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AN UNCONVENTIONAL QUADRATURE METHOD FOR LOGARITHMIC-KERNEL INTEGRAL EQUATIONS ON CLOSED CURVES

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ABSTRACT. A new, fully discrete method is proposed for the logarithmic-kernel integral equation of the first kind on a smooth closed curve. The method uses two levels of numerical quadrature: a trapezoidal rule for the integral containing the logarithmic singularity; and a special quadrature rule for the outer integral, which compensates, in part, for the errors in the first integral. A convergence and stability analysis is given, and the predicted orders of convergence verified in a numerical example. A numerical experiment suggests that the method can be useful even for a curve with corners.

1. Introduction. In this paper we propose and analyze a fully discrete method for the approximate solution of

$$-\frac{1}{\pi}\int_\Gamma \log|t-s|z(s)\,dl_s = g(t), \quad t\in\Gamma, \tag{1.1}$$

where z is an unknown function, dl_s the element of arc-length, $|t-s|$ the Euclidean distance between $t,s\in\Gamma$, and Γ a smooth simple closed curve in the plane. The curve is assumed to have transfinite diameter (or conformal radius) different from 1, in which case (1.1) has a unique solution.

If we assume that Γ can be parametrized by a 1-periodic C^∞ function $\nu:\mathbf{R}\to\Gamma$, with $|\nu'(x)|\neq 0$, then (1.1) can be written

$$-\int_0^1 2\log|\nu(x)-\nu(y)|u(y)\,dy = f(x), \quad x\in[0,1], \tag{1.2}$$

or

$$Lu = f, \tag{1.3}$$

where

$$u(x) = z(\nu(x))|\nu'(x)|/(2\pi), \quad x\in[0,1], \tag{1.4}$$

and

$$f(x) = g(\nu(x)), \quad x \in [0,1]. \tag{1.5}$$

The method is applied to the equation in the form (1.2). The method is described in the next section and then discussed in relation to other methods in Section 3. A convergence result for an important special case of the method is stated in Section 4. The theoretical analysis begins in Section 5. Arising out of that analysis there emerges a convergence theorem, stated and proved in Section 6. Before one can make use of the general theorem, one must show that particular versions of the method are 'stable' and of appropriate 'order.' That task is taken up in Section 7, one fruit of this being the result already stated in Section 4. Finally, some numerical examples are discussed in Section 8. The numerical examples include ones for which Γ is not smooth, so that the theory does not strictly apply. Even in this case, the experiment suggests that the method is of practical value, provided the curve is parametrized in an appropriate way in the vicinity of a corner.

2. The approximation. The first step, possibly surprising, is to approximate Lu, defined by (1.2) and (1.3), by a trapezoidal rule. Thus, with N a positive integer and $h = 1/N$, Lu is approximated by $L_h u$, defined by

(2.1)
$$L_h u(x) = -h \sum_{k=0}^{N-1} 2 \log|\nu(x) - \nu(kh)| u(kh), \quad x \in [0,1],\ Nx \notin \mathbf{Z}.$$

Needless to say, this is not a good approximation for most values of x; but the damage can be repaired later.

The proposed method is reminiscent of the Petrov-Galerkin method in that we require a 'test' space S_h, here chosen to be the space of 1-periodic smoothest splines of order r (i.e., degree $\le r-1$), with the uniformly spaced knots

$$\{kh : 0 \le k \le N-1\}. \tag{2.2}$$

If $r = 2$, then S_h is the space of continuous piecewise-linear functions with these knots. If $r = 4$, then S_h is the space of C^2 cubic splines.

Instead of the inner product

$$(v, w) = \int_0^1 v(x)\overline{w(x)}\, dx, \tag{2.3}$$

as used in the Galerkin method, the method makes use of a discrete inner product

$$(v, w)_h = Q_h(v\bar{w}), \tag{2.4}$$

where

$$Q_h g = h \sum_{k=0}^{N-1} \sum_{j=1}^{J} w_j g((k + \xi_j)h), \tag{2.5}$$

with

$$0 < \xi_1 < \xi_2 < \cdots < \xi_J < 1,$$

and

$$\sum_{j=1}^{J} w_j = 1, \quad w_j > 0, \quad \text{for } 1 \le j \le J. \tag{2.6}$$

Thus, Q_h is a composite quadrature rule, obtained by copying onto each subinterval $[kh, (k+1)h]$ the J-point quadrature rule for the unit interval

$$Qg = \sum_{j=1}^{J} w_j g(\xi_j). \tag{2.7}$$

We shall have much to say about the choice of the quadrature rule. For now, it suffices to say that the rules to be developed here are quite different from conventional quadrature rules.

The method may now be described as: find u_h such that

$$(L_h u_h, \chi)_h = (f, \chi)_h \quad \forall\, \chi \in S_h. \tag{2.8}$$

Remark . Since $L_h u_h$ depends on the values of u_h only on the discrete set (2.2), it is clear that (2.8) can determine u_h only on that set. If values of u_h are needed at intermediate points, then some form of interpolation is needed. For theoretical purposes, we shall, in Section 4, define u_h by trigonometric interpolation between the points of (2.2).

In practice, one must choose a basis $\{v_0, \ldots, v_{N-1}\}$ for S_h. For example, in the important case $r = 2$, one may choose v_k to be the hat-function centered at kh, i.e.,

$$v_k(x) = \begin{cases} 1 - |x - kh|/h, & \text{if } |x - kh| \leq h, \\ 0, & \text{otherwise.} \end{cases} \tag{2.9}$$

In general, the 1-periodic B-splines centered at kh, for $k = 0, \ldots, N-1$, form a convenient basis.

Given a basis, the approximation becomes: find u_h such that

$$\sum_{k=0}^{N-1} a_{l,k} u_h(kh) = (f, v_l)_h, \quad l = 0, \ldots, N-1, \tag{2.10}$$

where
(2.11)
$$\begin{aligned} a_{l,k} &= -h(2 \log |\nu(\cdot) - \nu(kh)|, v_l)_h \\ &= -h^2 \sum_{k'=0}^{N-1} \sum_{j=1}^{J} w_j 2 \log |\nu((k' + \xi_j)h) - \nu(kh)| \overline{v_l((k' + \xi_j)h)}. \end{aligned}$$

Remark . A complete specification of the method requires both a value for r and a specific choice for the quadrature rule Q. To fix ideas, we mention now that the following choice turns out to be particularly interesting. We take for the spline space $r = 2$ (i.e., piecewise-linear functions), and for the quadrature rule

$$\begin{aligned} &J = 2, \\ &\xi_1 = \frac{1}{6}, \qquad \xi_2 = \frac{5}{6}, \\ &w_1 = \frac{1}{2}, \qquad w_2 = \frac{1}{2}. \end{aligned} \tag{2.12}$$

We shall see that this choice gives a stable method and an $O(h^3)$ order of uniform convergence, if u is sufficiently regular.

An interesting special case of a different kind is that in which $J = 1$. In this case, the quadrature rule (2.5) is a (shifted) trapezoidal rule, and there is only one parameter to be chosen, namely $\xi = \xi_1$. In this case, it is easily seen that (2.8) is mathematically equivalent (provided the matrix $\{v_l((k+\xi)h)\}_{l,k=0}^{N-1}$ is nonsingular) to

$$L_h u_h((k+\xi)h) = f((k+\xi)h), \quad k = 0, \dots, N-1. \tag{2.13}$$

Thus, in this situation, the test space S_h becomes irrelevant, and the method may be described simply as: replace Lu by the trapezoidal rule approximation $L_h u$, as in (2.1), and then collocate at the set

$$\{(k+\xi)h : 0 \leq k \leq N-1\}. \tag{2.14}$$

If $\xi = 1/2$ we shall see (in Theorem 7.1) that the method with $r = 2$ and $J = 1$ is unstable. If $0 < \xi < 1/6$ or $1/6 < \xi < 1/2$, then it turns out that the method is stable but yields only $O(h)$ convergence, even if u is smooth. But for one special ξ value, namely $\xi = 1/6$, the convergence is $O(h^2)$ if u is smooth. All of these properties are developed in Section 7. At the end of that section we try to explain the special quality of the abscissa $\xi = 1/6$ in the 1-point rule and also in the 2-point rule (2.12).

3. Related methods. Several existing methods for the numerical solution of (1.1) have some relation to the present method, though none is really similar.

