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A QUADRATURE METHOD FOR CAUCHY INTEGRAL EQUATIONS WITH WEAKLY SINGULAR PERTURBATION KERNEL

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ABSTRACT. The authors study the mean weighted convergence of the quadrature method for solving integral equations over the arc $(-1,1)$ with Cauchy kernel and with a perturbation kernel not necessarily regular. Error estimates in uniform norm are also given.

1. Introduction. Many problems in aerodynamics and elasticity lead to a singular integral equation with Cauchy kernel of the form

$$a(x)u(x)+\frac{b(x)}{\pi}\int_{-1}^{1}\frac{u(t)}{t-x}\,dt+\int_{-1}^{1}k(x,t)u(t)\,dt=f(x) \tag{1.1}$$

on the interval $(-1,1)$ (see, e.g., [**1, 16, 19**]). The first integral in (1.1) is to be interpreted as the Cauchy principal value. Hereby a, b and f are given Hölder continuous functions, and k is a given smooth or weakly singular kernel function.

The problem we are interested in is to find an approximation to the unknown solution u by using projection methods (like collocation or Galerkin schemes) or quadrature procedures with orthogonal polynomials as trial functions. There is already a considerable literature on this subject in the case of regular kernel k (see, e.g., the surveys [**9, 6–8, 12, 22, 23, 13–15, 24**] and the references given by the same authors). In most of these papers the following strategy is employed. For given functions a and b, one introduces two sets of orthogonal polynomials which are denoted by $\{p_n\}$ and $\{q_n\}$, where $Dp_n = q_{n-\chi}$ with D being the dominant part of Equation (1.1) and χ the index of D (see Section 2). For a given value of n, we use Gauss-type quadrature rules based on the zeros of p_n and collocate at the zeros of $q_{n-\chi}$. In

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the case of constant coefficients a and b, the polynomials p_n and q_n turn out to be Jacobi polynomials corresponding to the weight functions $w(t) = w^{(\alpha,\beta)}(t) = (1-t)^\alpha(1+t)^\beta$, $-1 < \alpha,\ \beta < 1$, and $1/w(t)$, respectively. If, for example, $k \in C^{r+\lambda}([-1,1]^2)$ and $f \in C^{r+\lambda}[-1,1]$, where r is a nonnegative integer and $0 < \lambda \le 1$, and $|\alpha| = |\beta| = 1/2$, $\chi = 0$ or $\chi = 1$, then in [**14**] the corresponding quadrature method is proved to converge in the weighted space $L^2_w(-1,1)$ with the rate $O(n^{-r-\lambda})$ as $n \to \infty$.

In the present paper we prove that the quadrature method, under certain additional assumptions for a and b, converges in the space $L^2_w(-1,1)$ with the error bound $O(n^{-r-\lambda}\log n)$, provided only that k is a kernel of the form $k(x,t) = [h(x,t) - h(x,x)]/(t-x)$, where $h \in C^{r+\lambda}([-1,1]^2)$, $f \in C^{r+\lambda}[-1,1]$, and α, β are arbitrary numbers satisfying $-1 < \alpha,\ \beta < 1$ and $\chi = 0$ or $\chi = 1$. Moreover, error estimates in uniform norms are given. The crucial point in our analysis are bounds of the quadrature error for the perturbation kernel k which are founded on thorough estimates for the distances between the zeros of the orthogonal polynomials p_n and q_n (Section 3). Further ideas of Junghanns and Silbermann [**14**] and Elliott [**5**–**8**] are used. In Sections 2 and 3 the quadrature method is studied when a and b are real constants and in Section 4 when a and b are real-valued functions.

2. Singular integral equations with constant coefficients. Consider the singular integral equation with Cauchy kernel of the form

$$au(x) + \frac{b}{\pi}\int_{-1}^{1} \frac{u(t)}{t-x}\,dt + \int_{-1}^{1} k(x,t)u(t)\,dt = f(x), \tag{2.1}$$

$x \in (-1,1)$, where a and b are given real constants such that $a^2+b^2 > 0$, and k is a regular or weakly singular kernel function,

$$k(x,t) = \frac{h(x,t) - h(x,x)}{t-x} \tag{2.1$'$}$$

with $h \in C^\lambda([-1,1]^2)$, $0 < \lambda \le 1$.

Notice that if k is a weakly singular kernel of the form

$$k(x,t) = \frac{m(x,t)}{|x-t|^\mu}, \quad 0 \le \mu < 1,$$

with $m \in C^{\nu}([-1,1]^2)$, then the representation $(2.1)'$ holds, where $h(x,t) = (t-x)|t-x|^{-\mu} m(x,t) \in C^{\lambda}([-1,1]^2)$ with $\lambda = \min(1-\mu,\nu)$ (see, e.g., [**19**]).

The quadrature method under consideration, like most of the polynomial approximation methods for solving equation (2.1), is essentially based on the well-known relation (see, e.g., [**9, 5, 6, 13**–**15**])

$$D(p_n^{(\alpha,\beta)}) = -\frac{2^{-\chi} b}{\sin(\pi\alpha_0)}\, p_{n-\chi}^{(-\alpha,-\beta)}, \tag{2.2}$$

where

$$(Dv)(t) = aw(x)v(x) + \frac{b}{\pi}\int_{-1}^{1} \frac{w(t)v(t)}{t-x}\,dt, \quad -1 < x < 1, \tag{2.3}$$

is the dominant part of Equation (2.1), and $\{p_n^{(\alpha,\beta)}\}_{n\in\mathbf{N}}$ is the sequence of orthonormal Jacobi polynomials with respect to the weight function $w(x) = w^{(\alpha,\beta)}(x) = (1-x)^{\alpha}(1+x)^{\beta}$, that is, $p_n^{(\alpha,\beta)}$ is a polynomial of degree n with positive leading coefficient and $\int_{-1}^{1} p_n^{(\alpha,\beta)} p_m^{(\alpha,\beta)} w = \delta_{n,m}$ ($p_{n-\chi}^{(-\alpha,-\beta)} \equiv 0$ for $n-\chi < 0$). The numbers α, β and χ involved are defined as follows:

$$a + ib = (a^2+b^2)^{1/2} e^{i\pi\alpha_0}, \qquad 0 < |\alpha_0| < 1,$$
$$-1 < \alpha := \mu - \alpha_0, \qquad \beta := \nu + \alpha_0 < 1,$$

μ and ν are integers, $\chi := -(\alpha+\beta) = -(\mu+\nu)$.

Denote by $L_w^2 = L_w^2(-1,1)$ the Hilbert space of all complex-valued functions on $(-1,1)$ which are quadratic integrable with respect to the weight w. The space L_w^2 is equipped with the scalar product

$$(u,v)_w = \frac{1}{\pi}\int_{-1}^{1} u(t)\overline{v(t)} w(t)\,dt$$

and the norm $||u||_w = [(u,u)_w]^{1/2}$. As an immediate consequence of (2.2), the operator D defined by (2.3) and acting from L_w^2 to $L_{1/w}^2$ is Fredholm with index χ (see also [**11, 18**]). Moreover, D is invertible if $\chi = 0$, invertible from the right if $\chi = 1$ (in this case $D(\mathbf{P}_{n-1}) = \mathbf{P}_{n-2}$

with $\mathbf{P}_{n-1}$ being the set of all real polynomials of degree $\le n$) and invertible from the left if $\chi = -1$ (in which case $D(\mathbf{P}_{n-1}) + \mathbf{C} = \mathbf{P}_n$).

