_{\mathfrak{o}1}^- = (1/2) l^{\mathfrak{o}} r_+ (v_{\mathfrak{o}1}^- + v_1^+)$ or yet, from (3.7), (3.8),

$$l^{\mathfrak{o}} r_- v_{\mathfrak{o}1}^- = \frac{1}{2}\, l^{\mathfrak{o}} r_+ A^{-1}(v_{\mathfrak{o}0}^- - v_0^+) = -\frac{1}{2}\, l^{\mathfrak{o}} r_+ A^{-1}\varphi_0^- \tag{3.12}$$

where $A^{-1} = \mathcal{F}^{-1} t \mathcal{F}$ (see (3.8)). Moreover, if J denotes the reflection operator in $H_s(\mathbf{R})$ $(s \in \mathbf{R})$, defined by (for smooth functions in a dense subspace)

$$J\varphi(x) = \varphi(-x), \quad x \in \mathbf{R} \tag{3.13}$$

we get, from (3.12) by restriction to $\mathbf{R}^-$ together with the identity $r_- l^{\mathfrak{o}} r_+ = -r_- J$:

$$r_- v_{\mathfrak{o}1}^- = -\frac{1}{2}\, r_- l^{\mathfrak{o}} r_+ A^{-1}\varphi_0^- = \frac{1}{2}\, r_- J A^{-1}\varphi_0^- .$$

Furthermore, as $JA^{-1} = A^{-1}J$ due to the fact that t is an even function, we finally obtain

$$r_- v_{\mathfrak{o}1}^- = \frac{1}{2}\, r_- A^{-1} J \varphi_0^- . \tag{3.14}$$

Going back to the system of equations (3.10), by taking the restriction to $\mathbf{R}^-$ and using the boundary condition (2.16) and (3.14), we have

$$r_- C \begin{pmatrix} \varphi_0^- \\ \varphi_1^- \end{pmatrix} = \begin{pmatrix} f' \\ \frac{1}{2}\, r_- A^{-1} J \varphi_0^- \end{pmatrix} \tag{3.15}$$

from which we can derive a system of pseudodifferential equations for the ansatz $(\varphi_0^-, \varphi_1^-)^T$. It is convenient to start by representing C in the matrix operator form (see (3.11))

$$C = -\frac{1}{2}\begin{bmatrix} A_\lambda^{-1} & -A_\lambda^{-1}A \\ A^{-1} & I \end{bmatrix} : H_{1/2}^-(\mathbf{R}) \oplus H_{-1/2}^-(\mathbf{R}) \to H_{-1/2}(\mathbf{R}) \oplus H_{-1/2}(\mathbf{R}) \tag{3.16}$$

where A is the operator defined by (3.8), I stands for the identity operator in $H_{-1/2}(\mathbf{R})$ and A_λ denotes the bounded operator

$$A_\lambda = \mathcal{F}^{-1}\frac{1}{t+\lambda}\mathcal{F} : H_{-1/2}(\mathbf{R}) \to H_{1/2}(\mathbf{R}) \tag{3.17}$$

(since $t + \lambda \neq 0$ on $\mathbf{R}$) with inverse given by

$$A_\lambda^{-1} = \mathcal{F}^{-1}(t+\lambda)\mathcal{F}. \tag{3.18}$$

Then a straightforward computation yields

$$r_- \begin{bmatrix} A_\lambda^{-1} & -A_\lambda^{-1}A \\ A^{-1}(I+J) & I \end{bmatrix}\begin{pmatrix} \varphi_0^- \\ \varphi_1^- \end{pmatrix} = \begin{pmatrix} -2f' \\ 0 \end{pmatrix}. \tag{3.19}$$

Thus, every solution v to the boundary value problem (2.14)–(2.18) is such that their traces v_0^+ and $v_{\mathfrak{o}0}^-$ on $y = \pm 0$ determine a solution $(\varphi_0^-, \varphi_1^-)^T$ to the above system of pseudodifferential equations of Wiener-Hopf-Hankel type by (3.5).

Conversely, it is easy to recognize that if $(\varphi_0^-, \varphi_1^-)^T$ is a solution of (3.19), then $(v_0^+, v_{\mathfrak{o}0}^-)^T$ given by (3.5) are the traces on $y = \pm 0$ of a solution to the boundary-transmission problem (2.14)–(2.18), which is given by the representation formula in Theorem 3.1.

We summarize these results in the following equivalence theorem.

Theorem 3.3. *Problem $\mathcal{P}_\lambda^0$ is uniquely solvable if and only if the system of pseudodifferential equations* (3.19) *is uniquely solvable. Moreover, we have*:

(i) *If $(\varphi_0^-, \varphi_1^-)^T$ is a solution to* (3.19), *then the restriction to Ω of the function v in* (3.1) *with $(v_0^+, v_{\mathfrak{o}0}^-)$ given by* (3.5) *is a solution to Problem $\mathcal{P}_\lambda^0$.*

(ii) *If $v \in H_1(\Omega)$ is a solution to Problem $\mathcal{P}^0_\lambda$, then its trace v_0^+ on $y = +0$ and the odd extension $v_{\mathfrak{o}0}^-$ of its trace v_0^- on Γ_1' are such that* (*see* (3.5), (3.6))

$$\begin{pmatrix} \varphi_0^- \\ \varphi_1^- \end{pmatrix} = B^{-1} \begin{pmatrix} v_0^+ \\ v_{\mathfrak{o}0}^- \end{pmatrix} = \mathcal{F}^{-1} \begin{bmatrix} 1 & -1 \\ -t & -t \end{bmatrix} \mathcal{F} \begin{pmatrix} v_0^+ \\ v_{\mathfrak{o}0}^- \end{pmatrix} \tag{3.20}$$

is a solution to (3.19).