Ruotsalainen and Saranen [**9**] have proposed a Petrov-Galerkin method, in which the trial functions are Dirac delta functions and the test functions smoothest splines. Their use of delta functions as trial functions is equivalent, from the point of view of the linear system that results, to the use of the trapezoidal rule in (2.1). Their linear system is nevertheless different, because they use the exact inner product (2.3) instead of the discrete inner product (2.4). At a theoretical level, the difference of view influences the whole analysis, and through it the very design of the quadrature rules typified by (2.12). Related to this is the different nature of the theoretical results: the delta-function approach in [**9**] requires little in the way of regularity of the solution u, but obtains convergence estimates only in negative norms; whereas the present results (for example, those in the next section) obtain uniform estimates for the error, but require substantial regularity of the solution to get high orders of convergence.

Another method based on the Petrov-Galerkin method with a trial space of delta functions, but this time with a trigonometric function test space, is that of Cheng and Arnold [**7**]. That paper also proposes a fully discrete method, based on the use of the trapezoidal rule in the outer integral of the Petrov-Galerkin method.

Another fully discrete method, based this time on a trigonometric choice for both trial and test space, combined with a trapezoidal rule approximation of the integrals, is that of Atkinson [**4**].

The fully discrete methods mentioned above, those of [**7**] and [**4**], differ from the present method in a significant way, in that in both cases the quadrature approximation is applied only to the part of the integral operator L (with smooth kernel) that represents the departure of the curve Γ from a circle; whereas the principal part of the operator L (that is, the part which is appropriate to the case of a circle) is handled exactly. While both methods are undoubtedly effective for smooth curves, this exact treatment of the principal part may make it difficult to extend the methods to curves which are not smooth (such as, for instance, the curve in the second and third examples in Section 8).

In a different direction, the present method is related to the qualocation method of [**12, 13, 6**]. In those papers detailed Fourier series arguments, in the manner of [**10, 3**], were used to design a method similar to, but with a higher order of convergence than, the collocation method. Here similar Fourier series arguments are used to find quadrature rules Q which, when used in the manner described in the preceding section, lead to high orders of convergence. The present quadrature rules are different from those which occur in the qualocation method [**12, 13, 6**] but are also quite different from conventional quadrature rules because of the different purposes the rules aim to achieve.

4. Convergence theorem—special case. A convergence analysis requires u_h to be defined everywhere, not just on the discrete set $\{kh\}_{k=0}^{N-1}$. Given the values of u_h on this set, we choose to define its values elsewhere by trigonometric interpolation, as follows. Define

$$\Lambda_h = \left\{ m \in \mathbf{Z} : -\frac{N}{2} < m \leq \frac{N}{2} \right\} \tag{4.1}$$

and

$$T_h = \operatorname{span} \{ e^{i2\pi mx} : m \in \Lambda_h \}. \tag{4.2}$$

Then u_h is chosen to be the unique element of T_h which has the prescribed values on the set (2.2). Its explicit formula, as is easily verified, is

$$u_h(x) = h \sum_{m \in \Lambda_h} \sum_{k=0}^{N-1} u_h(kh) e^{i2\pi m(x-kh)}. \tag{4.3}$$

We shall measure errors usually in Sobolev norms $|| \cdot ||_s$, defined for $s \in \mathbf{R}$, by

$$||v||_s^2 = |\hat{v}(0)|^2 + \sum_{m \in \mathbf{Z}^*} |m|^{2s} |\hat{v}(m)|^2, \tag{4.4}$$

where $\hat{v}(m)$ is the Fourier coefficient

$$\hat{v}(m) = \int_0^1 v(x) e^{-i2\pi mx} \, dx, \quad m \in \mathbf{Z}, \tag{4.5}$$

and $\mathbf{Z}^*$ denotes $\mathbf{Z} \backslash \{0\}$. The Sobolev space H^s is the closure of C_p^∞ (the set of 1-periodic C^∞ functions) with respect to the norm $|| \cdot ||_s$. Occasionally, we also use the uniform norm

$$||v|| = \max_{x \in \mathbf{R}} |v(x)|, \tag{4.6}$$

for v in C_p, the space of 1-periodic continuous functions.

Theorem 4.1. *Let $r = 2$ and let Q be the 2-point quadrature rule with abscissae and weights given by* (2.12). *If f is continuous and* 1-*periodic, then* (2.8) *has a unique solution $u_h \in T_h$ for all h sufficiently small. If $u \in H^t$, and if*

$$s > \frac{1}{2}, \quad s + \frac{1}{2} < t \leq s + 3, \tag{4.7}$$

then, for h sufficiently small,

$$||u_h - u||_s \leq Ch^{t-s}||u||_t.$$

(In this paper C is a generic constant, which may take different values at its different occurrences.) Later in this paper, we shall state and prove a more general result, Theorem 6.1, in which the quadrature rule is not specified, but certain conditions must be satisfied. Theorem 4.1 then follows as a special case, with the help of results obtained in Section 7. Theorem 4.1 is quoted here as a separate result both because of its greater transparency and also because it is arguably the most important special case.

Corollary 4.2. *Let u_h be as in Theorem* 4.1. *If $u \in H^t$ with $t > 7/2$, then for all h sufficiently small,*

$$||u_h - u|| \leq Ch^3||u||_t.$$

Thus the method yields uniform errors of order $O(h^3)$, for u sufficiently smooth.

Proof of Corollary 4.2. With s chosen as $t-3$, Theorem 4.1 yields

$$||u_h - u||_s \leq Ch^3||u||_t.$$

The result then follows from the well-known imbedding of H^s in C_p for $s > 1/2$. $\square$

If Q is replaced by the 1-point quadrature rule with $\xi = 1/6$, as described at the end of Section 2, then Theorem 4.1 still holds except that (4.7) is replaced by

$$s > \frac{1}{2}, \quad s + \frac{1}{2} < t \leq s + 2. \tag{4.8}$$

Thus, the uniform error bound in Corollary 4.2 is now replaced by

$$||u_h - u|| \leq Ch^2||u||_t,$$

provided $t > 5/2$. In other words, the method now yields errors of order $O(h^2)$ under appropriate smoothness conditions on u.

Other special cases (including ones with higher orders of convergence) can be constructed by using the general Theorem 6.1 and the results in Section 7.

5. Analysis. For the present, we allow the quadrature rule (2.7) to be general and proceed by applying Fourier analysis to the approximate method (2.8), where u_h is the trigonometric interpolant given by (4.3).

We may begin with the well-known Fourier series for the $(\log\circ\sin)$ function,

$$\begin{aligned} -\log(2|\sin\pi t|) &= \sum_{m=1}^{\infty}\frac{1}{m}\cos 2\pi m t \\ &= \frac{1}{2}\sum_{m\in\mathbf{Z}^*}\frac{1}{|m|}e^{i2\pi m t}, \quad t\notin\mathbf{Z}, \end{aligned} \tag{5.1}$$

where, as always when dealing with nonabsolutely convergent Fourier series, we shall understand symmetric partial sums, i.e.,

$$\sum_m = \lim_{L\to\infty}\sum_{|m|\le L}. \tag{5.2}$$

On multiplying (5.1) by two and adding one to each side, we obtain

$$-2\log(2e^{-\frac{1}{2}}|\sin\pi t|) = \sum_{m\in\mathbf{Z}}\frac{1}{\tilde m}e^{i2\pi m t}, \quad t\notin\mathbf{Z}, \tag{5.3}$$

where

$$\tilde m = \begin{cases} 1, & \text{if } m = 0, \\ |m|, & \text{if } m \ne 0. \end{cases}$$

Now define an integral operator A with (5.3) as convolutional kernel, that is,

$$Au(x) = -\int_0^1 2\log(2e^{-\frac{1}{2}}|\sin\pi(x-y)|)u(y)\,dy. \tag{5.4}$$

Then it follows from (5.3) that

$$(Au)^\wedge(m) = \frac{1}{\tilde{m}}\hat{u}(m). \tag{5.5}$$

Thus, as is well known, A is a pseudodifferential operator of order -1. Moreover, A is a particular case of the operator L defined by (1.2) and (1.3); in fact, it is L for the case of a circle of radius $e^{-1/2}$.

For the general case of a smooth curve Γ, we may write

$$L = A + B, \tag{5.6}$$

where

$$Bu(x) = -\int_0^1 2\log\left|\frac{\nu(x)-\nu(y)}{2e^{-\frac{1}{2}}\sin\pi(x-y)}\right| u(y)\,dy. \tag{5.7}$$

Because $\nu \in C^\infty$, the kernel of the operator B is smooth, and from this it follows that B is a smoothing operator, in the sense that

$$B : H^s \to H^t \quad \forall\, s,t \in \mathbf{R}, \quad s \geq 0. \tag{5.8}$$

For this reason, B can be treated by a perturbation argument and so plays only a secondary role in the analysis.

