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THE METHOD OF LINES FOR PARABOLIC PARTIAL INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. We present a method of lines approximation of the solution of a particular nonlinear Volterra partial integro-differential equation. Discretization in space of this equation leads to a system of stiff integro-differential equations. In a second step, this system is integrated in time by the implicit Euler method.

The concept of the logarithmic norm, introduced in the theory of numerical methods for stiff ordinary differential equations, plays an important role in the convergence analysis.

1. Introduction. We consider the nonlinear, parabolic-type Volterra partial integro-differential equation (VPIDE)

(1.1)
$$u_t(x,t) = g(x,t) + \sum_{i=0}^{2} a_i(x,t)\frac{\partial^i u}{\partial x^i}(x,t) + \int_0^t b(x,t,s,u(x,s))\,ds, \qquad 0 < x < 1, \quad 0 < t \le T,$$

where $u_t := \partial u/\partial t$. The functions g, a_i, $i = 0,1,2$, and b are continuous on $D := \{(x,t) \in \mathbf{R}^2 : x \in I, t \in J\}$ and $I \times S \times \mathbf{R}$, respectively, with $I := [0,1]$, $J := [0,T]$ and $S := \{(t,s) \in J \times J : s \le t\}$. We assume that the function b is continuously differentiable and therefore satisfies a Lipschitz condition with respect to its last variable, i.e., there exists a constant $L \ge 0$ such that for all $u, v \in \mathbf{R}$, $x \in I$ and $(t,s) \in S$,

$$|b(x,t,s,u) - b(x,t,s,v)| \le L|u - v|. \tag{1.2}$$

We exclude the case $a_1 = a_2 \equiv 0$. To equation (1.1) we associate the following initial and boundary conditions:

$$u(x,0) = \phi(x), \quad x \in I, \tag{1.3}$$
$$u(0,t) = u(1,t) = 0, \quad t \in J, \tag{1.4}$$

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where ϕ is a continuous function defined on I and satisfying $\phi(0) = \phi(1) = 0$.

A survey of the theory and applications of linear and nonlinear VPIDEs can be found in [**7**]. Moreover, we refer to [**2, 5**] where existence, unicity and asymptotic behavior of the solution of a VPIDE are given.

In Section 2, we formulate the method of lines for the numerical solution of (1.1). The error of the discrete scheme is studied in Section 3. Finally, Section 4 contains some numerical results.

2. The method of lines. The method of semidiscretization reduces the given equation (partial differential equation (PDE) or VPIDE) in a first step, usually space discretization, to a system of ordinary differential equations (ODEs) or Volterra integro-differential equations (VIDEs) which, in a second step, is numerically integrated in time by well-known methods like Runge-Kutta or collocation schemes. In this paper, we discuss the method of lines (MOL). In [**6**] a general framework for the convergence analysis of the MOL applied to PDEs was set up. Certain stability concepts of the theory of nonlinear stiff ODEs proved to be of particular importance in their error analysis.

The MOL consists of two steps: space discretization and time integration. In a first step, the derivatives with respect to the space variable x occurring in (1.1) are approximated by finite difference schemes like:

(2.1a)
$$u_x(x,t) = \frac{1}{2h^{(x)}}(u(x+h^{(x)},t) - u(x-h^{(x)},t)) + \mathcal{O}(h^{(x)^2}),$$
(2.1b)
$$u_x(x,t) = \frac{1}{h^{(x)}}(u(x+h^{(x)},t) - u(x,t)) + \mathcal{O}(h^{(x)}),$$
(2.2)
$$u_{xx}(x,t) = \frac{1}{h^{(x)^2}}(u(x+h^{(x)},t) - 2u(x,t) + u(x-h^{(x)},t)) + \mathcal{O}(h^{(x)^2}).$$

In order to formulate the semidiscretization process, we introduce a (uniform) partition of the interval $I = [0,1]$ with mesh points denoted by x_j, $j = 0,\dots,M$, where $x_j = jh^{(x)}$, $j = 0,\dots,M$, and $h^{(x)} := M^{-1}$. Along the lines $x = x_j$, $j = 0,\dots,M$, the exact solution

u of (1.1) is then approximated by functions $U_j(t)$, $j = 0, \ldots, M$, i.e., $U_j(t) \approx u(x_j, t)$, $j = 0, \ldots, M$, $t \in J$. For $j = 1, \ldots, M-1$, we have