From now on, we shall concentrate our attention on system (3.19). For later convenience, we start by giving an alternative form to that system of equations. Therefore, let us consider its second equation and write it in the form

$$A^{-1}(I+J)\varphi_0^- + \varphi_1^- = \varphi_1^+ \tag{3.21}$$

where φ_1^+ is a distribution in $H_{-1/2}(\mathbf{R})$ whose support is contained in $\overline{R}^+$, i.e., $\varphi_1^+ \in H^+_{-1/2}(\mathbf{R})$. Moreover, by applying the reflection operator J to both sides of (3.22) and noting that $J^2 = I$ and $JA^{-1} = A^{-1}J$, we also have:

$$A^{-1}(J+I)\varphi_0^- + J\varphi_1^- = J\varphi_1^+. \tag{3.22}$$

Subtracting both sides of equations (3.21) and (3.22), we get

$$\varphi_1^- + J\varphi_1^+ = J\varphi_1^- + \varphi_1^+ \tag{3.23}$$

where the left-hand side is an element of $H^-_{-1/2}(\mathbf{R})$ and the right-hand side is an element of $H^+_{-1/2}(\mathbf{R})$. Then, because of $H^+_{-1/2}(\mathbf{R}) \cap H^-_{-1/2}(\mathbf{R}) = \{0\}$ (see [**4**]), it follows from (3.23) that $\varphi_1^- = -J\varphi_1^+$ or yet $\varphi_1^+ = -J\varphi_1^-$. So, substituting this result in (3.21), we conclude that the second of equations (3.19) can be written in the equivalent form:

$$A^{-1}(I+J)\varphi_0^- + (I+J)\varphi_1^- = 0. \tag{3.24}$$

Similarly, by introducing an auxiliary distribution $\psi^+ \in H^+_{-1/2}(\mathbf{R})$, we can write the first of equations (3.19) as

$$A_\lambda^{-1}\varphi_0^- - A_\lambda^{-1}A\varphi_1^- = -2l^{\mathfrak{o}} f' + \psi^+. \tag{3.25}$$

Furthermore, applying J to both sides of the above equation, we also get

$$A_\lambda^{-1} J\varphi_0^- - A_\lambda^{-1} A J\varphi_1^- = 2l^\circ f' + J\psi^+. \tag{3.26}$$

Let us introduce the vector

$$\underset{\sim}{\varphi}^+ = (J\varphi_0^-, J\varphi_1^-, \psi^+)^T \in H^+_{1/2}(\mathbf{R}) \times H^+_{-1/2}(\mathbf{R}) \times H^+_{-1/2}(\mathbf{R}) \tag{3.27}$$

and write equations (3.26), (3.25) and (3.24) in the matrix form

(3.28)
$$\begin{bmatrix} A_\lambda^{-1} & -A_\lambda^{-1}A & 0 \\ 0 & 0 & I \\ A^{-1} & I & 0 \end{bmatrix} \underset{\sim}{\varphi}^+ = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} l^\circ f' + \begin{bmatrix} 0 & 0 & I \\ A_\lambda^{-1} & -A_\lambda^{-1}A & 0 \\ -A^{-1} & -I & 0 \end{bmatrix} J\underset{\sim}{\varphi}^+$$

where J is now defined element-wise. Noting that

$$\begin{bmatrix} 0 & 0 & I \\ A_\lambda^{-1} & -A_\lambda^{-1}A & 0 \\ -A^{-1} & -I & 0 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} 0 & A_\lambda & -A \\ 0 & -A^{-1}A_\lambda & -I \\ 2I & 0 & 0 \end{bmatrix}$$

it follows easily from the above expressions that the system of equations (3.19) has the equivalent representation:

(3.29)
$$\frac{1}{2} \begin{bmatrix} -I & -A & A_\lambda \\ -A^{-1} & -I & -A^{-1}A_\lambda \\ 2A_\lambda^{-1} & -2A_\lambda^{-1}A & 0 \end{bmatrix} \underset{\sim}{\varphi}^+ = \begin{pmatrix} A_\lambda \\ -A^{-1}A_\lambda \\ 2 \end{pmatrix} l^\circ f' + J\underset{\sim}{\varphi}^+$$

with $\underset{\sim}{\varphi}^+$ defined by (3.27). It is clear that after applying the Fourier transform to both sides of (3.29), we get a particular Riemann-Hilbert problem for the real line involving $\mathcal{F}\underset{\sim}{\varphi}^+ = \underset{\sim}{\hat{\varphi}}^+$ and the reflected vector $J\underset{\sim}{\hat{\varphi}}^+$, thus having an additional symmetry condition. We shall use this fact in the next section to solve explicitly system (3.9) for the particular case $\lambda = 0$, corresponding to the Dirichlet/Neumann boundary value problem $\mathcal{P}_0^0$ (see (2.11)). The general case ($\lambda \neq 0$), corresponding to the impedance problem $\mathcal{P}_\lambda^0$, will be discussed in Section 5.