We recall that the first step in the approximate method is to replace L by the trapezoidal rule approximation L_h, defined by (2.1). Correspondingly, we may write

$$L_h = A_h + B_h, \tag{5.9}$$

where

$$\begin{aligned} A_h u(x) &= -h\sum_{k=0}^{N-1} 2\log(2e^{-\frac{1}{2}}|\sin\pi(x-kh)|)u(kh) \\ &= h\sum_{k=0}^{N-1}\sum_{m\in\mathbf{Z}} \frac{1}{\tilde{m}} e^{i2\pi m(x-kh)}u(kh), \quad Nx \notin \mathbf{Z}, \end{aligned} \tag{5.10}$$

and

$$B_h u(x) = -h\sum_{k=0}^{N-1} 2\log\left|\frac{\nu(x)-\nu(kh)}{2e^{-\frac{1}{2}}\sin\pi(x-kh)}\right| u(kh), \quad Nx \notin \mathbf{Z}. \tag{5.11}$$

The approximate method may now be expressed as: find $u_h \in T_h$ such that

$$
\begin{aligned}
((A_h + B_h)u_h, \chi)_h &= (f, \chi)_h \\
&= ((A + B)u, \chi)_h \quad \forall \chi \in S_h.
\end{aligned} \tag{5.12}
$$

For the remainder of this section, we shall consider the special case $B = 0$ or, in other words, the case in which Γ is a circle of appropriate radius. In this case the approximate method is: find $u_h \in T_h$ such that

$$
(A_h u_h, \chi)_h = (Au, \chi)_h \quad \forall \chi \in S_h. \tag{5.13}
$$

Equation (5.13) may be analyzed by Fourier series techniques. From (4.3) and (4.5), we have

$$
\hat{u}_h(m) = \begin{cases} h \sum_{k=0}^{N-1} e^{-i2\pi mkh} u_h(kh), & m \in \Lambda_h, \\ 0, & m \notin \Lambda_h. \end{cases}
$$

Since the sum on the right is unchanged if m is replaced by $m + \alpha N$ for $\alpha \in \mathbf{Z}$, we conclude that

$$
h \sum_{k=0}^{N-1} e^{-i2\pi mkh} u_h(kh) = \hat{u}_h(m_0(m, N)), \quad m \in \mathbf{Z},
$$

where $m_0(m, N)$ is the unique element of Λ_h which differs from m by a multiple of N. From the latter and (5.10), we now obtain

$$
A_h u_h(x) = \sum_{m \in \mathbf{Z}} \frac{1}{\tilde{m}} \hat{u}_h(m_0(m, N)) e^{i2\pi mx}. \tag{5.14}
$$

On writing $m = \mu + lN$ in the latter sum, where $\mu \in \Lambda_h$ and $l \in \mathbf{Z}$, we obtain

$$
A_h u_h(x) = \sum_{\mu \in \Lambda_h} \frac{1}{\tilde{\mu}} \hat{u}_h(\mu) e^{i2\pi\mu x} \sum_{l \in \mathbf{Z}} \frac{\tilde{\mu}}{(\mu + lN)^{\sim}} e^{i2\pi lNx},
$$

and on separating out the $\mu = 0$ and $l = 0$ terms,

$$
\begin{aligned}
A_h u_h(x) = \hat{u}_h(0) &\left[1 + h \sum_{l \in \mathbf{Z}^*} \frac{1}{|l|} e^{i2\pi lNx} \right] \\
&+ \sum_{\mu \in \Lambda_h^*} \frac{1}{|\mu|} \hat{u}_h(\mu) e^{i2\pi\mu x} \left[1 + |\mu| h \sum_{l \in \mathbf{Z}^*} \frac{1}{|l + \mu h|} e^{i2\pi lNx} \right],
\end{aligned}
$$

or

$$
\begin{aligned}
A_h u_h(x) = {} & \hat{u}_h(0)[1 + hG_1(Nx)] \\
& + \sum_{\mu \in \Lambda_h^*} \frac{1}{|\mu|} \hat{u}_h(\mu) e^{i2\pi\mu x}[1 + \Gamma_1(Nx, \mu h)],
\end{aligned} \tag{5.15}
$$

where $\Lambda_h^* = \Lambda_h \backslash \{0\}$, and where for $\alpha > 0$, $\xi \in \mathbf{R}$ and $\eta \in [-1/2, 1/2]$, we define

$$
G_\alpha(\xi) = 2 \sum_{l=1}^{\infty} \frac{1}{l^\alpha} \cos 2\pi l \xi, \tag{5.16}
$$

$$
F_\alpha^+(\xi, \eta) = \sum_{l \in \mathbf{Z}^*} \frac{1}{|l + \eta|^\alpha} e^{i2\pi l \xi}, \tag{5.17}
$$

and

$$
\Gamma_\alpha(\xi, \eta) = |\eta|^\alpha F_\alpha^+(\xi, \eta). \tag{5.18}
$$

The function F_α^+, and the similarly defined

$$
F_\alpha^-(\xi, \eta) = \sum_{l \in \mathbf{Z}^*} \frac{\operatorname{sign} l}{|l + \eta|^\alpha} e^{i2\pi l \xi}, \tag{5.19}
$$

have appeared before in Fourier analyses of boundary integral methods [**8, 11, 3, 13, 6**], and their properties have been studied in [**5**]. Here we use the notation of [**6**].

The next step towards analyzing (5.13) is to define a suitable basis for S_h. We follow [**6**] in defining the basis to be $\{\psi_\mu : \mu \in \Lambda_h\}$, where

$$
\psi_\mu(x) = \begin{cases} 1, & \text{if } \mu = 0, \\ \sum_{m \equiv \mu} (\frac{\mu}{m})^r e^{i2\pi m x}, & \text{if } \mu \in \Lambda_h^*, \end{cases} \tag{5.20}
$$

and where, here and elsewhere, $m \equiv \mu$ means $m \equiv \mu \pmod N$. (That ψ_μ is a periodic spline of order r follows from the characterization of such splines through the recurrence relation [**2**]

$$
m^r \hat{v}(m) = \mu^r \hat{v}(\mu) \quad \text{if } m \equiv \mu
$$

between their Fourier coefficients.) The particular virtue of this basis lies in the transformation property

$$\psi_\mu(x+h) = e^{i2\pi\mu h}\psi_\mu(x), \quad \mu \in \Lambda_h,\ x \in \mathbf{R}, \tag{5.21}$$

expressing the fact that ψ_μ behaves under translations by $h = 1/N$ in exactly the same way as the exponential function ϕ_μ, where ϕ_m is defined by

$$\phi_m(x) := e^{i2\pi mx}, \quad m \in \mathbf{Z},\ x \in \mathbf{R}. \tag{5.22}$$

In terms of the functions $F_\alpha^\pm$ defined above, we may write the basis function (5.20) as

$$\psi_\mu(x) = \begin{cases} 1, & \text{if } \mu = 0, \\ \phi_\mu(x)[1+\Delta(Nx,\mu h)], & \text{if } \mu \in \Lambda_h^*, \end{cases} \tag{5.23}$$

where

$$\Delta(\xi,\eta) = \eta^r F_r^\pm(\xi,\eta), \tag{5.24}$$

with the + or − sign holding when r is even or odd, respectively. It may be noted that, if Δ is replaced by zero, then the test space S_h reduces to the trigonometric space T_h defined by (4.2).

The equation (5.13) is now equivalent to

$$(A_h u_h, \psi_\mu)_h = (Au, \psi_\mu)_h, \quad \mu \in \Lambda_h. \tag{5.25}$$

The right side of the equation can easily be worked out by using

(5.26)
$$(\phi_m, \psi_\mu)_h = \begin{cases} 0, & \text{if } m \not\equiv \mu, \\ \sum_j w_j e^{i2\pi m\xi_j h}, & \text{if } m \equiv \mu = 0, \\ \sum_j w_j e^{i2\pi(m-\mu)\xi_j h}[1+\overline{\Delta(\xi_j,\mu h)}], & \text{if } m \equiv \mu \in \Lambda_h^*, \end{cases}$$

which follows easily from the definitions and the transformation property (5.21).