Let $t_k = t_{m,k}$, $k = 1, \dots, m$, and $x_j = x_{m,j}$, $j = 1, \dots, m-\chi$, be the zeros of the polynomials $p_m^{(\alpha,\beta)}$ and $p_{m-\chi}^{(-\alpha,-\beta)}$, respectively, i.e.,

$$p_m^{(\alpha,\beta)}(t_k) = 0, \qquad p_{m-\chi}^{(-\alpha,-\beta)}(x_j) = 0.$$

Choose now a Gauss-type quadrature formula with the weight w and the nodes t_k such that

$$\int_{-1}^{1} g(t)w(t)\,dt = \sum_{k=1}^{m} \lambda_{m,k} g(t_k), \quad g \in \mathbf{P}_{2m-1}, \tag{2.4}$$

where $\lambda_{m,k} = \lambda_{m,k}(w)$ stands for the Christoffel numbers. Then

$$(\mathbf{D}v_m)(x_j) = \frac{b}{\pi} \sum_{k=1}^{m} \lambda_{m,k} \frac{v_m(t_k)}{t_k - x_j}, \quad j = 1, \dots, m-\chi, \tag{2.5}$$

holds for any polynomial v_m of degree $m-1$ (see, e.g., [**15**, Lemma 1.15]; cf. also Lemma 4.2).

We consider the following quadrature method for the approximate solution of Equation (2.1). Determine $\xi_k = \xi_{m,k}$, $k = 1, \dots, m$, such that

$$\sum_{k=1}^{m} \lambda_{m,k} \left[\frac{b/\pi}{t_k - x_j} + k(x_j, t_k) \right] \xi_k = f(x_j), \quad j = 1, \dots, m-\chi. \tag{2.6}$$

In the case $\chi = 1$, it is necessary to give an additional condition in order to define the solution of (2.1) uniquely, e.g.,

$$\int_{-1}^{1} u(t)\,dt = 0 \tag{2.7}$$

which can be approximated by

$$\sum_{k=1}^{m} \lambda_{m,k} \xi_k = 0. \tag{2.8}$$

In practice, the most frequent case is that of an unknown function u which is unbounded at both the endpoints 1 and -1. Then the representation

$$u(t) = (1-t)^{\alpha}(1+t)^{\beta}v(t)$$

holds, where v is continuous, provided the given functions $f(x)$ and $k(x,t)$ are smooth enough, and α, β are real numbers satisfying $-1 < \alpha$, $\beta < 1$.

Given $0 < \lambda \leq 1$, an integer $r \geq 0$, and a subset $A \subseteq [-1,1]^2$, we denote by $C^{r+\lambda}(A)$ the class of all functions on A whose r-th derivatives belong to $\mathrm{Lip}_{\lambda}(A)$. The main results of the present paper are as follows.

Theorem 2.1. *Assume* $-1 < \alpha$, $\beta < 1$, $\chi = 0$ *or* $\chi = 1$, $k \in C^{r+\lambda}([-1,1]^2)$ *and* $f \in C^{r+\lambda}[-1,1]$, $r \geq 0$, $0 < \lambda \leq 1$. *If the problem* (2.1) (*for* $\chi = 0$) *or* (2.1), (2.7) (*for* $\chi = 1$) *has a unique solution* $u = wv$, $v \in L^2_w$, *then the system of equations* (2.6) *or* (2.6), (2.8), *respectively, is uniquely solvable for all sufficiently large* m *and*

$$||v - v_m||_w = O\left(\frac{1}{m^{r+\lambda}}\right), \tag{2.9}$$

where

$$v_m(t) = \sum_{k=1}^{m} \frac{p_m^{(\alpha,\beta)}(t)}{(t-t_k)p_m^{(\alpha,\beta)\prime}(t_k)}\xi_k$$

is the Lagrange interpolation polynomial corresponding to the solution of (2.6) *or* (2.6), (2.8), *respectively.*

Theorem 2.2. *Assume that* $\chi = 0$ *and* $|\alpha| \leq 1/2$ *or that* $\chi = 1$ *and* $-1 < \alpha < 0$, $h \in C^{r+\lambda}([-1,1]^2)$ (*recall* (2.1)′) *and* $f \in C^{r+\lambda}[-1,1]$, *where* $r \geq 0$ *and* $0 < \lambda \leq 1$. *Then the assertion of Theorem* 2.1 *remains true with the error bound*

$$||v - v_m||_w = O\left(\frac{\log m}{m^{r+\lambda}}\right). \tag{2.10}$$

Theorem 2.3. *Assume the hypotheses of Theorem* 2.2 *are satisfied. If* $r + \lambda > \gamma$, *where* $\gamma := \max(\alpha,\beta) + 1$, *then* $v \in C[-1,1]$, *and*

$$\max_{-1\leq t\leq 1} |v(t) - v_m(t)| = O\left(\frac{\log m}{m^{r+\lambda-\gamma}}\right). \tag{2.11}$$

Further, if $r + \lambda > 1/2$, then v is continuous on any closed set $\Delta \subset (-1,1)$ and

$$\max_{\Delta} |v(t) - v_m(t)| = O\left(\frac{\log m}{m^{r+\lambda-1/2}}\right). \tag{2.12}$$

Remark . Theorems 2.1 and 2.3 (under additional assumptions) were proved in [**14**] for the case $|\alpha| = |\beta| = 1/2$.

Note that Theorem 2.3 guarantees, for example, the uniform convergence in $[-1,1]$ and estimate (2.11) in the following particular cases:

1) $r \geq 1$ and $\chi = 0$, if $r + \lambda > \gamma = 1 + |\alpha|$,
2) $r \geq 1$ and $\chi = 1$, if $r + \lambda > \gamma = 1 + \max(\alpha, -1-\alpha)$,
3) $r = 0$ and $\chi = 1$, if $\lambda > \gamma = 1 + \max(\alpha, -1-\alpha)$.

3. The proof of the main results. In the following the symbol "$\mathcal{C}$" stands for some positive constant taking a different value each time it is used. It will always be clear what variables and indices the constants are independent of. If A and B are two expressions depending on some variables, then we write

$$A \sim B \quad \text{iff} \quad |AB^{-1}| \leq \mathcal{C} \quad \text{and} \quad |A^{-1}B| \leq \mathcal{C}$$

uniformly for the variables in consideration.

In order to prove the theorems of the preceding section, we need the following auxiliary results. In particular, we shall use the following lemma that can be found in [**2**].