4. The mixed problem $\mathcal{P}_0^0$. Taking $\lambda = 0$ in (3.29) we have (see (3.17)):

$$\frac{1}{2}\begin{bmatrix} -I & -A & A \\ -A^{-1} & -I & -I \\ 2A^{-1} & -2I & 0 \end{bmatrix} \underset{\sim}{\varphi}^{+} = \begin{pmatrix} A_\lambda \\ -1 \\ 2 \end{pmatrix} l^{o} f' + J\underset{\sim}{\varphi}^{+}. \tag{4.1}$$

From the last two equations we can eliminate the auxiliary distribution ψ^+ and consequently we reduce (4.1) by a two by two system. For this, let us multiply by two the second equation and add it with the third. We get

$$-2J\varphi_1^- - \psi^+ = 2\varphi_1^- + J\psi^+ \tag{4.2}$$

where the left- and right-hand sides are in $H^+_{-1/2}(\mathbf{R})$ and $H^-_{-1/2}(\mathbf{R})$, respectively. Therefore,

$$\psi^+ = -2J\varphi_1^- . \tag{4.3}$$

Substituting this result in the first and the third of equations (4.1), we obtain

$$\frac{1}{2}\begin{bmatrix} -I & -3A \\ -A^{-1} & I \end{bmatrix} \begin{pmatrix} J\varphi_0^- \\ J\varphi_1^- \end{pmatrix} = \begin{pmatrix} A \\ -1 \end{pmatrix} l^{o} f' + \begin{pmatrix} \varphi_0^- \\ \varphi_1^- \end{pmatrix}. \tag{4.4}$$

Moreover, we can reduce this system to a scalar equation of different type. Indeed, from the second of equations (4.4), it holds that

$$J\varphi_0^- = A(J-2I)\varphi_1^- + 2Al^{o} f', \tag{4.5}$$

and thus,

$$\varphi_0^- = A(I-2J)\varphi_1^- - 2Al^{o} f', \tag{4.6}$$

expressing φ_0^- in terms of φ_1^- and the data f'. Putting (4.5) into the first of equations (4.4), we obtain precisely the same equation, which we write in the more suggestive form

$$-2AJ\varphi_1^- + A\varphi_1^- = 2Al^{o} f' + \varphi_0^- . \tag{4.7}$$

This equation is reducible to a Hankel equation in $L_2^+(\mathbf{R})$ (for the unknown $J\varphi_1^-$), as we show in the next proposition.

Let us introduce some notation. By $t_\pm$ we denote the square root functions

$$t_\pm(\xi) = (\xi \pm k_0)^{1/2} = |\xi \pm k_0|^{1/2}\, e^{\frac{i}{2}\arg(\xi\pm k_0)}, \quad \xi \in \mathbf{R} \tag{4.8}$$

with branch cuts $\Gamma^\pm$, respectively, and $\arg(\xi - k_0) \in\,]-3\pi/2, \pi/2]$, $\arg(\xi + k_0) \in\,]-\pi/2, 3\pi/2]$, such that $t = t_- t_+$ holds (see (3.3)). Also, let A_+ denote the invertible convolution operator (cf. [**4**])

$$A_+ = \mathcal{F}^{-1}\frac{1}{t_+}\,\mathcal{F} : H^+_{-1/2}(\mathbf{R}) \rightarrow L_2^+(\mathbf{R}) \tag{4.9}$$

with inverse given by

$$A_+^{-1} = \mathcal{F}^{-1} t_+^{-1} \mathcal{F} : L_2^+(\mathbf{R}) \rightarrow H^+_{-1/2}(\mathbf{R}). \tag{4.10}$$

We have:

Proposition 4.1 (Lifting to L_2^+). *Equation* (4.7) *is equivalent to the Hankel equation in* $L_2^+(\mathbf{R})$*:*

$$-2\varphi^+ - \mathcal{P}^+\mathcal{K}_c J\varphi^+ = f^+ \tag{4.11}$$

where $\mathcal{P}^+ : L_2(\mathbf{R}) \rightarrow L_2^+(\mathbf{R})$ *is the usual projection operator* (*multiplication by the characteristic function of* $\mathbf{R}^+$), $\mathcal{K}_c$ *the pseudodifferential operator*

$$\mathcal{K}_c = \mathcal{F}^{-1} k \mathcal{F} : L_2(\mathbf{R}) \rightarrow L_2(\mathbf{R}) \quad \textit{with symbol } k = -i\,\frac{t_-}{t_+} \tag{4.12}$$

and f^+ *denotes the function*

$$f^+ = 2\mathcal{P}^+ A_+ l^\circ f' \tag{4.13}$$

in the sense that there is a bijective (*and bicontinuous*) *relation between the solutions of* (4.7) *and* (4.11), *given by*

$$\varphi^+ = A_+ J\varphi_1^- \qquad (\varphi_1^- = J A_+^{-1}\varphi^+). \tag{4.14}$$

Proof. Suppose that φ_1^- is a solution to equation (4.7) for a given distribution f'. Then, by use of the Fourier transformation, we have (see (3.8))

$$-2\,\frac{1}{t}\,J\hat{\varphi}_1^- + \frac{1}{t}\,\hat{\varphi}_1^- = 2\,\frac{1}{t}\,\widehat{l^o f'} + \hat{\varphi}_0^- \tag{4.15}$$

where $\hat{\varphi}_1^- = \mathcal{F}\varphi_1^-$, etc. Now, let φ^+ be the function given by (4.14), whose Fourier transform is (see (4.9)) $\hat{\varphi}^+ = (1/t_+)J\hat{\varphi}_1^-$ which yields

$$J\hat{\varphi}_1^- = t_+\hat{\varphi}^+ \quad \text{and} \quad \hat{\varphi}_1^- = it_- J\hat{\varphi}^+ \tag{4.16}$$

since $Jt_+ = it_- J$ (see (4.8)). Substituting these results in (4.15) we get (by noting that $t = t_- t_+$)

$$-2\hat{\varphi}^+ + i\,\frac{t_-}{t_+}\,J\hat{\varphi}^+ = 2\,\frac{1}{t_+}\,\widehat{l^o f'} + t_-\hat{\varphi}_0^-.$$