For $u \in H^t$ with $t > -1/2$, the Fourier series for Au (using (5.5)) is

$$Au(x) = \sum_{m\in\mathbf{Z}} \frac{1}{\tilde{m}}\hat{u}(m)e^{i2\pi mx}, \tag{5.27}$$

which converges absolutely, since by an application of the Cauchy-Schwarz inequality,

$$\left|\sum_{m\in\mathbf{Z}}\frac{1}{\tilde{m}}\hat{u}(m)e^{i2\pi mx}\right| = \left|\sum_{m\in\mathbf{Z}}\frac{1}{\tilde{m}^{1+t}}\tilde{m}^t\hat{u}(m)e^{i2\pi mx}\right|$$
$$\leq \left(\sum_{n\in\mathbf{Z}}\frac{1}{\tilde{n}^{2+2t}}\right)^{\frac{1}{2}}\left(\sum_{m\in\mathbf{Z}}\tilde{m}^{2t}|\hat{u}(m)|^2\right)^{\frac{1}{2}} = C||u||_t.$$

From (5.27) and (5.26), we obtain

$$(5.28)\quad (Au,\psi_\mu)_h$$
$$= \begin{cases} \hat{u}(0) + \sum_j w_j \sum'_{m\equiv 0} \frac{1}{|m|}\hat{u}(m)e^{i2\pi m\xi_j h}, & \text{if } \mu = 0, \\ \frac{1}{|\mu|}\sum_j w_j \sum_{m\equiv\mu} |\frac{\mu}{m}|\hat{u}(m)e^{i2\pi(m-\mu)\xi_j h}[1+\overline{\Delta(\xi_j,\mu h)}], & \text{if } \mu\in\Lambda_h^*, \end{cases}$$

where $\sum'_{m\equiv\mu}$ denotes the sum over all values of m congruent to, but different from, μ.

In a similar way, from (5.15) and (5.21), together with the fact that $G_\alpha(\xi)$ and $\Gamma_\alpha(\xi,\eta)$ are 1-periodic in ξ, we obtain

$$(5.29)\qquad (A_h u_h,\psi_\mu)_h = \begin{cases} \hat{u}_h(0)d_h, & \text{if } \mu = 0, \\ \frac{1}{|\mu|}\hat{u}_h(\mu)D(\mu h), & \text{if } \mu\in\Lambda_h^*, \end{cases}$$

where

$$(5.30)\qquad d_h = 1 + h\sum_j w_j G_1(\xi_j),$$

$$(5.31)\qquad D(\eta) = \sum_j w_j[1+\Gamma_1(\xi_j,\eta)][1+\overline{\Delta(\xi_j,\eta)}].$$

Values of $\hat{u}_h(\mu)$ can now be determined (uniquely) by equating the two sides of (5.25), provided only that d_h and $D(\mu h)$ are different from zero. The former is certainly bounded away from zero if h is sufficiently small, but the latter may not be. We therefore introduce a concept of stability:

Definition. The method is *stable* if

$$\inf\left\{|D(\eta)| : \eta \in \left[-\frac{1}{2}, \frac{1}{2}\right]\right\} > 0.$$

Conditions under which the method is stable are explored in Section 7. For the present, we simply assume that it is so. On equating the two sides of (5.25) and then solving for $\hat{u}_h(\mu)$, we find

$$\hat{u}_h(\mu) - \hat{u}(\mu) = \begin{cases} -\frac{e_h}{d_h}\hat{u}(0) + r_h, & \text{if } \mu = 0, \\ -\frac{E(\mu h)}{D(\mu h)}\hat{u}(\mu) + R_h(\mu), & \text{if } \mu \in \Lambda_h^*, \end{cases} \tag{5.32}$$

where

$$e_h = d_h - 1 = h\sum_j w_j G_1(\xi_j), \tag{5.33}$$

$$\begin{aligned} E(\eta) &= D(\eta) - \sum_j w_j[1 + \overline{\Delta(\xi_j, \eta)}] \\ &= \sum_j w_j \Gamma_1(\xi_j, \eta)[1 + \overline{\Delta(\xi_j, \eta)}], \end{aligned} \tag{5.34}$$

$$r_h = \frac{1}{d_h}\sum_j w_j \sum_{m\equiv 0}{}' \frac{1}{|m|}\hat{u}(m)e^{i2\pi m\xi_j h}, \tag{5.35}$$

(5.36)
$$R_h(\mu) = \frac{1}{D(\mu h)}\sum_j w_j \sum_{m\equiv\mu}{}' \left|\frac{\mu}{m}\right| \hat{u}(m)e^{i2\pi(m-\mu)\xi_j h}[1 + \overline{\Delta(\xi_j, \mu h)}].$$

The terms r_h and $R_h(\mu)$ in the error expression depend only on the higher Fourier coefficients of u (specifically, only on $\hat{u}(m)$ with $|m| \geq N/2$) and can, therefore, be made small by imposing appropriate conditions on u. The first terms in (5.32), on the other hand, determine the rate of convergence of the error independently of u, provided $\hat{u}(\mu) \neq 0$. That rate of convergence depends on the behavior of e_h and $E(\mu h)$ when h is small. Thus, we say:

Definition. The method is of *order* p if p is the least nonnegative number such that

$$e_h = O(h^p) \quad \text{as } h \to 0$$

and

$$E(\eta) = O(|\eta|^p) \quad \text{as } \eta \to 0.$$

6. Convergence theorem—general case. We now state a convergence theorem for the case of the general equation (1.3), where Γ is a C^∞ curve. The theorem is expressed in terms of the notions of stability and order defined in the preceding section.

Theorem 6.1. *Let u be the unique solution of* (1.3). *Moreover, let $u_h \in T_h$ be determined by* (2.8) *and assume the method to be stable and of order $p > 1/2$. If f is continuous and* 1*-periodic, then u_h exists and is unique for all h sufficiently small. Further, if $u \in H^t$, and if*

$$s > \frac{1}{2}, \quad s + \frac{1}{2} < t \leq s + p, \tag{6.1}$$

then, for h sufficiently small,

$$||u_h - u||_s \leq C h^{t-s} ||u||_t.$$

Corollary 6.2. *Let the method and $u_h \in T_h$ be as in Theorem* 6.1. *If $u \in H^t$ with $t > p + 1/2$, then, for all h sufficiently small,*

$$||u_h - u|| \leq C h^p ||u||_t.$$

The corollary follows by the same argument as Corollary 4.2.

Proof of Theorem 6.1. We first prove the result for the case in which Γ is a circle of radius $e^{-1/2}$. In this case the operator B defined by (5.7) vanishes, and $L = A$. In this part of the proof, the conditions (6.1), which govern s and t in the theorem, can be weakened to

$$-1 \leq s \leq t \leq s + p, \quad t > -\frac{1}{2}. \tag{6.2}$$

The analysis in the preceding section establishes, under the assumption that the method is stable, that the nonvanishing Fourier coefficients of u_h are uniquely determined and differ from the Fourier coefficients of the exact solution by the expression (5.32).

By definition,

$$||u_h - u||_s^2 = |\hat{u}_h(0) - \hat{u}(0)|^2 + |u_h - u|_s^2, \tag{6.3}$$

where

$$|g|_s^2 = \sum_{m\in\mathbf{Z}^*} |m|^{2s} |\hat{g}(m)|^2.$$

Now, from (5.32),

(6.4)

$$\begin{aligned}
|\hat{u}_h(0) - \hat{u}(0)|^2 &= |-(e_h/d_h)\hat{u}(0) + r_h|^2 \le 2|(e_h/d_h)\hat{u}(0)|^2 + 2|r_h|^2 \\
&\le C|e_h|^2|\hat{u}(0)|^2 + C\Big(\sum_j w_j \sum_{m\equiv 0}{}' |m|^{-1}|\hat{u}(m)|\Big)^2 \\
&\le Ch^{2p}|\hat{u}(0)|^2 + C\Big(\sum_{m\equiv 0}{}' |m|^{-1-t}|m|^t|\hat{u}(m)|\Big)^2 \\
&\le Ch^{2p}|\hat{u}(0)|^2 + C\sum_{n\equiv 0}{}' |n|^{-2-2t} \sum_{m\equiv 0}{}' |m|^{2t}|\hat{u}(m)|^2 \\
&\le Ch^{2p}|\hat{u}(0)|^2 + C\sum_{l\in\mathbf{Z}^*} |lN|^{-2-2t}|u|_t^2 \\
&\le Ch^{2(t-s)}|\hat{u}(0)|^2 + Ch^{2+2t}\sum_{l=1}^{\infty} l^{-2-2t}|u|_t^2 \\
&\le Ch^{2(t-s)}|\hat{u}(0)|^2 + Ch^{2(t-s)}|u|_t^2 \\
&= Ch^{2(t-s)}||u||_t^2,
\end{aligned}$$

in the course of which we have used the stability property $|d_h|^{-1} \le C$ for h sufficiently small and the conditions (6.2), as well as, in the first line, $(a+b)^2 \le 2a^2 + 2b^2$. Thus, the first term of (6.3) satisfies the desired inequality.