Lemma 3.1 (Jackson). *For any function $f \in C^r([-1,1]^2)$, $r \geq 0$, and for any positive integer n, there exists an algebraic polynomial $P_n(x,y)$ of degree n in x and y separately such that*

$$|f(x,y) - P_n(x,y)| \leq \mathcal{C} n^{-r} \Omega_r\left(f; \frac{1}{n}\right), \quad -1 \leq x, y \leq 1, \tag{3.1}$$

where

$$\Omega_r(f;\delta) = \max_{0 \leq i \leq r} \omega(f_i^{r-i};\delta), \qquad f_i^{r-i} = \frac{\partial^r f}{\partial x^{r-i} \partial y^i}, \quad \delta > 0,$$

and

$$\omega(f_i^{r-i};\delta)=\max_{h_1+h_2\le\delta}|f_i^{r-i}(x+h_1,y+h_2)-f_i^{r-i}(x,y)|,\quad h_1,h_2\ge 0.$$

Let $k(x,t)\in C^r([-1,1]^2)$. For any continuous function f, we define the operator K by

$$(Kf)(x)=\int_{-1}^{1}k(x,t)f(t)w(t)\,dt. \tag{3.2}$$

If $L_m^{(1)}g$ denotes the Lagrange polynomial interpolating the bounded function g on the zeros t_k, $k=1,2,\dots,m$ of $p_m^{(\alpha,\beta)}$, then we set

$$\overline{G}_m(x)=\int_{-1}^{1}L_{m,t}^{(1)}\{k(x,t)v_m(t)\}w^{(\alpha,\beta)}(t)\,dt,$$

where $L_{m,t}^{(1)}$ is the interpolating operator $L_m^{(1)}$ acting on the function $k(x,t)v_m(t)$ with respect to the variable t. Obviously,

$$\overline{G}_m(x)=\sum_{k=1}^{m}\lambda_{m,k}(w^{(\alpha,\beta)})k(x,t_k)v_m(t_k). \tag{3.3}$$

Further, denoting by $L_{m-\chi}^{(2)}g$, $\chi=-(\alpha+\beta)\in\{0,1\}$, the Lagrange polynomial interpolating g on the zeros x_j, $j=1,2,\dots,m-\chi$, of $p_{m-\chi}^{(-\alpha,-\beta)}$, for any polynomial $v_m\in\mathbf{P}_{m-1}$ we define the operator K_m by

$$\begin{aligned}(K_mv_m)(x)&=(L_{m-\chi}^{(2)}\overline{G}_m)(x)=\sum_{j=1}^{m-\chi}l_{m-\chi,j}(x)\overline{G}_m(x_j)\\ &=\sum_{j=1}^{m-\chi}l_{m-\chi,j}(x)\Bigg[\sum_{k=1}^{m}\lambda_{m,k}(w^{(\alpha,\beta)})k(x_j,t_k)v_m(t_k)\Bigg],\end{aligned} \tag{3.4}$$

where $l_{m-\chi,j}(x)$ are the fundamental Lagrange polynomials corresponding to the knots x_j, $j=1,2,\dots,m-\chi$.

Notice that if $k(x,t)$ is a polynomial of degree $m-2$ in the variables x and t separately, then by well-known properties of the Gaussian rules and of the Lagrange polynomials, we have

$$(K_m v_m)(x) = \int_{-1}^{1} k(x,t) v_m(t) w^{(\alpha,\beta)}(t)\, dt = (K v_m)(x), \tag{3.5}$$

or, that is the same $(K - K_m)v_m = 0$.

Therefore, for any function $k(x,t)$ and for any polynomial Q of degree $m-2$ in the variables x and t separately, we get

$$\begin{aligned}[(K-K_m)v_m](x) = \int_{-1}^{1} &[k(x,t) - Q(x,t)] v_m(t) w^{(\alpha,\beta)}(t)\, dt \\ &- \sum_{j=1}^{m-\chi} l_{m-\chi,j}(x) \bigg\{ \sum_{k=1}^{m} \lambda_{m,k}(w^{(\alpha,\beta)})[k(x_j,t_k) \\ &- Q(x_j,t_k)] v_m(t_k) \bigg\}.\end{aligned} \tag{3.6}$$

Finally, setting

$$||g||_\infty = \sup_{[-1,1]^2} |g(x,y)|,$$

we can state the following

Lemma 3.2. *Assume $k \in C^r([-1,1]^2)$, $r \geq 0$ and $\alpha, \beta \in (-1,1)$. Then*

$$||(K-K_m)v_m||_{w^{(-\alpha,-\beta)}} \leq \mathcal{C} ||r_m k||_\infty ||v_m||_{w^{(\alpha,\beta)}}, \tag{3.7}$$

where $r_m k$ is the remainder of the Jackson polynomial corresponding to the function k (see Lemma 3.1), and $\mathcal{C} = 2\sqrt{2}[B(1+\alpha,1+\beta)B(1-\alpha,1-\beta)]^{1/2}$ with B the Euler function.

Proof. Let $r_m k = k - Q$, with Q the Jackson polynomial (Lemma

3.1) corresponding to the function k. Then by (3.6), we have

$$[(K-K_m)v_m](x) = \int_{-1}^{1}(r_mk)(x,t)v_m(t)w^{(\alpha,\beta)}(t)\,dt - \sum_{j=1}^{m-\chi} l_{m-\chi,j}(x)\left[\sum_{k=1}^{m}\lambda_{m,k}(w^{(\alpha,\beta)})(r_mk)(x_j,t_k)v_m(t_k)\right]$$
$$= \int_{-1}^{1}(r_mk)(x,t)v_m(t)w^{(\alpha,\beta)}(t)\,dt - \sum_{j=1}^{m-\chi} l_{m-\chi,j}(x)G_m(x_j),$$

where

$$G_m(x) = \sum_{k=1}^{m}\lambda_{m,k}(w^{(\alpha,\beta)})(r_mk)(x,t_k)v_m(t_k).$$

Since

$$|G_m(x)| \le ||r_mk||_\infty \sqrt{\sum_{k=1}^{m}\lambda_{m,k}(w^{(\alpha,\beta)})}\sqrt{\sum_{k=1}^{m}\lambda_{m,k}(w^{(\alpha,\beta)})v_m^2(t_k)}$$

and $v_m \in \mathbf{P}_{m-1}$, we can write

$$|G_m(x)| \le ||r_mk||_\infty ||v_m||_{w^{(\alpha,\beta)}}\left[\int_{-1}^{1} w^{(\alpha,\beta)}(x)\,dx\right]^{1/2}. \tag{3.8}$$

On the other hand, we also have

$$\left|\int_{-1}^{1}(r_mk)(x,t)v_m(t)w^{(\alpha,\beta)}\,dt\right| \le ||r_mk||_\infty ||v_m||_{w^{(\alpha,\beta)}}\left[\int_{-1}^{1} w^{(\alpha,\beta)}(x)\,dx\right]^{1/2}. \tag{3.9}$$

Therefore,

$$
\begin{aligned}
||(K-K_m)v_m||^2_{w^{(-\alpha,-\beta)}} &\le 2\int_{-1}^{1}\left[\int_{-1}^{1}(r_mk)(x,t)v_m(t)w^{(\alpha,\beta)}(t)\,dt\right]^2 \\
&\qquad\qquad\cdot w^{(-\alpha,-\beta)}(x)\,dx \\
&\quad + 2\int_{-1}^{1}(L^{(2)}_{m-\chi}G_m)^2(x)w^{(-\alpha,-\beta)}(x)\,dx \\
&=: \mathcal{I}_1+\mathcal{I}_2.
\end{aligned}
\tag{3.10}
$$