Then, taking the inverse Fourier transformation and making use of (4.9) and (4.12), we have

$$-2\varphi^+ - \mathcal{K}_c J\varphi^+ = 2A_+ l^o f' + \psi^-$$

with $\psi^- = \mathcal{F}^{-1}t_-\hat{\varphi}_0^- \in L_2^-(\mathbf{R})$, i.e., ψ^- is an L_2-function supported in $\overline{R}^-$. Applying the operator $\mathcal{P}^+$ to both sides of the last equation, we see that φ^+ satisfies the Hankel equation (4.11). Reciprocally, it is now obvious that any solution φ^+ of (4.11) gives a solution φ_1^- of equation (4.7) through (see (4.16)) $\varphi_1^- = \mathcal{F}^{-1}it_-\mathcal{F}\hat{\varphi}^+ = J\mathcal{F}^{-1}t_+\mathcal{F}\varphi^+ = JA_+^{-1}\varphi^+$. □

Following the notation already used in [**18, 20**], let us write equation (4.11) in the form

$$\mathcal{K}_{-2}\varphi^+ = f^+ \tag{4.17}$$

with

$$\mathcal{K}_{-2} = -2I - \mathcal{K} \tag{4.18}$$

where I now denotes the identity operator on $L_2^+(\mathbf{R})$ and $\mathcal{K}$ is the Hankel operator

$$\mathcal{K} = \mathcal{P}^+\mathcal{K}_c J_{|_{L_2^+(\mathbf{R})}} : L_2^+(\mathbf{R}) \to L_2^+(\mathbf{R}). \tag{4.19}$$

A study of this class of Hankel equations can be found in [**18, 19**]. In particular, equation (4.17) was already considered in [**20**, Section 4 and Section 5]. Therefore, we will omit the details and will now present only the essential steps to show the invertibility of operator $\mathcal{K}_{-2}$.

Let $T : [L_2^+(\mathbf{R})]^2 \to [L_2^+(\mathbf{R})]^2$ be the Wiener-Hopf operator associated with $\mathcal{K}_{-2}$, whose presymbol G is given by (see [**18**, Theorem 3.2] or [**20**, Theorem 2.1])

$$G = \frac{1}{2}\begin{bmatrix} -Jk & 1 \\ 4 - kJk & k \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -i\,\frac{t_+}{t_-} & 1 \\ 3 & -i\,\frac{t_-}{t_+} \end{bmatrix}. \tag{4.20}$$

It was proved in [**18**] that the invertibility of T implies the invertibility of $\mathcal{K}_{-2}$. Furthermore, as it is well known, the invertibility of T is equivalent to the existence of a canonical generalized factorization relative to L_2 of the matrix-valued function G (see [**2**, **11**]), $G = G_- G_+$, and its inverse is given by

$$T^{-1} = \mathcal{F}^{-1} G_+^{-1} P_2^+ G_-^{-1} \mathcal{F} : [L_2^+(\mathbf{R})]^2 \to [L_2^+(\mathbf{R})]^2 \tag{4.21}$$

with P_2^+ denoting the projection operator from $[\widehat{L_2}(\mathbf{R})]^2$ onto $[\widehat{L_2^+}(\mathbf{R})]^2$ along $[\widehat{L_2^-}(\mathbf{R})]^2$. Moreover, it follows from Theorem 3.3 in [**18**] that the inverse of $\mathcal{K}_{-2}$ is expressed in terms of T^{-1} by

$$\mathcal{K}_{-2}^{-1} = \Pi\, T^{-1} \mathcal{A} \tag{4.22}$$

where $\Pi : [L_2(\mathbf{R})]^2 \to L_2(\mathbf{R})$ and $\mathcal{A} : L_2^+(\mathbf{R}) \to [L_2^+(\mathbf{R})]^2$ are the operators defined by

$$\Pi \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = f_1, \qquad \mathcal{A} f^+ = \frac{1}{2}\begin{pmatrix} 0 \\ (-2I + \mathcal{K}) f^+ \end{pmatrix} \tag{4.23}$$

with $\mathcal{K}$ defined in (4.19).

The existence of a canonical generalized factorization for G, as well as the explicit factors were obtained in [**20**] (see also [**17**] for a general discussion of matrices of that type). In fact, we have (see (4.20))

$$G = \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{\sqrt{3}}{2} & 0 \end{bmatrix} \begin{bmatrix} 1 & \frac{i}{\sqrt{3}}\frac{t_-}{t_+} \\ \frac{i}{\sqrt{3}}\frac{t_+}{t_-} & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & -1 \end{bmatrix} = C_1 S C_2 \tag{4.24}$$

and the generalized factorization $S = S_- S_+$ was presented in [**20**, equation (4.29)] (see also [**7**]):

$$S_\pm = \left(\frac{4}{3}\right)^{1/4} \begin{bmatrix} \cosh\left(\frac{1}{3}\log\gamma_\pm\right) & \frac{t_-}{t_+}\sinh\left(\frac{1}{3}\log\gamma_\pm\right) \\ \frac{t_+}{t_-}\sinh\left(\frac{1}{3}\log\gamma_\pm\right) & \cosh\left(\frac{1}{3}\log\gamma_\pm\right) \end{bmatrix}, \tag{4.25}$$

where

$$\gamma_+ = \frac{t_+ - t_-}{\sqrt{2k_0}}, \qquad \gamma_- = i\,\frac{t_+ + t_-}{\sqrt{2k_0}} \tag{4.26}$$

and log represents the principal branch of the logarithm. Thus, it follows that

$$G = G_- G_+ = (C_1 S_-)(S_+ C_2) \tag{4.27}$$

is a canonical generalized factorization of G.

Consequently, the Wiener-Hopf operator T is invertible, with its inverse given explicitly by (4.21) and factors $G_\pm$ given above. Then the invertibility of $\mathcal{K}_{-2}$ follows with inverse (4.22).