Because $\hat{u}_h(m) = 0$ for $m \notin \Lambda_h$, the second term of (6.3) may be written as

(6.5)

$$|u_h - u|_s^2 = \sum_{\mu\in\Lambda_h^*} |\mu|^{2s}|\hat{u}_h(\mu) - \hat{u}(\mu)|^2 + \sum_{m\notin\Lambda_h} |m|^{2s}|\hat{u}(m)|^2 = U + T,$$

where, from (5.32),
(6.6)

$$
\begin{aligned}
U &= \sum_{\mu\in\Lambda_h^*} |\mu|^{2s}\left| -\frac{E(\mu h)}{D(\mu h)}\hat{u}(\mu) + R_h(\mu)\right|^2 \\
&\le C \sum_{\mu\in\Lambda_h^*} |\mu|^{2s}|E(\mu h)\hat{u}(\mu)|^2 + 2\sum_{\mu\in\Lambda_h^*} |\mu|^{2s}|R_h(\mu)|^2 \\
&\le C \sum_{\mu\in\Lambda_h^*} |\mu|^{2s}|\mu h|^{2p}|\hat{u}(\mu)|^2 + C\sum_{\mu\in\Lambda_h^*} |\mu|^{2s}\left(\sideset{}{'}\sum_{m\equiv\mu}\left|\frac{\mu}{m}\right||\hat{u}(m)|\right)^2 \\
&= Ch^{2(t-s)} \sum_{\mu\in\Lambda_h^*} |\mu h|^{2(p+s-t)}|\mu|^{2t}|\hat{u}(\mu)|^2 \\
&\quad + C\sum_{\mu\in\Lambda_h^*} |\mu|^{2(s+1)}\left(\sideset{}{'}\sum_{m\equiv\mu}|m|^{-1-t}|m|^t|\hat{u}(m)|\right)^2 \\
&\le Ch^{2(t-s)} \sum_{\mu\in\Lambda_h^*} |\mu|^{2t}|\hat{u}(\mu)|^2 + C\sum_{\mu\in\Lambda_h^*} |\mu|^{2(s+1)}\sideset{}{'}\sum_{n\equiv\mu}|n|^{-2-2t} \\
&\quad \cdot \sideset{}{'}\sum_{m\equiv\mu}|m|^{2t}|\hat{u}(m)|^2 \\
&\le Ch^{2(t-s)}|u|_t^2 + C\sum_{\mu\in\Lambda_h^*} |\mu|^{2(s+1)}\sum_{l\in\mathbf{Z}^*}|lN+\mu|^{-2-2t}\sideset{}{'}\sum_{m\equiv\mu}|m|^{2t}|\hat{u}(m)|^2 \\
&= Ch^{2(t-s)}|u|_t^2 + Ch^{2(t-s)}\sum_{\mu\in\Lambda_h^*} |\mu h|^{2(s+1)}\sum_{l\in\mathbf{Z}^*}|l+\mu h|^{-2-2t} \\
&\quad \cdot \sideset{}{'}\sum_{m\equiv\mu}|m|^{2t}|\hat{u}(m)|^2 \\
&\le Ch^{2(t-s)}|u|_t^2 + Ch^{2(t-s)}\sum_{\mu\in\Lambda_h^*}\sideset{}{'}\sum_{m\equiv\mu}|m|^{2t}|\hat{u}(m)|^2 \\
&\le Ch^{2(t-s)}|u|_t^2,
\end{aligned}
$$

in which, in addition to the stability property $|D(\mu h)^{-1}| \le C$ and the conditions (6.2), we have used $|\mu h| \le 1/2$,

$$
\sup\left\{\sum_{l\in\mathbf{Z}^*}|l+\eta|^{-2-2t} : \eta\in\left[-\frac{1}{2},\frac{1}{2}\right]\right\} < \infty,
$$

and

$$\sup\left\{|\Delta(\xi,\eta)| : \xi \in \mathbf{R},\ \eta \in \left[-\frac{1}{2},\frac{1}{2}\right]\right\} < \infty.$$

(For the latter, see [**6**, Lemma 1(iv)].)

It only remains to show that the term

$$T = \sum_{m \notin \Lambda_h} |m|^{2s}|\hat{u}(m)|^2$$

in (6.5) also satisfies the desired inequality. Because $t \geq s$ and $|m| \geq N/2$ for $m \notin \Lambda_h$,

$$\begin{aligned} T &= \sum_{m \notin \Lambda_h} |m|^{2(s-t)}|m|^{2t}|\hat{u}(m)|^2 \\ &\leq \sum_{m \notin \Lambda_h} |N/2|^{2(s-t)}|m|^{2t}|\hat{u}(m)|^2 \\ &\leq Ch^{2(t-s)}|u|_t^2. \end{aligned} \tag{6.7}$$

Combining terms, we obtain

$$||u_h - u||_s^2 \leq Ch^{2(t-s)}||u||_t^2. \tag{6.8}$$

The result is now proved for the case $B = 0$.

We now turn to the general case $L = A + B$. In this case the method is: find $u_h \in T_h$ such that

$$((A_h + B_h)u_h, \chi)_h = ((A+B)u, \chi)_h \quad \forall\, \chi \in S_h. \tag{6.9}$$

Assume initially that a solution u_h of this equation exists. Then u_h satisfies

$$(A_h u_h, \chi)_h = (A(u + A^{-1}(Bu - B_h u_h)), \chi)_h \quad \forall\, \chi \in S_h.$$

In other words, u_h is the solution of the equation (5.13) which we have analyzed already, if the exact solution is $u + A^{-1}(Bu - B_h u_h)$. Applying the result in the theorem to this special case, we obtain

$$||u_h - u - A^{-1}(Bu - B_h u_h)||_s \leq Ch^{t-s}||u + A^{-1}(Bu - B_h u_h)||_t. \tag{6.10}$$

To proceed further, we revert to the stronger conditions on s and t imposed in the theorem, i.e.,

$$s > \frac{1}{2}, \quad s + \frac{1}{2} < t \leq s + p.$$

We also make use of the following properties of B_h established in the appendix. The first is that because $s > 1/2$ $(I + A^{-1}B_h)^{-1}$ exists as a uniformly bounded linear operator on H^s for all h sufficiently small; i.e., there exists C such that

$$||(I + A^{-1}B_h)^{-1}||_s \leq C. \tag{6.11}$$

Other needed properties are that for $\sigma \in \mathbf{R}$ and $\nu > 1/2$ there exists C (possibly depending on σ and ν, but not depending on h or v) such that

$$||A^{-1}(B_h - B)v||_\sigma \leq Ch^\nu ||v||_\nu, \tag{6.12}$$

and

$$||A^{-1}B_h v||_\sigma \leq C||v||_\nu. \tag{6.13}$$

We may now write

$$u_h - u = (I + A^{-1}B_h)^{-1}[u_h - u - A^{-1}(Bu - B_h u_h) - A^{-1}(B_h - B)u].$$

Using (6.10)–(6.13), we obtain, because $s > 1/2$, $t - s > 1/2$,

$$\begin{aligned}
||u_h - u||_s &\leq ||(I + A^{-1}B_h)^{-1}||_s[||u_h - u - A^{-1}(Bu - B_h u_h)||_s \\
&\qquad + ||A^{-1}(B_h - B)u||_s] \\
&\leq C[h^{t-s}||u + A^{-1}(Bu - B_h u_h)||_t + h^{t-s}||u||_{t-s}] \\
&\leq Ch^{t-s}[||u||_t + ||A^{-1}(B - B_h)u||_t + ||A^{-1}B_h(u - u_h)||_t \\
&\qquad + ||u||_{t-s}] \\
&\leq Ch^{t-s}[||u||_t + ||u - u_h||_s].
\end{aligned}$$

Thus, for h sufficiently small, we have

$$||u_h - u||_s \le Ch^{t-s}||u||_t, \tag{6.14}$$

and the estimate in the theorem is now proved.