In view of (3.9) we deduce

$$
\mathcal{I}_1 \le 2\int_{-1}^{1}w^{(\alpha,\beta)}(x)\,dx\int_{-1}^{1}w^{(-\alpha,-\beta)}(x)\,dx||v_m||^2_{w^{(\alpha,\beta)}}||r_mk||^2_{\infty}.
\tag{3.11}
$$

In order to estimate $\mathcal{I}_2$, we use the Gaussian formula corresponding to the weight $w^{(-\alpha,-\beta)}$. So, by (3.8),

$$
\begin{aligned}
\mathcal{I}_2 &= 2\sum_{j=1}^{m-\chi}\lambda_{m-\chi,j}(w^{(-\alpha,-\beta)})G^2_m(x_j) \\
&\le 2||r_mk||^2_{\infty}||v_m||^2_{w^{(\alpha,\beta)}}\int_{-1}^{1}w^{(\alpha,\beta)}(x)\,dx\sum_{j=1}^{m}\lambda_{m,j}(w^{(-\alpha,-\beta)}) \\
&\le 2\int_{-1}^{1}w^{(\alpha,\beta)}(x)\,dx\int_{-1}^{1}w^{(-\alpha,-\beta)}(x)\,dx||v_m||^2_{w^{(\alpha,\beta)}}||r_mk||^2_{\infty}.
\end{aligned}
\tag{3.12}
$$

Finally, combining (3.11) and (3.12) with (3.10), we obtain (3.7). □

In order to state a lemma similar to the previous one with $k(x,t)$ replaced by $k(x,t)=[h(x,t)-h(x,x)]/(t-x)$, where $h\in C^r([-1,1]^2)$, we need some other preliminary results.

Lemma 3.3. *Let* $\{p_m^{(\alpha,\beta)}\},\{p_m^{(-\alpha,-\beta)}\}$, $\alpha,\beta\in(-1,1)$, *be the sequences of Jacobi polynomials corresponding to the weights* $w^{(\alpha,\beta)}$ *and* $w^{(-\alpha,-\beta)}$, *respectively. Denote by* $t_k=t_{m,k}=\cos\tau_{m,k}$, $k=1,2,\dots,m$, *and* $x_j=x_{m,j}=\cos\theta_{m,j}$, $j=1,2,\dots,m$, *the zeros of*

$p_m^{(\alpha,\beta)}$ *and* $p_m^{(-\alpha,-\beta)}$*, respectively. If* $\alpha+\beta=0$*, then*

$$\min_{j,k} |\tau_{m,k} - \theta_{m,j}| \sim m^{-1}. \tag{3.13}$$

Furthermore, if $\alpha+\beta=-1$*, then*

$$\min_{j,k} |\tau_{m+1,k} - \theta_{m,j}| \sim m^{-1}. \tag{3.14}$$

The proof of the previous lemma is due to the authors and it can be found in [**17**]. Now, assume that $k(x,t) = [h(x,t) - h(x,x)]/(t-x)$. Then the function $\overline{G}_m$ in (3.3) becomes

$$\overline{G}_m(x) = \sum_{k=1}^{m} \lambda_{m,k}(w^{(\alpha,\beta)}) \frac{h(x,t_k) - h(x,x)}{t_k - x} v_m(t_k), \tag{3.15}$$

and, by retaining the previous notations, (3.4) becomes

$$\begin{aligned}(K_m v_m)(x) = (L^{(2)}_{m-\chi}\overline{G}_m)(x) &= \sum_{j=1}^{m-\chi} l_{m-\chi,j}(x)\overline{G}_m(x_j) \\ &= \sum_{j=1}^{m-\chi} l_{m-\chi,j}(x)\Bigg[\sum_{k=1}^{m} \lambda_{m,k}(w^{(\alpha,\beta)}) \\ &\qquad \cdot \frac{h(x_j,t_k) - h(x_j,x_j)}{t_k - x_j} v_m(t_k)\Bigg],\end{aligned}$$

for any $v_m \in \mathbf{P}_{m-1}$.

Similarly,

$$(Kv_m)(x) = \int_{-1}^{1} \frac{h(x,t) - h(x,x)}{t-x} v_m(t) w^{(\alpha,\beta)}(t)\, dt.$$

On the other hand, we remark that if $h(x,t)$ is a polynomial of degree $[m/2]-1$ in the variables t and x separately, then by (3.15) and (2.4),

$$\overline{G}_m(x) = \int_{-1}^{1} \frac{h(x,t) - h(x,x)}{t-x} v_m(t) w^{(\alpha,\beta)}(t)\, dt = (Kv_m)(x).$$

Since $\overline{G}_m$ is a polynomial of degree $m-2$ at most in x, we get $(L^{(2)}_{m-\chi}\overline{G}_m)(x) = \overline{G}_m(x)$. Thus,

$$(K - K_m)v_m = 0$$

in the case when $h(x,t)$ is a polynomial of degree $[m/2]-1$ in the variables x and t separately.

Therefore, denoting by $P(x,t)$ the Jackson polynomial (Lemma 3.1) corresponding to the function $h(x,t)$ of degree $n = [m/2]-1$ in the variables x and t separately and setting $(r_nh)(x,t) = h(x,t) - P(x,t)$, we can write

$$\begin{aligned}[(K - K_m)v_m](x) = &\int_{-1}^{1} \frac{(r_nh)(x,t) - (r_nh)(x,x)}{t-x} v_m(t) w^{(\alpha,\beta)}(t)\,dt \\ &- \sum_{j=1}^{m-\chi} l_{m-\chi,j}(x) \sum_{k=1}^{m} \lambda_{m,k}(w^{(\alpha,\beta)}) \\ &\qquad \cdot \frac{(r_nh)(x_j,t_k) - (r_nh)(x_j,x_j)}{t_k - x_j} v_m(t_k).\end{aligned}$$

Finally, denoting by $(Hg)(x)$ the finite Hilbert transform on $[-1,1]$ of the given function g, i.e.,

$$(Hg)(x) = \lim_{\varepsilon\to 0^+} \int_{|x-t|\geq\varepsilon} \frac{g(t)}{t-x}\,dt, \quad -1 < x < 1,$$

and setting

(3.16)
$$F_m(x) = \sum_{k=1}^{m} \lambda_{m,k}(w^{(\alpha,\beta)}) \frac{(r_nh)(x,t_k) - (r_nh)(x,x)}{t_k - x} v_m(t_k), \quad x \neq t_k,$$

we come to the more convenient formula,

$$\begin{aligned}[(K-K_m)v_m](x) = &(H(r_nh)v_m w^{(\alpha,\beta)})(x) - (r_nh)(x,x)(Hv_m w^{(\alpha,\beta)})(x) \\ &- (L^{(2)}_{m-\chi} F_m)(x).\end{aligned} \tag{3.17}$$