$$U_j(0) = \phi(x_j), \tag{2.3}$$

and for $j = 0$ and $j = M$,

$$U_0(t) = U_M(t) = 0, \quad t \in J. \tag{2.4}$$

This choice is natural in view of (1.3) and (1.4). At each interior mesh line (x_j, t), $j = 1, \ldots, M-1$, $t \in J$, the approximation of $u(x_j, t)$ is required to satisfy

$$\begin{aligned} U_j'(t) = g(x_j,t) + a_0(x_j,t)U_j(t) + a_1(x_j,t)D_1^{(j)}(h^{(x)}, U(t)) \\ + a_2(x_j,t)D_2^{(j)}(h^{(x)}, U(t)) + \int_0^t b(x_j,t,s,U_j(s))\,ds, \end{aligned} \tag{2.5}$$

where $U(t) := (U_1(t), \ldots, U_{M-1}(t))^\top$ and where $D_1^{(j)}(h^{(x)}, U(t))$ and $D_2^{(j)}(h^{(x)}, U(t))$ denote finite difference approximations of $u_x(x_j, t)$ and $u_{xx}(x_j, t)$, respectively. In matrix notation, (2.5) and (2.3) read

$$U'(t) = G(t) + A(t)U(t) + \int_0^t B(t,s,U(s))\,ds, \quad t \in J, \qquad U(0) = \Phi. \tag{2.6}$$

Here we have set

$$G(t) := (g(x_1,t), \ldots, g(x_{M-1},t))^\top, \quad \Phi := (\phi(x_1), \ldots, \phi(x_{M-1}))^\top,$$

$$\begin{aligned} A(t) := \operatorname{diag}(a_0(x_1,t), \ldots, a_0(x_{M-1},t)) \\ + \operatorname{diag}(a_1(x_1,t), \ldots, a_1(x_{M-1},t))C_1 \\ + \operatorname{diag}(a_2(x_1,t), \ldots, a_2(x_{M-1},t))C_2, \end{aligned} \tag{2.7}$$

$$B(t,s,U(s)) : \begin{pmatrix} b(x_1,t,s,U_1(s)) \\ \vdots \\ b(x_{M-1},t,s,U_{M-1}(s)) \end{pmatrix}. \tag{2.8}$$

The matrices C_1 and C_2 characterize the finite difference schemes used for the discretization of u_x and u_{xx}, respectively. For the discretizations (2.1a), (2.1b) and (2.2), C_1 and C_2 are

$$C_1 = \frac{1}{2h^{(x)}} \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}, \tag{2.9a}$$

$$C_1 = \frac{1}{h^{(x)}} \begin{pmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \\ & & & & -1 \end{pmatrix}, \tag{2.9b}$$

and

$$C_2 = \frac{1}{h^{(x)^2}} \begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix}, \tag{2.10}$$

respectively.

The system (2.6) has a unique continuous solution $U(t)$ on J since $G(t)$, $A(t)$ and $B(t,s,U)$ are continuous on J and $S \times \mathbf{R}$, respectively, and B satisfies a Lipschitz condition with Lipschitz constant L : for all $U, V \in \mathbf{R}^{M-1}$ and $(t,s) \in S$,

$$||B(t,s,U) - B(t,s,V)|| \leq L||U - V||, \tag{2.11}$$

with $|| \cdot ||$ denoting some (standard) norm in $\mathbf{R}^{M-1}$. This trivially follows from (1.2). Equation (2.6) represents a system of (nonlinear) stiff Volterra integro-differential equations (VIDEs) which will be solved numerically in a second discretization step. Its dimension is $M-1$ and increases as the mesh diameter $h^{(x)}$ diminishes. The same holds for the stiffness parameter $(h^{(x)})^{-2}$ (or possibly $(h^{(x)})^{-1}$) of (2.6).

In order to get a fully discrete scheme, we now discretize the interval $J = [0, T]$ by a uniform partition $\{t_n : n = 0, \ldots, N\}$, with $t_n = nh^{(t)}$,

$n = 0, \dots, N$, and stepsize $h^{(t)} := TN^{-1} > 0$. The extension to nonuniform partitions (variable stepsize) is obvious. Let V_n denote the numerical approximation at $t_n = nh^{(t)}$ of the exact solution $U(t)$ of (2.6) obtained by the *implicit Euler method*:

$$\begin{aligned} V_{n+1} = V_n + h^{(t)}[G(t_{n+1}) + A(t_{n+1})V_{n+1}] \\ + h^{(t)^2} \sum_{i=0}^{n} B(t_{n+1}, t_{i+1}, V_{i+1}), \end{aligned} \tag{2.12}$$

for $n \geq 0$ with $V_0 := \Phi$.