As an immediate corollary of Proposition 4.1, we see that equation (4.7) is uniquely solvable and, moreover, using (4.13), (4.14) and (4.17), we have that

$$\varphi_1^- = JA_+^{-1}\mathcal{K}_{-2}^{-1}f^+ = 2JA_+^{-1}\mathcal{K}_{-2}^{-1}\mathcal{P}^+A_+l^\circ f'. \tag{4.28}$$

The function φ_0^- was given in terms of φ_1^- and f' by formula (4.6). Then it is clear that the system of pseudodifferential equations of Wiener-Hopf-Hankel type (3.19) is uniquely solvable, and consequently we obtain from Theorem 3.3 the following result:

Theorem 4.2. *For every $f' \in H_{-1/2}(\mathbf{R}^-)$, Problem $\mathcal{P}_0^0$ has a unique solution that can be obtained by restricting to Ω the function v given by the representation formula* (3.1), *with $(v_0^+, v_{\circ 0}^-)^T$ given by* (3.5) *where*

$$\varphi_0^- = A(I-2J)\varphi_1^- - 2Al^\circ f' \tag{4.29}$$

$$\varphi_1^- = 2JA_+^{-1}\mathcal{K}_{-2}^{-1}\mathcal{P}^+A_+l^\circ f' \tag{4.30}$$

Remark. We would like to point out that the existence and uniqueness of a solution to Problem $\mathcal{P}_0^0$ was already known from [**12**]. However, in [**12**] the result was established by variational methods, in a nonconstructive approach.

5. The impedance problem $\mathcal{P}_\lambda^0$. In Section 3 we reduced the boundary-value problem $\mathcal{P}_\lambda^0$ to a system of pseudodifferential equations of Wiener-Hopf-Hankel type (see Theorem 3.3 and (3.19)). Moreover, we have proved that this system can be transformed into an equivalent 3×3 system of functional equations (see (3.29)) which can be described in the Fourier space as a special Riemann-Hilbert problem. Actually, taking the Fourier transforms on both sides of (3.29) and using (3.8) and (3.17), we obtain

$$G\hat{\underset{\sim}{\varphi}}^+ = \begin{pmatrix} \frac{1}{t+\lambda} \\ -\frac{t}{t+\lambda} \\ 2 \end{pmatrix} \widehat{l^\circ f'} + \widehat{J\underset{\sim}{\varphi}^+} \tag{5.1}$$

where the hat stands for the Fourier transformation and G denotes now the matrix-valued function:

$$G = \frac{1}{2}\begin{bmatrix} -1 & -\frac{1}{t} & \frac{1}{t+\lambda} \\ -t & -1 & -\frac{t}{t+\lambda} \\ 2(t+\lambda) & -2\,\frac{t+\lambda}{t} & 0 \end{bmatrix}. \tag{5.2}$$

One first idea to solve (5.1) could be to lift it to $[L_2^+(\mathbf{R})]^3$ by the use of Bessel potential operators and try to get a generalized canonical factorization for the lifted symbol G_0. But, unfortunately, it turns out that G_0 is L_2-singular (see [**2**]) and, therefore, such a factorization does not exist. So it is natural to exploit more carefully the symmetry properties of equations (5.1) to obtain a further simplification.

With this objective, let us decompose $\underset{\sim}{\varphi}^+$ into its even and odd components

$$\underset{\sim}{\varphi}^+ = \underset{\sim}{\varphi}_e^+ + \underset{\sim}{\varphi}_o^+. \tag{5.3}$$

From (5.1) we obtain two separated systems

$$G\hat{\underset{\sim}{\varphi}}_e^+ = \hat{\underset{\sim}{\varphi}}_e^+ \tag{5.4}$$

and

$$G\underset{\sim}{\hat{\varphi}}{}^{+}_{\mathfrak{o}} = \begin{pmatrix} \frac{1}{t+\lambda} \\ -\frac{t}{t+\lambda} \\ 2 \end{pmatrix} \widehat{l^{\mathfrak{o}} f'} - \underset{\sim}{\hat{\varphi}}{}^{+}_{\mathfrak{o}} \tag{5.5}$$

because $\mathcal{F}$ maps even functionals into even, and odd into odd ones.

The algebraic solution of (5.4) and (5.5) is an eigenvalue problem. We must study

$$\begin{aligned} G - I &= \frac{1}{2}\begin{bmatrix} -3 & -\frac{1}{t} & \frac{1}{t+\lambda} \\ -t & -3 & -\frac{t}{t+\lambda} \\ 2(t+\lambda) & -2\,\frac{t+\lambda}{t} & -2 \end{bmatrix}, \\ G + I &= \frac{1}{2}\begin{bmatrix} 1 & -\frac{1}{t} & \frac{1}{t+\lambda} \\ -t & 1 & -\frac{t}{t+\lambda} \\ 2(t+\lambda) & -2\,\frac{t+\lambda}{t} & 2 \end{bmatrix} \end{aligned} \tag{5.6}$$

where I now represents the identity matrix. It is easily seen that $\operatorname{rank}(G-I) = 2$ and $\operatorname{rank}(G+I) = 1$, i.e., ± 1 are eigenvalues of G, independent of ξ and λ. We simply get the eigenvectors

$$\begin{pmatrix} \frac{1}{t+\lambda} \\ \frac{-t}{t+\lambda} \\ 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{1}{t} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{t+\lambda} \\ 0 \\ -1 \end{pmatrix}.$$

Then, the algebraic solution of (5.4) and (5.5) lead us to the form of the solutions of (5.1)

$$\underset{\sim}{\hat{\varphi}}{}^{+} = \begin{pmatrix} \frac{2}{t+\lambda} \\ 0 \\ 0 \end{pmatrix} \widehat{l^{\mathfrak{o}} f'} + \begin{pmatrix} \frac{1}{t} \\ 1 \\ 0 \end{pmatrix} \hat{\varphi}_{\mathfrak{o}1} + \begin{pmatrix} \frac{1}{t+\lambda} \\ 0 \\ -1 \end{pmatrix} \hat{\varphi}_{\mathfrak{o}2} + \begin{pmatrix} \frac{1}{t+\lambda} \\ -\frac{t}{t+\lambda} \\ 2 \end{pmatrix} \hat{\varphi}_{e} \tag{5.7}$$

where $\varphi_{\mathfrak{o}1}, \varphi_{\mathfrak{o}2}$ and φ_e are unknown scalar odd and even distributions in $H_{-1/2}(\mathbf{R})$, respectively.