It remains only to prove the existence and uniqueness of the solution to the approximate equation (2.8). To prove uniqueness, assume that $u_h \in T_h$ is a solution for the case in which $f = 0$. Since u is zero in this case, it follows from (6.14) that $||u_h||_s = 0$ and, hence, $u_h = 0$. Thus, the solution, if it exists, is unique. But (2.8) is a linear system of N equations in N unknowns, which has full rank because of the just established uniqueness property. Thus, (2.8) has a solution for every right-hand side, and the proof is complete. □

Remark . It is clear from the proof of Theorem 6.1 that the method and the theoretical results extend almost trivially to the more general equation

$$-\frac{1}{\pi}\int_\Gamma \log|t-s|z(s)\,dl_s + \int_\Gamma m(t,s)z(s)\,dl_s = g(t), \quad t \in \Gamma, \tag{6.15}$$

if $m \in C^\infty(\mathbf{R}\times\mathbf{R})$, provided that the solution of this equation is unique. After parametrizing Γ as before, equation (1.3) is replaced now by $(L+K)u = f$, where K, which maps H^s into H^t for all $s \ge 0$ and all $t \in \mathbf{R}$, is defined in the obvious way. And in the approximation we similarly replace L_h by $L_h + K_h$, where K_h is the trapezoidal approximation to K (cf. (2.1)). Then, in the proof of Theorem 6.1 we need only replace B by $B + K$ and B_h by $B_h + K_h$. As these changes have no affect on the argument, the theorem stands as before.

7. Stability and order. The stability and order of the method, we recall from Section 5, depend on the properties of the functions

$$D(\eta) = \sum_j w_j[1+\Gamma_1(\xi_j,\eta)][1+\overline{\Delta(\xi_j,\eta)}], \quad \eta \in \left[-\frac{1}{2},\frac{1}{2}\right], \tag{7.1}$$

$$E(\eta) = \sum_j w_j\Gamma_1(\xi_j,\eta)[1+\overline{\Delta(\xi_j,\eta)}], \quad \eta \in \left[-\frac{1}{2},\frac{1}{2}\right], \tag{7.2}$$

and

$$e_h = h\sum_j w_j G_1(\xi_j), \tag{7.3}$$

where Γ_α, Δ and G_α are defined by (5.18), (5.24) and (5.16).

From the definitions, we obtain

$$\Gamma_\alpha(\xi,-\eta) = \overline{\Gamma_\alpha(\xi,\eta)}, \qquad \Delta(\xi,-\eta) = \overline{\Delta(\xi,\eta)}, \tag{7.4}$$

and from these follow

$$D(-\eta) = \overline{D(\eta)}, \qquad E(-\eta) = \overline{E(\eta)}. \tag{7.5}$$

Thus, when studying the stability and order of the method, we may restrict attention to $\eta \geq 0$.

Let $G_\alpha^\pm$ and $H_\alpha^\pm$ denote the real and imaginary parts of $F_\alpha^\pm$, so that, from (5.17) and (5.19),

$$G_\alpha^\pm(\xi,\eta) = \sum_{l=1}^\infty \left[\frac{1}{(l+\eta)^\alpha} \pm \frac{1}{(l-\eta)^\alpha}\right]\cos 2\pi l\xi, \tag{7.6}$$

$$H_\alpha^\pm(\xi,\eta) = \sum_{l=1}^\infty \left[\frac{1}{(l+\eta)^\alpha} \mp \frac{1}{(l-\eta)^\alpha}\right]\sin 2\pi l\xi. \tag{7.7}$$

Then, it follows from (7.1) and the definitions of the quantities therein that

$$\begin{aligned}\operatorname{Re} D(\eta) = \sum_j w_j[&(1+\eta G_1^+(\xi_j,\eta))(1+\eta^r G_r^\sigma(\xi_j,\eta)) \\ &+ \eta^{1+r} H_1^+(\xi_j,\eta) H_r^\sigma(\xi_j,\eta)], \quad \eta \geq 0,\end{aligned} \tag{7.8}$$

where σ is $+$ if r is even and $-$ if r is odd. Now we make use of properties of $G_\alpha^\pm$ and $H_\alpha^\pm$ proved in [**5**]. (A convenient summary, in the present notation, is given in the appendix of [**6**].) There it is shown, for all $\alpha > 0$, that

$$1 + \eta^\alpha G_\alpha^+(\xi,\eta) \geq 0, \quad \xi \in (0,1), \quad \eta \in \left[0, \frac{1}{2}\right], \tag{7.9}$$

with equality if and only if $(\xi, \eta) = (1/2, 1/2)$;

$$1 + \eta^{\alpha} G_{\alpha}^{-}(\xi, \eta) \geq 0, \quad \xi \in [0, 1], \quad \eta \in \left[0, \frac{1}{2}\right], \tag{7.10}$$

with equality if and only if $(\xi, \eta) = (0, 1/2)$;

$$H_{\alpha}^{+}(\xi, \eta) \leq 0, \quad \xi \in \left[0, \frac{1}{2}\right], \quad \eta \in \left[0, \frac{1}{2}\right], \tag{7.11}$$

$$H_{\alpha}^{-}(\xi, \eta) \geq 0, \quad \xi \in \left[0, \frac{1}{2}\right], \quad \eta \in \left[0, \frac{1}{2}\right], \tag{7.12}$$

together with

$$H_{\alpha}^{\pm}(\xi, \eta) = -H_{\alpha}^{\pm}(1 - \xi, \eta).$$

With the aid of these results and (7.8), we obtain the following stability result for the case of splines of even order (e.g., piecewise-linear functions).

Theorem 7.1. *Let the order r of the splines be even. Then the method is stable unless $J = 1$ and $\xi_1 = 1/2$, in which case it is unstable.*

Remark . We have no result for the case of splines of odd order. Numerical experiments suggest that such methods can be unstable.

Proof. For the case of splines of even order, it follows from (7.9)–(7.12) that each term of the expression (7.8) for $\operatorname{Re} D(\eta)$ is nonnegative. Moreover, each term is continuous in η (see [**6**, Lemma A.2]), and the expression $(1 + \eta G_1^{+}(\xi, \eta))(1 + \eta^r G_r^{+}(\xi, \eta))$ can vanish only if $\xi = 1/2$. Thus, provided $\xi = 1/2$ is not the only quadrature point, $\operatorname{Re} D(\eta)$ is bounded away from zero, and the method is stable.

On the other hand, if $J = 1$ and $\xi_1 = 1/2$, then it is easily verified that $D(\eta)$ is real. Thus, from (7.9) $D(1/2) = 0$ so that in this case the method is unstable. □

It may be noted that the stability theorem holds also if Δ is replaced by zero, thus it holds for the case $S_h = T_h$, in which the test space consists of trigonometric polynomials.

The following theorem shows that the order of the method, as defined at the end of Section 5, depends sensitively on the choice of the abscissae $\{\xi_j\}$ and the weights $\{w_j\}$ in the quadrature rule Q. (The quadrature rule Q, given by (2.7), is said to be symmetric if $\xi_j = 1 - \xi_{J-j+1}$ and $w_j = w_{J-j+1}$.)

Theorem 7.2. (a) *Every choice of the quadrature rule Q yields a method of order ≥ 1.*

(b) *The method is of order ≥ 2 if and only if*

$$\sum_j w_j G_1(\xi_j) = 0. \tag{7.13}$$

(c) *Assume $r \geq 2$. If the quadrature rule Q is symmetric and* (7.13) *holds, then the method is of order ≥ 3.*

Proof. Since

$$\Gamma_1(\xi, \eta) = |\eta| F_1^+(\xi, \eta)$$

and

$$\Delta(\xi, \eta) = \eta^r F_r^\sigma(\xi, \eta),$$

it follows from (7.2) that

$$E(\eta) = |\eta| \sum_j w_j F_1^+(\xi_j, \eta)[1 + \eta^r \overline{F_r^\sigma(\xi_j, \eta)}]. \tag{7.14}$$

Thus, $E(\eta) = O(|\eta|)$ as $\eta \to 0$ for every choice of the quadrature rule Q. Since $e_h = O(h)$ as $h \to 0$, the order of the method is always at least 1, thus proving part (a). On the other hand, it follows from (7.3) that unless (7.13) holds the order cannot be higher than 1.

For fixed $\xi \in (0,1)$ and $\alpha > 0$, it is known (see [**6**, Lemma A.2]) that the real and imaginary parts of F_α^+ have the expansion

$$G_\alpha^+(\xi, \eta) = \sum_{k=0}^\infty \binom{-\alpha}{2k} G_{\alpha+2k}(\xi) \eta^{2k}, \tag{7.15}$$

$$H_\alpha^+(\xi, \eta) = \sum_{k=1}^\infty \binom{-\alpha}{2k-1} H_{\alpha+2k-1}(\xi) \eta^{2k-1}, \tag{7.16}$$

where $\binom{-\alpha}{j}$ denotes $(-\alpha)(-\alpha-1)\cdots(-\alpha-j+1)/j!$, and

$$H_\alpha(\xi) = 2\sum_{l=1}^{\infty} \frac{1}{l^\alpha} \sin 2\pi l\xi,$$

with the power series uniformly convergent for $\eta \in [-1/2, 1/2]$.