3. Error analysis. In order to study the *discretization error*

$$E_n := V_n - u_{h^{(x)}}(t_n), \quad n = 1, \dots, N, \tag{3.1}$$

where $u_{h^{(x)}}(t)$ denotes the restriction of $u(x, t)$ to the lines $x = x_j$, $j = 1, \dots, M-1$, i.e.,

$$u_{h^{(x)}}(t) = (u(x_1, t), \dots, u(x_{M-1}, t))^\top, \quad t \in J,$$

we first have to introduce the *space truncation error*

$$\begin{aligned} \alpha(t) := u'_{h^{(x)}}(t) - G(t) - A(t)u_{h^{(x)}}(t) \\ - \int_0^t B(t, s, u_{h^{(x)}}(s))\, ds, \quad t \in J, \end{aligned} \tag{3.2}$$

where $u'_{h^{(x)}} := du_{h^{(x)}}/dt$, and the *total truncation error*

$$\begin{aligned} \beta(t_{n+1}) := u_{h^{(x)}}(t_{n+1}) - u_{h^{(x)}}(t_n) - h^{(t)}[G(t_{n+1}) + A(t_{n+1})u_{h^{(x)}}(t_{n+1})] \\ - h^{(t)^2} \sum_{i=0}^{n} B(t_{n+1}, t_{i+1}, u_{h^{(x)}}(t_{i+1})). \end{aligned} \tag{3.3}$$

The truncation errors α and β measure how "well" the system of VIDEs (2.6) and the fully discrete scheme (2.12), respectively, approximate the given VPIDE (1.1). The space truncation error α essentially consists of the error terms of the finite difference formulas used to approximate the derivatives w.r.t. x (cf. (2.1) and (2.2)). The space discretization is said to be *consistent* if

$$||\alpha(t)|| \to 0 \quad \text{as } h^{(x)} \to 0 \quad \text{for all } t \in J,$$

where $||\cdot||$ is a (standard) norm in $\mathbf{R}^{M-1}$. The total truncation error β contains, in addition to α, the local error term of the implicit Euler method for (2.6).

As shown in [**6**], the concept of the logarithmic matrix norm is a key to realistic bounds for $||E_n||$. Let $\mu[\cdot]$ denote the logarithmic matrix norm associated to the given vector norm $||\cdot||$ in $\mathbf{R}^{M-1}$. For a given matrix A it is defined as

$$\mu[A] := \lim_{\Delta\to 0} \frac{||I+\Delta A|| - 1}{\Delta}.$$

If $||\cdot||$ is an inner product norm, $\mu[A]$ can be written as

$$\mu[A] = \max_{\xi\neq 0} \frac{\langle A\xi, \xi\rangle}{||\xi||^2},$$

and $\mu[A]$ is the smallest possible one-sided Lipschitz constant of A. For the standard norms $||\cdot||_1, ||\cdot||_2$ and $||\cdot||_\infty$, the corresponding logarithmic norms are known explicitly; with $A = (a_{ij})$, one has $\mu_1[A] = \max_j(a_{jj} + \sum_{i\neq j}|a_{ij}|)$, $\mu_\infty[A] = \max_i(a_{ii} + \sum_{j\neq i}|a_{ij}|)$ and $\mu_2[A] = \lambda_{\max}[(A+A^\top)/2]$. (For more details, see, e.g., [**1**, pp. 27–31].)

Let ν be a finite (possibly negative) constant such that

$$\mu[A(t)] \leq \nu < \infty, \quad \text{for all } t\in J, \quad \text{uniformly in } h^{(x)}. \tag{3.4}$$

In other words, the functions a_i, $i = 0, 1, 2$, and the finite difference schemes used to approximate the derivatives of u with respect to x are assumed to be such that (3.4) holds for some norm. Usually in applications, once the discretization schemes are fixed, an adequate norm is chosen to determine a ν in (3.4) which is independent of $h^{(x)}$. In particular, if a_1 and a_2 are nonnegative and if we use (2.1b) for the discretization of u_x, then $\mu_\infty[A(t)] \leq \max\{a_0(x_j, t) : j = 1, \ldots, M-1, t\in J\}$, (for $\mu[A+B] \leq \mu[A] + \mu[B]$, see [**1**, p. 31]).