Now we must look for a "plus" functional of this form, imposing that the restriction to $\mathbf{R}^-$ of $\underset{\sim}{\varphi}{}^{+}$ is the zero vector

$$r_{-}\underset{\sim}{\varphi}{}^{+} = \underset{\sim}{0}. \tag{5.8}$$

This leads to the following system of equations, now written in operator form, by the use of (3.8) and (3.17)

$$r_- \begin{bmatrix} A & A_\lambda & A_\lambda \\ 1 & 0 & -A^{-1}A_\lambda \\ 0 & -1 & 2 \end{bmatrix} \begin{pmatrix} \varphi_{\mathfrak{o}1} \\ \varphi_{\mathfrak{o}2} \\ \varphi_e \end{pmatrix} = \begin{pmatrix} 2r_- A_\lambda l^{\mathfrak{o}} f' \\ 0 \\ 0 \end{pmatrix}. \tag{5.9}$$

We can replace the unknowns in terms of φ_e by writing the last two equations as

$$\begin{cases} r_-\varphi_{\mathfrak{o}1} = r_- A^{-1}A_\lambda\varphi_e \\ r_-\varphi_{\mathfrak{o}2} = 2r_-\varphi_e, \end{cases}$$

and, thus,

$$\begin{cases} \varphi_{\mathfrak{o}1} = l^{\mathfrak{o}} r_- A^{-1}A_\lambda\varphi_e \\ \varphi_{\mathfrak{o}2} = 2l^{\mathfrak{o}} r_-\varphi_e, \end{cases} \tag{5.10}$$

which gives us from the first of equations (5.9) the following second-order integral equation of Wiener-Hopf type:

$$r_- A l^{\mathfrak{o}} r_- A^{-1}A_\lambda\varphi_e + 2r_- A_\lambda l^{\mathfrak{o}} r_-\varphi_e + r_- A_\lambda\varphi_e = 2r_- A_\lambda l^{\mathfrak{o}} f'. \tag{5.11}$$

For what follows, a precise discussion of the functional spaces turns out to be important. It is immediately seen from the last equation that if it has a solution, then the odd extension onto $\mathbf{R}$ of $r_- A_\lambda\varphi_e$ must belong to $H_{1/2}(\mathbf{R})$, or equivalently,

$$r_- A_\lambda\varphi_e \in \tilde{H}_{1/2}(\mathbf{R}^-). \tag{5.12}$$

Remark. From (3.27) and the last of equations (5.7), we have $2\varphi_e - \varphi_{\mathfrak{o}2} = \psi^+$ and, consequently,

$$\varphi_e = \frac{1}{4}\,(I+J)\psi^+.$$

Moreover, it follows from (3.25) and (3.26) that

$$A_\lambda\varphi_e = \frac{1}{4}\,[(\varphi_0^- + J\varphi_0^-) - A(\varphi_1^- + J\varphi_1^-)]$$

where φ_0^-, φ_1^- are the jumps *ansatzs* defined in (3.4). Using (3.7) and the fact that v_{o0}^- and v_{o1}^- are odd functions (distributions), we have yet $A_\lambda \varphi_e = (v_0^+ + Jv_0^+)/2$, and, therefore,

$$r_- A_\lambda \varphi_e = \frac{1}{2}\,(r_- v_0^+ + r_- J v_0^+).$$

The compatibility condition (5.12) can now be understood as being a natural one, due to the fact that $r_- v_0^+$ (and then also $r_- J v_0^+$) must be a function in $\tilde{H}_{1/2}(\mathbf{R}^-)$. Indeed, $r_- v_0^+$ is the trace of $v \in H_1(\Omega)$ on Γ_1, which has a zero trace on Γ_2 (see (2.12)). Thus, $(r_- v_0^+, 0)$ is the trace of v on the Lipschitz boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$, and this implies that $r_- v_0^+ \in \tilde{H}_{1/2}(\mathbf{R}^-)$ (taking Γ_1 as a copy of $\mathbf{R}^-$, see [**20**] for details).

We are now going to show that by incorporating in (5.11) the compatibility condition (5.12), we get a scalar equivalent equation (of higher complexity) involving only the sum of a Wiener-Hopf operator with a Hankel operator. For this purpose, define

$$\varphi^- = r_- A_\lambda \varphi_e \in \tilde{H}_{1/2}(\mathbf{R}^-) \tag{5.13}$$

and note that

$$r_- A^{-1} A_\lambda \varphi_e = r_- \varphi_e - \lambda \varphi^- \tag{5.14}$$

where we used the identity $A^{-1}A_\lambda = I - \lambda A_\lambda$. From (5.11), (5.13) and (5.14), we readily obtain

$$r_-[(A + 2A_\lambda) l^o r_- \varphi_e + (I - \lambda A) l^o \varphi^-] = 2 r_-[A_\lambda l^o f'] \tag{5.15}$$

where the functions in brackets are odd functions in $H_{1/2}(\mathbf{R})$. This allows us to use the odd extension operator in both sides of (5.15), getting

$$(A + 2A_\lambda) l^o r_- \varphi_e + (I - \lambda A) l^o \varphi^- = 2 A_\lambda l^o f'$$