We also recall some inequalities which are useful in the following. The Christoffel numbers defined by (2.4) can be expressed for any $\alpha, \beta > -1$ as

$$\lambda_{m,k}(w^{(\alpha,\beta)}) = \lambda_m(w^{(\alpha,\beta)}, t_k),$$

where

$$\lambda_m(w^{(\alpha,\beta)},t)=\left(\sum_{j=0}^{m-1}[p_j^{(\alpha,\beta)}(t)]^2\right)^{-1}. \tag{3.18}$$

The following properties hold (see [**21**, p. 673])

(3.19)

$$\begin{cases}\lambda_m(w^{(\alpha,\beta)},t)\sim(1/m)(\sqrt{1-t}+m^{-1})^{2\alpha+1}(\sqrt{1+t}+m^{-1})^{2\beta+1},\\ \lambda_{m,k}(w^{(\alpha,\beta)})\sim(\sqrt{1-t_k^2}/m)w^{(\alpha,\beta)}(t_k).\end{cases}$$

Let

$$\sigma_m(w^{(\alpha,\beta)};x)=\sum_{|\theta-\tau_k|\sim m^{-1}}\frac{\lambda_{m,k}(w^{(\alpha,\beta)})}{|x-t_k|}, \tag{3.20}$$

where $|x|\le 1$, $x=\cos\theta$, and $t_k=\cos\tau_k$, $k=1,2,\dots,m$, are the zeros of $p_m^{(\alpha,\beta)}$. Then,

$$\sigma_m(w^{(\alpha,\beta)};x)\le \mathcal{C}w^{(\rho,\sigma)}(x)\log m,\quad |x|\le 1, \tag{3.21}$$

where $\rho=\min(0,\alpha)$, $\sigma=\min(0,\beta)$, $\alpha,\beta>-1$. (See [**4**].)

Finally, we shall use the following inequality

$$\sum_{|\theta-\tau_k|\sim m^{-1}}\frac{(1\pm t_k)^a}{m|x-t_k|}\le\mathcal{C}(\sqrt{1\pm x}+m^{-1})^{2a-1}\log m, \tag{3.22}$$

where $|a|\le 1/2$, $|x|\le 1$ and θ, τ_k, t_k are defined as above. (See [**3**].) Now we are able to prove the following

Lemma 3.4. *Assume* $-(\alpha+\beta)=\chi=0$ *and* $|\alpha|\le 1/2$ *or* $\chi=1$ *with* $-1<\alpha<0$, *and let* $h\in C^r([-1,1]^2)$. *Then*

$$||(K-K_m)v_m||_{w^{(-\alpha,-\beta)}}\le\mathcal{C}||r_nh||_\infty||v_m||_{w^{(\alpha,\beta)}}\log m, \tag{3.23}$$

where $v_m\in\mathbf{P}_{m-1}$, r_nh *is the error corresponding to the function* h *of the Jackson polynomial of degree* $n=[m/2]-1$, *and the constant* $\mathcal{C}$ *is independent of* m *and* h.

Proof. In view of (3.17), we have

$$\begin{aligned}||(K-K_m)v_m||_{w^{(-\alpha,-\beta)}} &\le \{||r_n||_\infty ||Hv_m w^{(\alpha,\beta)}||_{w^{(-\alpha,-\beta)}} \\ &\quad + ||Hv_m r_n w^{(\alpha,\beta)}||_{w^{(-\alpha,-\beta)}}\} \\ &\quad + ||L^{(2)}_{m-\chi}F_m||_{w^{(-\alpha,-\beta)}} \\ &=: I_1 + I_w,\end{aligned} \tag{3.24}$$

where $r_n = r_n h$.

To bound I_1 we recall that for any continuous function g it results

$$||Hw^{(\alpha,\beta)}g||_{w^{(-\alpha,-\beta)}} \le \mathcal{C}||g||_{w^{(\alpha,\beta)}}, \quad -1<\alpha,\beta<1. \tag{3.25}$$

Inequality (3.25) follows from Theorem 3.1 in [**18**, p. 53]. Then we get

$$I_1 \le 2\mathcal{C}||r_n||_\infty ||v_m||_{w^{(\alpha,\beta)}}. \tag{3.26}$$

To bound I_2 we make use of the Gaussian rule and write

$$I_2^2 = ||L^{(2)}_{m-\chi}F_m||^2_{w^{(-\alpha,-\beta)}} = \sum_{j=1}^{m-\chi}\lambda_{m-\chi,j}(w^{(-\alpha,-\beta)})F_m^2(x_j).$$

Recalling (3.16), we have

$$|F_m(x)| \le 2||r_n||_\infty \left[\sum_{k=1}^{m}\lambda_{m,k}(w^{(\alpha,\beta)})\frac{v_m^2(t_k)}{|x-t_k|}\right]^{1/2}\left[\sum_{k=1}^{m}\frac{\lambda_{m,k}(w^{(\alpha,\beta)})}{|x-t_k|}\right]^{1/2}, \qquad x\ne t_k.$$

Then, by Lemma 3.3 and the definition of $\sigma_m(w^{(\alpha,\beta)};x)$ (cf. (3.20)), we deduce

$$\begin{aligned}I_2^2 &\le 4||r_n||^2_\infty \sum_{j=1}^{m-\chi}\lambda_{m-\chi,j}(w^{(-\alpha,-\beta)})\sum_{k=1}^{m}\lambda_{m,k}(w^{(\alpha,\beta)})\frac{\sigma_m(w^{(\alpha,\beta)};x_j)}{|x_j-t_k|}v_m^2(t_k) \\ &= 4||r_n||^2_\infty \sum_{k=1}^{m}\lambda_{m,k}(w^{(\alpha,\beta)})v_m^2(t_k)\sum_{j=1}^{m-\chi}\lambda_{m-\chi,j}(w^{(-\alpha,-\beta)})\frac{\sigma_m(w^{(\alpha,\beta)};x_j)}{|x_j-t_k|} \\ &= 4||r_n||^2_\infty ||v_m^2||^2_{w^{(\alpha,\beta)}}\sum_{j=1}^{m-\chi}\lambda_{m-\chi,j}(w^{(-\alpha,-\beta)})\frac{\sigma_m(w^{(\alpha,\beta)};x_j)}{|x_j-t_k|}.\end{aligned}$$

Therefore, it remains to prove that the behavior of the sum in the last inequality is $O(\log^2 m)$.

At first, assume that $\chi = 1$ with $-1 < \alpha$, $\beta < 0$. In this case, because of (3.21), we have

$$\sigma_m(w^{(\alpha,\beta)}; x_j) \leq \mathcal{C} w^{(\alpha,\beta)}(x_j) \log m.$$

Further, taking into account that

$$\lambda_{m-1,j}(w^{(-\alpha,-\beta)}) \sim (1/m) w^{(-\alpha,-\beta)}(x_j) \sqrt{1 - x_j^2}$$

(cf. (3.19)), we get

$$\begin{aligned}\sum_{j=1}^{m-1} \lambda_{m-1,j}(w^{(-\alpha,-\beta)}) \frac{\sigma_m(w^{(\alpha,\beta)}; x_j)}{|x_j - t_k|} &\leq \mathcal{C} \sum_{j=1}^{m-1} \frac{\sqrt{1 - x_j^2}}{m|x_j - t_k|} \log m \\ &\leq \mathcal{C} \log^2 m.\end{aligned}$$

The last inequality follows from (3.14) and (3.22).