In our error analysis, similarly to the B-convergence theory for ODEs (see [**3**]), we want to derive error bounds which are different from the conventional error estimates in the sense that they depend on the constant ν in (3.4) and on the Lipschitz constant L of B, rather than on $||A(t)||$ which may become arbitrarily large if $h^{(x)} \to 0$. We are thus seeking for an error estimate which is independent of all quantities that may become very large because of the stiffness of the problem.

Let the norm $||\cdot||_\infty$ of a vector function $e : J \to \mathbf{R}^{M-1}$ be defined by $||e||_\infty := \max\{||e(t)|| : t \in J\}$.

Theorem 3.1. *Let $||\cdot||$ be a given norm in $\mathbf{R}^{M-1}$. Assume that the space discretization for* (1.1) *is consistent and that* (3.4) *holds. Furthermore, assume that the exact solution u of* (1.1) *is sufficiently smooth, more precisely that u is as smooth with respect to x as required for the consistency and the desired accuracy of the space discretization and that u is continuously differentiable with respect to t and $\partial^2 u/\partial t^2$ is bounded on D. Then the approximation V_n converges to $u_{h^{(x)}}(t_n)$ as $h^{(x)} \to 0$ and $h^{(t)} \to 0$ and we have*

$$||E_n|| \le C||\alpha||_\infty + \tilde{C}h^{(t)}, \tag{3.5}$$

for $h^{(t)} \le h_0^{(t)}$ and for all $n = 1, \dots, N$, where C and $\tilde{C}$ are finite constants which do not depend on the grid spacings.

Proof. A simple Taylor expansion shows that

$$u_{h^{(x)}}(t_{n+1}) = u_{h^{(x)}}(t_n) + h^{(t)}u'_{h^{(x)}}(t_{n+1}) - \frac{h(t)^2}{2}u''_{h^{(x)}}(\xi_n),$$
$$t_n < \xi_n < t_{n+1}.$$

Using (3.2), we obtain

$$\begin{aligned} u_{h^{(x)}}(t_{n+1}) = u_{h^{(x)}}(t_n) &+ h^{(t)}[G(t_{n+1}) + A(t_{n+1})u_{h^{(x)}}(t_{n+1}) \\ &+ \sum_{i=0}^{n}\int_{t_i}^{t_{i+1}} B(t_{n+1}, s, u_{h^{(x)}}(s))\,ds] + h^{(t)}\alpha(t_{n+1}) \\ &- \frac{{h^{(t)}}^2}{2}u''_{h^{(x)}}(\xi_n), \end{aligned}$$

(3.6)

$$\begin{aligned} u_{h^{(x)}}(t_{n+1}) = u_{h^{(x)}}(t_n) &+ h^{(t)}[G(t_{n+1}) + A(t_{n+1})u_{h^{(x)}}(t_{n+1})] \\ &+ {h^{(t)}}^2\sum_{i=0}^{n} B(t_{n+1}, t_{i+1}, u_{h^{(x)}}(t_{i+1})) \end{aligned}$$

$$+ h^{(t)}\alpha(t_{n+1}) - \frac{{h^{(t)}}^2}{2} u''_{h^{(x)}}(\xi_n)$$
$$- \frac{{h^{(t)}}^3}{2} \sum_{i=0}^{n} \frac{dB}{ds}(t_{n+1}, \zeta_i, u_{h^{(x)}}(\zeta_i)),$$

with $t_i < \zeta_i < t_{i+1}$, where we approximated the integral terms by a one-point quadrature formula. By definition, (see (3.3)), the sum of the last three terms in (3.6) is equal to $\beta(t_{n+1})$, and we have

$$(3.7)\quad \tilde{\beta} := \max\{||\beta(t_n)|| : n = 1, \dots, N\} \leq h^{(t)}||\alpha||_\infty + {h^{(t)}}^2 \frac{M_2 + B_0}{2}$$

where $M_2 := \max\{|(\partial^2 u/\partial t^2)(x,t)| : (x,t) \in D\}$ and B_0 is a constant such that for all $t \in J$, $|| \int_0^t (dB/ds)(t, s, u_{h^{(x)}}(s))\, ds|| \leq B_0$. Subtraction of (3.6) from (2.12) yields

$$V_{n+1} - u_{h^{(x)}}(t_{n+1}) = V_n - u_{h^{(x)}}(t_n) + h^{(t)} A(t_{n+1})[V_{n+1} - u_{h^{(x)}}(t_{n+1})]$$
$$+ {h^{(t)}}^2 \sum_{i=0}^{n} [B(t_{n+1}, t_{i+1}, V_{i+1}) - B(t_{n+1}, t_{i+1}, u_{h^{(x)}}(t_{i+1}))] - \beta(t_{n+1}),$$

which is equivalent to

$$[I - h^{(t)} A(t_{n+1})]E_{n+1}$$
$$= E_n + {h^{(t)}}^2 \sum_{i=0}^{n} [B(t_{n+1}, t_{i+1}, V_{i+1}) - B(t_{n+1}, t_{i+1}, u_{h^{(x)}}(t_{i+1}))]$$
$$- \beta(t_{n+1}).$$