or yet

$$l^o \varphi^- = 2(I - \lambda A)^{-1} A_\lambda l^o f' - (I - \lambda A)^{-1}(A + 2A_\lambda) l^o r_- \varphi_e \tag{5.16}$$

with

$$(I - \lambda A)^{-1} = \mathcal{F}^{-1} \frac{t}{t - \lambda}\, \mathcal{F} : H_{1/2}(\mathbf{R}) \to H_{1/2}(\mathbf{R})$$

being a pseudodifferential operator (of order zero). A direct computation yields

$$(I-\lambda A)^{-1}A_\lambda = \mathcal{F}^{-1}\frac{t}{t^2-\lambda^2}\,\mathcal{F} : H_{-1/2}(\mathbf{R}) \to H_{1/2}(\mathbf{R}) \tag{5.17}$$

and

$$B_\lambda = (I-\lambda A)^{-1}(A+2A_\lambda) = \mathcal{F}^{-1}\frac{3t+\lambda}{t^2-\lambda^2}\,\mathcal{F} : H_{-1/2}(\mathbf{R}) \to H_{1/2}(\mathbf{R}). \tag{5.18}$$

Applying the restriction operator r_- to both sides of (5.16), and using (5.13), (5.17) and (5.18) we finally obtain

$$r_-A_\lambda\varphi_e + r_-B_\lambda l^{\circ} r_-\varphi_e = r_-\left(\mathcal{F}^{-1}\frac{2t}{t^2-\lambda^2}\,\mathcal{F}\right) l^{\circ} f'. \tag{5.19}$$

This equation can be written as an integral (or pseudodifferential) equation of Wiener-Hopf-Hankel type. For this objective, let φ_e (an even functional in $H_{-1/2}(\mathbf{R})$) be decomposed in the form

$$\varphi_e = (I+J)\phi^+ \quad \text{with} \quad \phi^+ = l_0 r_+\varphi_e \in H^+_{-1/2}(\mathbf{R}) \tag{5.20}$$

which also yields

$$l_{\circ} r_-\varphi_e = -(I-J)\phi^+. \tag{5.21}$$

Then, from (5.19), we get the Wiener-Hopf Hankel equation:

$$r_-(A_\lambda - B_\lambda)\phi^+ + r_-(A_\lambda + B_\lambda)J\phi^+ = \frac{1}{2}\,r_-(A_\lambda + B_\lambda)l^{\circ} f' \tag{5.22}$$

where $A_\lambda \mp B_\lambda$ denote the (invertible) convolution or pseudodifferential operators:

$$A_\lambda - B_\lambda = \mathcal{F}^{-1}\frac{-2}{t-\lambda}\,\mathcal{F} : H_{-1/2}(\mathbf{R}) \to H_{1/2}(\mathbf{R}) \tag{5.23}$$

and

$$A_\lambda + B_\lambda = \mathcal{F}^{-1}\frac{4t}{t^2-\lambda^2}\,\mathcal{F} : H_{-1/2}(\mathbf{R}) \to H_{1/2}(\mathbf{R}). \tag{5.24}$$

We summarize in the next theorem the results obtained so far (see Section 3, Theorem 3.3):

Theorem 5.1. *Problem $\mathcal{P}_\lambda^0$ is uniquely solvable if and only if the Wiener-Hopf-Hankel equation* (5.22)*is uniquely solvable. Moreover, if ϕ^+ is a solution to* (5.22)*, then a solution of Problem $\mathcal{P}_\lambda^0$ can be obtained by restriction to Ω of the function v given by the representation formula* (3.1)*, with $(v_0^+, v_{\mathfrak{o}0}^-)^T$ given by* (3.5) *and*

$$\begin{pmatrix}\varphi_0^- \\ \varphi_1^-\end{pmatrix} = -\begin{pmatrix}2A_\lambda \\ 0\end{pmatrix} l^{\mathfrak{o}} f' - \begin{pmatrix}A \\ 1\end{pmatrix}\varphi_{\mathfrak{o}1} - \begin{pmatrix}A_\lambda \\ 0\end{pmatrix}\varphi_{\mathfrak{o}2} + \begin{pmatrix}A_\lambda \\ -A^{-1}A_\lambda\end{pmatrix}\varphi_e \tag{5.25}$$

where

$$\varphi_e = (I+J)\phi^+ \tag{5.26}$$

$$\varphi_{\mathfrak{o}1} = l^{\mathfrak{o}} r_- A^{-1} A_\lambda (I+J)\phi^+ \tag{5.27}$$

and

$$\varphi_{\mathfrak{o}2} = 2 l^{\mathfrak{o}} r_- J \phi^+. \tag{5.28}$$

Proof. For necessity of the Wiener-Hopf-Hankel equation, it remains only to prove (5.25)–(5.28), which are obtained by direct computation, using (3.27), (5.7), (5.10) and (5.20). Sufficiency is proved by inspection. □

The resulting Wiener-Hopf-Hankel equation cannot be solved by the approach presented at the end of Section 4.