So, if $\chi = 1$ with $-1 < \alpha$, $\beta < 0$, then

$$I_2 \leq \mathcal{C} \log m ||r_n||_\infty ||v_m||_{w^{(\alpha,\beta)}}. \tag{3.27}$$

Now, let $\chi = 0$ and $-1/2 \leq \alpha \leq 1/2$. In this case, $-\alpha = \beta$. Assume for instance $0 \leq \alpha \leq 1/2$. Inequalities (3.13) and (3.21) allow us to write

$$\sigma_m(w^{(\alpha,-\alpha)}; x_j) \leq \mathcal{C}(1 + x_j)^{-\alpha} \log m.$$

Moreover, since (cf. (3.19))

$$\lambda_{m,j}(w^{(-\alpha,\alpha)}) \sim (1/m)(1 - x_j)^{-\alpha+1/2}(1 + x_j)^{\alpha+1/2},$$

and since $0 \leq \alpha \leq 1/2$, we also have

$$\lambda_{m,j}(w^{(-\alpha,\alpha)}) \sigma_m(w^{(\alpha,-\alpha)}; x_j) \leq (\mathcal{C}/m) \sqrt{1 + x_j} \log m.$$

Thus,

$$\begin{aligned}\sum_{j=1}^{m} \lambda_{m,j}(w^{(-\alpha,\alpha)}) \frac{\sigma_m(w^{(\alpha,-\alpha)}; x_j)}{|x_j - t_k|} &\leq \mathcal{C} \sum_{j=1}^{m} \frac{\sqrt{1 + x_j}}{m|x_j - t_k|} \log m \\ &\leq \mathcal{C} \log^2 m,\end{aligned}$$

where the last inequality follows from (3.13) and (3.22). Similar computations can be done in the case $-1/2 \le \alpha < 0$. Therefore, inequality (3.27) is still true when $\chi = 0$ with $|\alpha| \le 1/2$.

Finally, combining (3.26) and (3.27) with (3.24), we deduce (3.23). □

Proof of Theorems 2.1 *and* 2.2. Equation (2.1) can be written in the form

$$Dv + Kv = f, \tag{3.28}$$

where K is the operator defined by (3.2). Assume first $\chi = 1$.

We recall that $L_m^{(1)} f$ denotes the Lagrange interpolation polynomial of degree $m-1$ with the nodes t_k, $k = 1, 2, \dots, m$, and $L_{m-1}^{(2)} f$ denotes the interpolation polynomial of degree $m-2$ with the nodes x_j, $j = 1, 2, \dots, m-1$. By virtue of (2.5) the system (2.6) is equivalent to the operator equation

$$Dv_m + K_m v_m = L_{m-1}^{(2)} f, \tag{3.29}$$

where

$$(K_m v_m)(x) = L_{m-1}^{(2)} \int_{-1}^{1} L_{m,t}^{(1)} \{k(x,t) v_m(t)\} w(t)\, dt,$$

and $L_{m,t}^{(1)}$ stands for the interpolation operator $L_m^{(1)}$ applied to the function $k(x,t)v_m(t)$ with respect to the variable t.

Since the solution of (3.28) is unique in $L_{w,0}^2 := \{v \in L_w^2 : (v,1)_w = 0\}$ and the operator $D : L_{w,0}^2 \to L_{1/w}^2$ is invertible, we conclude that the operator $D + K : L_{w,0}^2 \to L_{1/w}^2$ has a bounded inverse, because $K : L_{w,0}^2 \to L_{1/w}^2$ is a compact operator (see, e.g., [**18**]). Provided (3.29) has a solution v_m, then

$$(D+K)v_m = (K - K_m)v_m + L_{m-1}^{(2)} f.$$

Thus,

$$\begin{aligned} ||v_m||_w &\le ||(D+K)^{-1}|| [||(K-K_m)v_m||_{1/w} + ||L_{m-1}^{(2)} f||_{1/w}] \\ &\le ||(D+K)^{-1}|| [\varepsilon_m ||v_m||_w + ||(D+K_m)v_m||_{1/w}], \end{aligned}$$

where $\varepsilon_m = O(m^{-r-\lambda})$ as $m \to \infty$ in the case of Theorem 2.1 and $\varepsilon_m = O(m^{-r-\lambda} \log m) \to 0$ in the case of Theorem 2.2 (see Lemmas 3.1, 3.2 and 3.4). Consequently, for all sufficiently large m, the estimate

$$\mathcal{C}||v_m||_w \leq ||(D + K_m)v_m||_{1/w}, \tag{3.30}$$

holds with a positive constant $\mathcal{C} \leq ||(D+K)^{-1}||^{-1} - \varepsilon_m$. Since $(D + K_m)v_m \in \operatorname{im} L^{(2)}_{m-1}$ for all $v_m \in \operatorname{im} L^{(1)}_m$, the estimate (3.30) implies the invertibility of the finite dimensional operator $D + K_m$: $\operatorname{im} L^{(1)}_m \to \operatorname{im} L^{(2)}_{m-1}$. Hence, (3.29) has a solution $v_m \in \operatorname{im} L^{(1)}_m$ for all sufficiently large m and $||v_m||_w \leq \mathcal{C}^{-1}||L^{(2)}_{m-1}f||_{1/w} \leq$ const, because of the well-known estimate

$$||f - L^{(2)}_{m-1}f||_{1/w} \leq \mathcal{C}(f)m^{-r-\lambda}, \tag{3.31}$$

(see, e.g., [**20**, Chap. VI, Section 2], [**14**, Conclusion 4.4]).

Estimates (2.9) and (2.10) are an immediate consequence of (3.31), Lemmas 3.1, 3.2, 3.4 and the equation

$$v - v_m = (D+K)^{-1}\{(K - K_m)v_m + (L^{(2)}_{m-1}f - f)\}.$$

Replacing $L^{(2)}_{m-1}$ by $L^{(2)}_m$ and $L^2_{w,0}$ by L^2_w and repeating the preceding argumentations we prove the assertions of Theorems 2.1 and 2.2 for $\chi = 0$. $\square$

Proof of Theorem 2.3. At first, we consider the expansion of the polynomial v_m of degree $m-1$ in the system $\{p^{(\alpha,\beta)}_m\}$

$$v_m(t) = \sum_{k=0}^{m-1} c_k p^{(\alpha,\beta)}_k(t), \quad |t| \leq 1,$$

$$c_k = \int_{-1}^{1} v_m(x) p^{(\alpha,\beta)}_k(x) w^{(\alpha,\beta)}(x)\, dx.$$