Since $r \geq 1$, it follows from (7.14) that

$$E(\eta) = |\eta| \sum_j w_j G_1(\xi_j) + O(|\eta|^2).$$

Thus, (7.13) is both a necessary and sufficient condition for the order to be two or greater, proving part (b).

If the rule is symmetric, then it is easily seen that E is real, so that, from (7.14),

$$\begin{aligned} E(\eta) = |\eta| \sum_j w_j [G_1^+(\xi_j,\eta) + \eta^r (G_1^+(\xi_j,\eta) G_r^\sigma(\xi_j,\eta) \\ + H_1^+(\xi_j,\eta) H_r^\sigma(\xi_j,\eta))]. \end{aligned} \tag{7.17}$$

If, in addition, $r \geq 2$, then

$$\begin{aligned} E(\eta) &= |\eta| \sum_j w_j G_1^+(\xi_j,\eta) + O(|\eta|^3) \\ &= |\eta| \sum_j w_j G_1(\xi_j) + O(|\eta|^3). \end{aligned}$$

Thus, if (7.13) holds, then $E(\eta) = O(|\eta|^3)$, and, therefore, the order is at least three, proving part (c). $\square$

If we note, from (5.1) and (5.16), that

$$G_1(\xi) = -2\log(2|\sin \pi\xi|),$$

then we can re-express (7.13) as the condition

$$\sum_j w_j \log(2|\sin \pi\xi_j|) = 0. \tag{7.13$'$}$$

There are, of course, many choices of the parameters $\{J, \xi_1, w_1, \dots, \xi_J, w_J\}$ which satisfy the condition. Here we consider just the two simplest cases:

Lemma 7.3 (a) *If $J = 1$, then the condition* (7.13) *is satisfied if and only if $\xi_1 = 1/6$ or $5/6$.*

(b) *If $J = 2$ and the rule Q is symmetric, then* (7.13) *is satisfied if and only if $\xi_1 = 1/6$, $\xi_2 = 5/6$.*

This follows immediately from (7.13′).

The special cases discussed in Section 4 (and, in particular, Theorem 4.1) now follow as special cases of Theorem 6.1 with the aid of Theorems 7.1 and 7.2 and Lemma 7.3.

We conclude this section by attempting to give some insight into the special virtue of the number 1/6 as an abscissa in the rule Q. Suppose that the exact solution u happens to be identically 1. Then, from (5.27), we have

$$Au(x) = 1, \quad x \in \mathbf{R}, \tag{7.18}$$

whereas, from (5.15),

$$\begin{aligned} A_h u(x) &= 1 + hG_1(Nx) \\ &= 1 - 2h \log(2|\sin \pi N x|), \quad x \in \mathbf{R}. \end{aligned} \tag{7.19}$$

For most values of x the trapezoidal rule approximation to the logarithmic integral Au is in error. However, (7.18) and (7.19) show that the trapezoidal rule is exact at the points

$$\left\{ \left(\frac{1}{6} + k \right) h : k \in \mathbf{Z} \right\} \cup \left\{ \left(\frac{5}{6} + k \right) h : k \in \mathbf{Z} \right\}.$$

A quadrature rule Q that uses 1/6 or 5/6 as abscissae is therefore at an initial advantage, at least for the case $u =$ constant. This observation is relevant even for nonconstant u, since as $h \to 0$ every smooth solution u looks more and more like a constant function on the scale given by the mesh size h.

8. Numerical examples. In the following examples we report not the values of the unknown function u in (1.2), or equivalently z in (1.1), but rather values of the potential

$$\begin{aligned}\phi(\tau) &= -\frac{1}{\pi}\int_\Gamma \log|\tau - s| z(s)\, dl_s \\ &= -\int_0^1 2\log|\tau - \nu(y)| u(y)\, dy\end{aligned} \tag{8.1}$$

at some point τ not on Γ. In addition to replacing u by the approximate solution u_h, we take the further step of approximating the integral in (8.1) by the trapezoidal rule with spacing h. Thus, the approximation is

$$\phi_h(\tau) = -h\sum_{k=0}^{N-1} 2\log|\tau - \nu(kh)| u_h(kh). \tag{8.2}$$

Note that this uses only the N values of u_h that come directly from the linear system (2.10), thus explicit trigonometric interpolation is not needed. Of course, this final trapezoidal rule approximation would not be appropriate for τ very near Γ.

In each of the examples the version of the method used in the calculations is that described in Theorem 4.1, except that in Examples $1'$ and $3'$ the parameters in the quadrature rule Q are varied from the theoretically ideal values.

Example 1. In this case Γ is the ellipse

$$\frac{x^2}{4} + \frac{y^2}{64} = 1, \tag{8.3}$$

and we solve (1.1) with the right-hand side

$$g(x,y) = (x+y)^2. \tag{8.4}$$

With the curve parametrized by

$$(x,y) = \nu(\sigma) = (2\cos 2\pi\sigma, 8\sin 2\pi\sigma), \quad \sigma \in [0,1], \tag{8.5}$$

we obtain the results shown in Table 1 for the potential at $\tau = (6/7, 8/7)$. (The errors e_h are calculated by reference to the 'exact' value, which is obtained from a careful calculation by a different method. The apparent rates of convergence are defined by $\alpha_h = \log_2(e_{2h}/e_h)$.)

After initial fluctuation, the numerical results in Table 1 settle down to the predicted $O(h^3)$ order of convergence. They also demonstrate satisfactory accuracy. Similar results are obtained for the potential at other points.

Example 1′. To assess the importance in practice of using the correct quadrature rule, Example 1 was repeated with the abscissae in the symmetric 2-point rule Q chosen to be

$$\xi_1 = \xi, \qquad \xi_2 = 1 - \xi, \tag{8.6}$$

for the values of ξ in Table 1′. The results in Table 1′ demonstrate convincingly the magical quality of the choice $\xi = 1/6$; all the other values yield, as expected, only an $O(h)$ rate of convergence and dramatically larger errors at $h = 1/256$.

In each of the remaining examples, Γ is the cardioid-like curve

$$(x, y) = r(\theta)(\cos\theta, \sin\theta), \quad \theta \in \left[0, \frac{3\pi}{2}\right], \tag{8.7}$$

where

$$r(\theta) = 2\theta\left(\frac{3\pi}{2} - \theta\right), \tag{8.8}$$

and the right-hand side of (1.1) is

$$g(x, y) = (x - 1)^2. \tag{8.9}$$

The interior point τ at which the potential ϕ is computed is taken to be $(17/70, 281/70)$. Because of the right-angled re-entrant corner at the origin, the theory of this paper is not strictly applicable. On the other hand, the algorithm is still available, and it seems interesting to explore experimentally whether it can be made to give good results.

Example 2. For the example described above, we initially ignore the effects of the inevitable singularity in the solution z at the right-angled corner. Thus, the curve is parametrized in the obvious way,

$$(x,y) = \nu(\sigma) = r\left(\frac{3\pi}{2}\sigma\right)\left(\cos\frac{3\pi}{2}\sigma, \sin\frac{3\pi}{2}\sigma\right), \quad \sigma \in [0,1], \tag{8.10}$$

so that the parameter is roughly proportional to arc-length in a neighborhood of the origin.

In this situation, we obtain the results shown in Table 2. The apparent order of convergence is $O(h^\alpha)$ with $\alpha = 1.4$ or less.

Example 3. The example is now repeated with a new parametrization,

$$(x,y) = \nu(\sigma) = r(s(\sigma))(\cos s(\sigma), \sin s(\sigma)), \quad \sigma \in [0,1], \tag{8.11}$$

where

$$s(\sigma) = \frac{3\pi}{2}\sigma^2(3 - 2\sigma), \quad \sigma \in [0,1]. \tag{8.12}$$

Note that for σ small, we have

$$s(\sigma) \approx C\sigma^2,$$

and that

$$s(1-\sigma) = \frac{3\pi}{2} - s(\sigma).$$

The effect is that the uniform partition with respect to the parameter, as in (2.1), results in a 'grading of the mesh' (with grading exponent 2) as the corner is approached. In effect, we are trying, by concentrating more break-points near the corner, to counteract the effects of the expected singularity.