But, in the sense of the ideas presented in [**18**], we can associate with equation (5.22) an equivalent Riemann-Hilbert problem in $[L_2^+(\mathbf{R})]^2$, whose solvability can be studied through the determination of a generalized (canonical) factorization [**2**] of its presymbol. More precisely, equation (5.22) can be written in the equivalent form

$$(A_\lambda - B_\lambda)\phi^+ + (A_\lambda + B_\lambda)J\phi^+ = \frac{1}{2}(A_\lambda + B_\lambda) l^{\mathfrak{o}} f' + \Psi^+ \tag{5.29}$$

where Ψ^+ is a suitable function in $H^+_{1/2}(\mathbf{R})$. By symmetrization (i.e., applying J to both sides of (5.29)) we get further

$$(5.30)\quad (A_\lambda - B_\lambda)J\phi^+ + (A_\lambda + B_\lambda)\phi^+ = -\frac{1}{2}\,(A_\lambda + B_\lambda)l^{\mathrm{o}} f' + J\psi^+.$$

Let

$$(5.31)\quad \underset{\sim}{\phi}^+ = (\phi^+, \psi^+) \in H^+_{-1/2}(\mathbf{R}) \times H^+_{1/2}(\mathbf{R}).$$

Using Fourier transformation and (5.23), (5.24), we can write equations (5.29) and (5.30) in the following matrix form (after some elementary computations)

$$(5.32)\quad G'\underset{\sim}{\hat{\phi}}^+ = \underset{\sim}{\hat{f}} + J\underset{\sim}{\hat{\phi}}^+$$

where G' and $\hat{f}$ are given by:

$$(5.33)\quad G' = \frac{1}{2t}\begin{bmatrix} t+\lambda & \frac{1}{2}\,(t^2-\lambda^2) \\ 2\,\frac{3t+\lambda}{t+\lambda} & -(t+\lambda) \end{bmatrix}, \qquad \underset{\sim}{\hat{f}} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{t+\lambda} \end{pmatrix} \widehat{l^{\mathrm{o}} f'}$$

From this system of equations we readily obtain an equivalent Riemann-Hilbert problem in $[L^+_2(\mathbf{R})]^2$. Indeed, following the usual lifting procedure [**10**], let us introduce the new ansatz vector (see (4.8))

$$(5.34)\quad \underset{\sim}{\phi}^+_0 = (\phi^+_0, \psi^+_0)^T = \mathcal{F}^{-1}\mathrm{diag}\,[t_+^{-1}, t_+]\mathcal{F}\underset{\sim}{\phi} \in [L^+_2(\mathbf{R})]^2.$$

Then a direct computation shows that (5.32) is equivalent to

$$(5.35)\quad G'_0\underset{\sim}{\hat{\phi}}^+_0 = \underset{\sim}{\hat{f}_o} + J\underset{\sim}{\hat{\phi}}^+_0$$

with

$$(5.36)\quad G'_0 = -i\begin{bmatrix} \frac{t_+}{t_-}\,\frac{t+\lambda}{2t} & \frac{t^2-\lambda^2}{4t^2} \\ -\frac{3t+\lambda}{t+\lambda} & \frac{t_-}{t_+}\,\frac{t+\lambda}{2t} \end{bmatrix}, \qquad \hat{f}_0 = \begin{pmatrix} \frac{i}{2t_-} \\ \frac{-it_-}{t+\lambda} \end{pmatrix} \widehat{l^{\mathrm{o}} f'}$$

To prove the existence and uniqueness of solution to the Riemann-Hilbert problem (5.35) we are led to obtaining a canonical generalized factorization [**2**] for its presymbol G_0'. This has been achieved for the case $\lambda = 0$, as in this case G_0' coincides (up to a constant nonsingular matrix) with the matrix-valued function G defined in (4.20). However, for $\lambda \neq 0$, the referred methods are not suitable to get (even a function-theoretic) factorization for G_0', due to the fact that G_0' is not of Daniel's or Krapkhov's form (see [**7**]). Nevertheless, the existence of a generalized factorization for G_0' (with a priori arbitrary but symmetric partial indices) can be easily established by using the well-known criteria for piecewise continuous matrix-valued functions (see [**2**, Chapter 8], [**11**, Chapter 5]). Indeed, we have

Lemma 5.2. *The matrix-valued function* G_0' *in* (5.36) *is* L_2*-nonsingular, with zero total index.*

Proof. Using the above cited criteria, we associate with G_0' the matrix-valued function $\tilde{G} : \dot{\mathbf{R}} \times [0,1] \to \mathbf{C}^{2\times 2}$ defined by

$$\begin{cases} \tilde{G}(\xi,\mu) = G_0'(\xi+0)\mu + (1-\mu)G_0'(\xi), & \xi \in \mathbf{R}, \quad \mu \in [0,1] \\ \tilde{G}(\infty,\mu) = G_0'(-\infty)\mu + (1-\mu)G_0'(\infty), & \mu \in [0,1]. \end{cases}$$

The first statement in the lemma is equivalent to

$$\tilde{g}(\xi,\mu) = \det \tilde{G}(\xi,\mu) \neq 0, \quad (\xi,\mu) \in \dot{\mathbf{R}} \times [0,1]$$

which easily follows by direct computation. Indeed, infinity being the only discontinuity point of G_0', we have

$$\tilde{g}(\xi,\mu) = \det G_0'(\xi) = -1 \neq 0, \quad \xi \in \mathbf{R}$$

and

$$\tilde{g}(\infty,\mu) = -\frac{1}{4}(1-2\mu)^2 - \frac{3}{4} \neq 0, \quad \mu \in [0,1].$$

The proof ends by noting that the curve $\tilde{g}$ has a zero winding number with respect to the origin. $\square$

As a final remark, we would like to point out that the method used so far to handle problem $\mathcal{P}_\lambda$ can also be used to deal with the boundary-value problem obtained by replacing the Dirichlet boundary condition on Γ_2 (see (2.4)) by a Neumann condition. In this situation, for $\lambda = 0$, we get also a Neumann condition on Γ_1, a problem which was already treated in [**20**]. In the case $\lambda \neq 0$, similar results to those obtained for Problem $\mathcal{P}_\lambda$ can be derived, with slight modifications (the compatibility condition is more evident here). Only the problem with two impedance conditions (one on Γ_1, the other on Γ_2) cannot directly be studied by the present method, due to the impossibility of using the reduction procedure followed in Section 2. So a new method is being developed to investigate this last problem, which will be the subject of the forthcoming paper.

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