Recalling (3.18), we get

$$|v_m(t)| \leq \sqrt{\sum_{k=0}^{m-1} c_k^2}\sqrt{\sum_{k=0}^{m-1} [p^{(\alpha,\beta)}_k(t)]^2}$$

$$= ||v_m||_{w^{(\alpha,\beta)}} \sqrt{\lambda_m^{-1}(w^{(\alpha,\beta)}; t)}.$$

Then, in virtue of (3.19), we can write

$$|v_m(t)| \leq \mathcal{C}\sqrt{m}||v_m||_{w^{(\alpha,\beta)}}(\sqrt{1-t}+m^{-1})^{-\alpha-1/2}(\sqrt{1+t}+m^{-1})^{-\beta-1/2}.$$

Therefore,

$$||v_m||_\infty \leq \mathcal{C}||v_m||_{w^{(\alpha,\beta)}}m^\gamma, \quad \gamma = 1+\max(\alpha,\beta). \tag{3.32}$$

If Δ denotes a closed subset of $(-1,1)$, then

$$\max_{t\in\Delta}|v_m(t)| \leq \mathcal{C}\sqrt{m}||v_m||_{w^{(\alpha,\beta)}}. \tag{3.33}$$

That being stated, if the assumptions of Theorem 2.2 are verified the estimate (2.10) holds and we have

$$v - v_m = \sum_{k=0}^{\infty}(v_{2^{k+1}m} - v_{2^k m}) \tag{3.34}$$

almost everywhere in $[-1,1]$.

On the other hand, applying (3.32) and (2.10), we obtain

$$\begin{aligned}||v_{2^{k+1}m} - v_{2^k m}||_\infty &\leq \mathcal{C}m^\gamma 2^{\gamma(k+1)}||v_{2^{k+1}m} - v_{2^k m}||_{w^{(\alpha,\beta)}}\\ &\leq \mathcal{C}\frac{\log m}{m^{r+\lambda-\gamma}}\,\frac{\log 2^{k+1}}{2^{(r+\lambda-\gamma)(k+1)}}.\end{aligned}$$

Therefore, if $r+\lambda-\gamma > 0$ the series $\sum_{k=0}^{\infty} 2^{-(r+\lambda-\gamma)(k+1)}\log 2^{k+1}$ converges. Consequently, the series in (3.34) converges uniformly in $[-1,1]$. Thus, we deduce that v is continuous and the estimate (2.11) holds.

Making use of (3.33), by similar computations we deduce that v is continuous in any closed subset Δ of $(-1,1)$ and the estimate (2.12) is valid provided $r+\lambda > 1/2$. □

4. Singular integral equations with variable coefficients. In this section a and b are assumed to be real-valued Hölder continuous functions on $[-1,1]$ satisfying

$$[r(x)]^2 := [a(x)]^2 + [b(x)]^2 > 0, \quad x \in [-1,1].$$

Define the continuous function $\alpha_0(x)$, $x \in [-1,1]$, by

$$a(x) - ib(x) = r(x)e^{i\pi\alpha_0(x)}, \quad -1 < \alpha_0(1) \le 1.$$

Further, we choose integers μ and ν such that

$$-1 < \alpha := \mu + \alpha_0(1), \qquad \beta := \nu - \alpha_0(-1) < 1.$$

Define now $\chi := -(\mu+\nu)$ and $\tilde{w}(t) = X(t)/r(t)$, where

$$X(t) := (1-t)^{\mu}(1+t)^{\nu} \exp \int_{-1}^{1} \alpha_0(x)(x-t)^{-1}\, dx.$$

Since $a, b \in \mathrm{Lip}_\lambda[-1,1]$, $0 < \lambda < 1$, the function $\tilde{w}$ admits the representation

$$\tilde{w}(t) = (1-t)^{\alpha}(1+t)^{\beta} w_0(t), \tag{4.1}$$

where $w_0 \in \mathrm{Lip}_\lambda[-1,1]$ is a positive (nonvanishing) function (see [**19**, Sections 26, 27], cf. also [**15**, Lemma 1.4]).

Following [**5, 6**], we shall assume that we can find a nonnegative function c defined on $[-1,1]$ such that

1^0. $B(x) := c(x)b(x)$ is a polynomial of degree R, say,

2^0. The functions $w := \tilde{w}/c$ and $\omega(t) := [X(t)r(t)c(t)]^{-1}$ are integrable,

3^0. If $B(x_0) = 0$, $-1 \le x_0 \le 1$, then $b(x_0) = 0$.

Multiplying Equation (1.1) by $c(x)$, we obtain the following equation

$$(D+K)v = f_0, \tag{4.2}$$

where

$$Dv = a\tilde{w}v + BSwv, \qquad (Su)(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{u(t)}{t-x}\, dt,$$

$$(Kv)(x) = \int_{-1}^{1} k_0(x,t)w(t)v(t)\, dt,$$

and

$$u = wv, \qquad f_0 = cf, \qquad k_0(x,t) = c(x)k(x,t).$$

The analysis of the quadrature method for the approximate solution of Equation (4.2) is essentially based on the following statement.

Lemma 4.1 (cf. [**5, 6, 15**]). *Let p_m be a polynomial of degree m. Then $q_m := Dp_m$ is a polynomial of degree $\le \max(m-\chi, R-1)$ (of degree $m-\chi$ if $m-\chi > R-1$). If p_m^w is an orthogonal polynomial with respect to the weight w of degree m and if $m-\chi > R-1$, then $q_m^\omega = Dp_m^w$ is an orthogonal polynomial with respect to ω of degree $m-\chi$. Moreover, the relation $||p_m^w||_\omega = ||q_m^w||_w$ holds.*

The following properties of the operator $D : L_w^2 \to L_\omega^2$ are well known (see, e.g., [**18, 6**, Section 2, **15**, Theorem 1.13]):

(i) $\dim\ker D = \max(0,\chi)$, $\dim\ker D^* = \max(0,-\chi)$, where $D^* = ac\omega I - SB\omega I$.

(ii) $\ker D = \operatorname{span}\{B, Bt, \dots, Bt^{\chi-1}\}$ if $\chi > 0$.

(iii) Let $\chi < 0$ and $f \in L_\omega^2$. Then the equation $Dv = f$ has a solution $v \in L_w^2$ if and only if $(t^j, f)_\omega = 0$, $j = 0, 1, \dots, -\chi-1$.

(iv) $D^{(-1)}D = I$ if $\chi \le 0$ and $DD^{(-1)} = I$ if $\chi \ge 0$ where $D^{(-1)} = ac\omega I - BS\omega I$.