In this case the results shown in Table 3 are obtained. They suggest (but certainly do not prove) that one can obtain close to $O(h^3)$ convergence even in the case of corners by parametrizing the curve in an appropriate way. Similar results, with the apparent orders of convergence differing but all close to three, were obtained at other points in the interior.

Example 3′. Example 3 is repeated with the abscissae of the rule Q given instead by (8.6). Notwithstanding the absence of a theory, the results in Table 3′ still firmly suggest that the value $\xi = 1/6$ has a special status.

In summary, for a smooth curve the numerical results confirm the predicted rates of convergence for this fully discrete method. For a curve with a corner and an appropriate choice of parametrization, the numerical results suggest that the same rate of convergence can be achieved. For the case of a corner, no theoretical results are yet available, and further research is needed.

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Appendix. Properties of B_h.

Let $T_h v$ denote the trapezoidal rule, with spacing $h = 1/N$, applied to the 1-periodic function v; i.e.,

$$T_h v = h \sum_{k=0}^{N-1} v(kh). \tag{A1}$$

The operator B_h is, by definition, obtained from the exact operator B by using this rule to approximate the integral in

$$Bw(x) = \int_0^1 b(x,y)w(y)\,dy, \tag{A2}$$

where $b(x,y)$ is the kernel in (5.7). Thus,

$$\begin{aligned} B_h w(x) &= h \sum_{k=0}^{N-1} b(x,kh)w(kh), \\ &= T_h[b(x,\cdot)w(\cdot)]. \end{aligned} \tag{A3}$$

In the following lemma Iv denotes the exact integral

$$Iv = \int_0^1 v(x)\,dx.$$

Lemma A1. *If $v \in H^\nu$ with $\nu > 1/2$, then*

$$|T_h v - Iv| \leq Ch^\nu ||v||_\nu.$$

Remark . The condition $\nu > 1/2$ is natural because only if $\nu > 1/2$ can we be sure that a function v in H^ν is continuous.

Proof. We may express v in terms of its Fourier series,

$$v(x) = \sum_{m \in \mathbf{Z}} \hat{v}(m) e^{i2\pi mx},$$

which converges absolutely, by an application of the Cauchy-Schwarz inequality, because $\nu > 1/2$. Thus,

$$\begin{aligned} T_h v &= \sum_{m \in \mathbf{Z}} \hat{v}(m) T_h \phi_m \\ &= \sum_{m \equiv 0} \hat{v}(m). \end{aligned}$$

Therefore,

$$\begin{aligned} |T_h v - Iv| &= \left| {\sum_{m \equiv 0}}' \hat{v}(m) \right| = \left| {\sum_{m \equiv 0}}' |m|^{-\nu} |m|^\nu \hat{v}(m) \right| \\ &\leq \left({\sum_{m \equiv 0}}' |m|^{-2\nu} \right)^{\frac{1}{2}} \left({\sum_{m \equiv 0}}' |m|^{2\nu} |\hat{v}(m)|^2 \right)^{\frac{1}{2}} \\ &= \left(\sum_{l \in \mathbf{Z}^*} |lN|^{-2\nu} \right)^{\frac{1}{2}} \left({\sum_{m \equiv 0}}' |m|^{2\nu} |\hat{v}(m)|^2 \right)^{\frac{1}{2}} \\ &\leq Ch^\nu ||v||_\nu. \qquad \square \end{aligned}$$

It follows from the lemma that, for fixed $x \in \mathbf{R}$, we have

$$|B_h w(x) - Bw(x)| \leq Ch^\nu ||[b(x, \cdot) w(\cdot)]||_\nu,$$

and because b is a C^∞ function on $\mathbf{R} \times \mathbf{R}$, it follows easily that

$$||B_h w - Bw||_{L_\infty} \leq Ch^\nu ||w||_\nu \tag{A4}$$

if $\nu > 1/2$. A similar argument shows that any derivative of $B_h w$ converges to the corresponding derivative of Bw in the fashion of (A4); the only change in (A2) and (A3) is that $b(x, y)$ is replaced by the appropriate partial derivative with respect to x in each expression. By the use of appropriate imbedding arguments, we then obtain the same order of convergence in any Sobolev norm. Thus, we obtain

Lemma A2. *Let $\tau \in \mathbf{R}$. If $v \in H^\nu$ with $\nu > 1/2$, then*

$$||B_h v - Bv||_\tau \leq Ch^\nu ||v||_\nu.$$

We may now prove the property (6.12). Noting that

$$||Aw||_\tau = ||w||_{\tau-1}, \quad \tau \in \mathbf{R},$$

(which follows immediately from (5.5) and the definition of the Sobolev norms), we have

$$\begin{aligned} ||A^{-1}(B_h - B)v||_\sigma &= ||(B_h - B)v||_{\sigma+1} \\ &\leq Ch^\nu ||v||_\nu, \end{aligned}$$

provided $\nu > 1/2$, which is just (6.12). From the triangle inequality, we also have

$$\begin{aligned} ||A^{-1}B_h v||_\sigma &\leq ||A^{-1}(B_h - B)v||_\sigma + ||A^{-1}Bv||_\sigma \\ &\leq Ch^\nu ||v||_\nu + ||Bv||_{\sigma+1} \\ &\leq Ch^\nu ||v||_\nu + C||v||_\nu \\ &\leq C||v||_\nu, \end{aligned}$$

where we have used (5.8). Thus, (6.13) is proved.

It remains to prove (6.11). This we do by means of the collectively compact operator approximation theory of [**1**]. Because $s > 1/2$, it follows from (6.13) that

$$||A^{-1}B_h v||_s \leq C||v||_s,$$

and

$$||A^{-1}B_h v||_{s+\varepsilon} \leq C||v||_s, \quad \varepsilon > 0.$$

From the first, it follows that $\{A^{-1}B_h\}$ is uniformly bounded in H^s. From the second, together with the compact imbedding of $H^{s+\varepsilon}$ in H^s, it follows that $\{A^{-1}B_h\}$ is collectively compact in H^s. In addition, from (6.12) we have $||A^{-1}(B_h - B)v||_s \to 0$ for all $v \in H^s$. Since $(I + A^{-1}B)^{-1}$ is a bounded operator on H^s, the conditions of the collectively compact theory [**1**] are satisfied, and it follows that $(I + A^{-1}B_h)^{-1}$ exists as a uniformly bounded set of operators on H^s, that is (6.11) holds, provided h is sufficiently small.

TABLE 1. Errors and apparent convergence rates for Example 1, i.e., for the ellipse (8.3) and the potential at the interior point $(6/7, 8/7)$. The exact value is $9.992797118847\ldots$.

No. of subintervals	Error	Apparent rate of convergence
8	-2.99	
16	2.06(-1)	
32	1.71(-2)	3.59
64	5.21(-4)	5.04
128	4.91(-5)	3.41
256	6.13(-6)	3.00

TABLE 1′. Errors and apparent convergence rates at $h = 1/256$ for Example 1 with the modified quadrature rule (8.6).

ξ	Error	Apparent rate of convergence
1/4	-1.16(-1)	1.00
1/5	-5.39(-2)	1.00
1/6	6.13(-6)	3.00
1/7	4.72(-2)	1.00
1/8	8.89(-2)	1.00

TABLE 2. Errors and apparent convergence rates for Example 2, i.e., for the cardioid-like curve parametrized as in (8.10) and the potential at the point $(17/70, 281/70)$. The exact value is $17.99584807\ldots$.

No. of subintervals	Error	Apparent rate of convergence
8	-1.54	
16	-4.70(-1)	1.71
32	-1.58(-1)	1.57
64	-5.70(-2)	1.47
128	-2.16(-2)	1.40
256	-8.38(-3)	1.36

TABLE 3. Errors and apparent convergence rates for Example 3, i.e., the same problem as in Table 2, but with the revised parametrization (8.11), (8.12).

No. of subintervals	Error	Apparent rate of convergence
8	-2.48(-2)	
16	1.19(-2)	
32	1.98(-3)	2.59
64	3.07(-4)	2.69
128	4.74(-5)	2.70
256	7.71(-6)	2.62

TABLE 3′. Errors and apparent convergence rates at $h = 1/256$ for Example 3 with the modified quadrature rule (8.6).

ξ	Error	Apparent rate of convergence
1/4	-7.02(-2)	1.00
1/5	-3.27(-2)	1.00
1/6	7.71(-6)	2.62
1/7	2.87(-2)	1.00
1/8	5.40(-2)	1.00

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