Given two sequences $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ of real numbers with $b_n \le 0$, $n = 1, 2, \dots$, define the polynomials p_n, $n = 0, 1, \dots$, by

$$p_n(t) = p_n^w(t) + a_n p_{n-1}^w(t) + b_n p_{n-2}^w(t), \tag{4.3}$$

where $p_{-1}^w = p_{-2}^w = 0$. Assuming that all zeros $t_{k,n}$ of $p_n(t)$ lie on $[-1,1]$, we conclude that $-1 \le t_{n,n} < t_{n-1,n} < \cdots < t_{1,n} \le 1$ (see [**10**]). For $q_n = Dp_n$, we have by Lemma 4.1,

$$q_n(x) = q_n^\omega(x) + a_n q_{n-1}^\omega(x) + b_n q_{n-2}^\omega(x), \tag{4.4}$$

if $n > \chi + R + 1$ (which will be assumed in the sequel). Thus, the zeros $x_{j,n}$, $j = 1, 2, \dots, n-\chi$, of $q_n(t)$ are simple and real (cf. [**10**]). Again suppose all zeros $x_{j,n}$ lie on $[-1,1]$. Under these conditions we associate to the polynomials (4.3) and (4.4) the Gauss-type quadrature rules $\mathcal{P}_n$ and $\mathcal{Q}_n$ defined by

$$\mathcal{P}_n(v) = \frac{1}{\pi}\int_{-1}^{1} w(t)L_n^w v(t)\,dt = \sum_{k=1}^{n} A_{k,n}^w v(t_{k,n}),$$
$$\mathcal{Q}_n(f) = \frac{1}{\pi}\int_{-1}^{1} \omega(x)L_{n-\chi}^\omega f(x)\,dx = \sum_{j=1}^{n-\chi} A_{j,n}^\omega f(x_{j,n}),$$

where L_n^w and $L_{n-\chi}^\omega$ are the corresponding interpolation operators, i.e.,

$$L_n^w v(t) = \sum_{k=1}^{n} \frac{p_n(t)v(t_{k,n})}{(t-t_{k,n})p_n'(t_{k,n})},$$
$$L_{n-\chi}^\omega f(x) = \sum_{j=1}^{n-\chi} \frac{q_n(x)f(x_{j,n})}{(x-x_{j,n})q_n'(x_{j,n})}.$$

Notice that (cf. [**10**]) $A_{k,n}^w > 0$, $k = 1,2,\dots,n$, $A_{j,n}^\omega > 0$, $j = 1,2,\dots,n-\chi$. The explicit form of Dv_n, $v_n \in \mathbf{P}_{n-1}$, at the points $x_{j,n}$, is given in the next lemma.

Lemma 4.2 (cf. [**5, 6, 15**]). *Let* $v_n \in \mathbf{P}_{n-1}$. *Then*

$$(Dv_n)(x_{j,n}) = \begin{cases} B(x_{j,n})\sum_{k=1}^{n} A_{k,n}^w v_n(t_{k,n})/(t_{k,n}-x_{j,n}), \\ \qquad \text{if } x_{j,n} \neq t_{k,n},\ k=1,2,\dots,n, \\ a(x_{j,n})\tilde{w}(x_{j,n})v_n(t_{l,n}), \\ \qquad \text{if } x_{j,n} = t_{l,n}. \end{cases}$$

We now consider the following quadrature method for the approximate solution of (4.2): Find $\xi_k = \xi_k^{(n)}$, $k = 1,2,\dots,n$, such that

$$\sum_{k=1}^{n} A_{k,n}^w \left[\frac{B(x_{j,n})}{t_{k,n}-x_{j,n}} + k_0(x_{j,n},t_{k,n})\right]\xi_k = f_0(x_{j,n}), \quad j = 0,1,\dots,n-\chi. \tag{4.5}$$

In the case $\chi > 0$ one needs the additional conditions

$$\int_{-1}^{1} w(t)t^m v(t)\,dt = 0, \quad m = 0,1,\dots,\chi-1, \tag{4.6}$$

in order to define the solution of (4.2) uniquely (cf. property (ii)). Let $L^2_{w,\chi}$ denote the subspace of all functions $v \in L^2_w$ satisfying (4.6) ($L^2_{w,0} = L^2_w$). By property (iv), the operator $D : L^2_{w,\chi} \to L^2_\omega$ is invertible if $\chi \geq 0$, where the inverse is given by $D^{-1} = D^{(-1)}$. Condition (4.6) will be approximated by

$$\sum_{k=1}^{n} A^w_{k,n} t^m_{k,n} \xi_k = 0, \quad m = 0, 1, \ldots, \chi - 1. \tag{4.7}$$

In view of Lemmas 4.1 and 4.2, Equations (4.5) and (4.7) can be rewritten as

$$Dv_n + K_n v_n = L^\omega_{n-\chi} f_0, \tag{4.8}$$

where

$$v_n(t) = \sum_{k=1}^{n} \xi_k \prod_{j=1, j\neq k}^{n} \frac{t - t_{j,n}}{t_{k,n} - t_{j,n}} \in L^2_{w,\chi}$$

and

$$K_n v_n = L^\omega_{n-\chi} \int_{-1}^{1} L^w_{n,t}[k_0(x,t) v_n(t)] w(t)\, dt.$$

Thus, we have the same situation as in the preceding section.

To apply the proof of Theorem 2.1 to (4.8), we suppose that the function c, involved in conditions 1^0 and 2^0, is of the form

$$c(t) = \prod_{j=1}^{N} |s_j - t|^{-\gamma_j} d(t), \quad -1 \leq t \leq 1, \tag{4.9}$$

where $-1 = s_N < s_{N-1} < \cdots < s_2 < s_1 = 1$, and d is both positive and continuous on $[-1, 1]$. Then $\tilde{w}(t)$ can be written as (4.1) where $-1 < \alpha$, $\beta \leq 0$ and w_0 is a positive Hölder continuous function on $[-1, 1]$ (see [**5**, (2.14), (2.15)]). Combining (4.9) and (4.1), it follows that

$$w(t) = (1-t)^{\gamma_1+\alpha}(1+t)^{\gamma_N+\beta} \prod_{j=2}^{N-1} |s_j - t|^{\gamma_j} w_1(t),$$

$$\omega(t) = (1-t)^{\gamma_1-\alpha}(1+t)^{\gamma_N-\beta} \prod_{j=2}^{N-1} |s_j - t|^{\gamma_j} \omega_1(t),$$

where w_1, ω_1 are positive and Hölder continuous on $[-1,1]$. In what follows we assume that

$$\chi = 0 \quad \text{and} \quad |\alpha| \le \frac{1}{2} \quad \text{or} \quad \chi = 1 \quad \text{and} \quad |\alpha|, |\beta| < 1, \tag{4.10}$$

and

$$\gamma_1 > -1+\alpha, \quad \gamma_N > -1+\beta, \quad \gamma_j > -1, \quad j = 2, 3, \dots, N-1. \tag{4.11}$$

Then w and ω are generalized Jacobi weight functions satisfying

$$w(t) = (1-t)^\alpha (1+t)^\beta \omega(t).$$

Repeating the proofs of Theorems 2.1 through 2.3, we obtain the following result.

Theorem 4.1. *Assume* $\chi \ge 0$, (4.10) *and* (4.11). *Suppose the conditions of Theorems* 2.1, 2.2 *or* 2.3 *are fulfilled with* k, h, f *replaced by* ck, ch, cf. *If* (1.1) *has a unique solution* $u \in L^2_{w,\chi}$, *then the system of equations* (4.5), (4.7) *is uniquely solvable for all sufficiently large* n *and the estimates* (2.9), (2.10) *or* (2.11), *respectively, hold.*

Remark . In order to determine an approximate solution of (1.1) in the case $\chi < 0$, one can apply the modified method studied in [**14**, Section 6, **15**, Section 2.2].

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