From 1.5.8, properties 7, 3 and 2, of [**1**, p. 31], together with (3.4), it follows that

$$\begin{aligned} ||[I - h^{(t)} A(t_{n+1})]E_{n+1}|| &\geq -\mu[-I + h^{(t)} A(t_{n+1})]||E_{n+1}|| \\ &= -(-1 + h^{(t)}\mu[A(t_{n+1})])||E_{n+1}|| \\ &= (1 - h^{(t)}\mu[A(t_{n+1})])||E_{n+1}|| \\ &\geq (1 - h^{(t)}\nu)||E_{n+1}||. \end{aligned}$$

This implies that

$$(1 - h^{(t)}\nu)||E_{n+1}|| \leq ||E_n|| + {h^{(t)}}^2 L \sum_{i=0}^{n} ||E_{i+1}|| + \tilde{\beta},$$

where we used the Lipschitz condition (2.11) for B. Hence, the sequence $\{||E_n|| : n = 1, \dots, N\}$ satisfies the inequality

$$||E_{n+1}|| \leq \frac{1}{1 - h^{(t)}\nu - {h^{(t)}}^2 L}[||E_n|| + {h^{(t)}}^2 L \sum_{i=1}^{n} ||E_i|| + \tilde{\beta}].$$

This only holds for $h^{(t)} \leq h_0^{(t)}$ with $h_0^{(t)}$ defined such that $h_0^{(t)}\nu + {h_0^{(t)}}^2 L < 1$. There exist positive constants c_1 and c_2 such that $(1 - h^{(t)}\nu - {h^{(t)}}^2 L)^{-1} \leq 1 + c_1 h^{(t)}$ and

$$||E_{n+1}|| \leq (1 + c_1 h^{(t)})||E_n|| + c_2 {h^{(t)}}^2 \sum_{i=1}^{n} ||E_i|| + (1 + c_1 h^{(t)})\tilde{\beta}.$$

It now follows from [**4**, Lemma 6] that

$$||E_n|| \leq \frac{R_1^n - R_2^n}{R_1 - R_2}(1 + c_1 h^{(t)})\tilde{\beta}, \quad n = 1, \dots, N, \tag{3.8}$$

(with $E_0 = 0$), where R_1 and R_2 are the roots of the characteristic polynomial

$$R^2 - (2 + c_1 h^{(t)} + c_2 {h^{(t)}}^2)R + (1 + c_1 h^{(t)}) = 0,$$

i.e.,

$$R_1 = 1 + h^{(t)}\frac{c_1}{2} + {h^{(t)}}^2\frac{c_2}{2} + \frac{h^{(t)}}{2}\sqrt{c_1^2 + 4c_2 + 2h^{(t)}c_1 c_2 + {h^{(t)}}^2 c_2^2},$$
$$R_2 = 1 + h^{(t)}\frac{c_1}{2} + {h^{(t)}}^2\frac{c_2}{2} - \frac{h^{(t)}}{2}\sqrt{c_1^2 + 4c_2 + 2h^{(t)}c_1 c_2 + {h^{(t)}}^2 c_2^2}.$$

We observe that $R_1 = 1 + \mathcal{O}(h^{(t)})$, $R_2 = 1 + \mathcal{O}(h^{(t)})$ and that R_1^n and R_2^n are bounded for $nh^{(t)} \leq T$. Furthermore, $R_1 - R_2 = h^{(t)}\sqrt{c_1^2 + 4c_2} + \mathcal{O}({h^{(t)}}^2)$. Together with (3.7) and (3.8), this proves that there exists a constant C, not depending on the stiffness of (2.6), such that

$$||E_n|| \leq C\frac{\tilde{\beta}}{h^{(t)}} \leq C\left(||\alpha||_\infty + h^{(t)}\frac{M_2 + B_0}{2}\right),$$

with $h^{(t)} \leq h_0^{(t)}$, and for all $n = 1, \dots, N$. $\square$

Remark 3.2. The steplength $h^{(t)}$ has to be sufficiently small for the implicit system (2.12) for V_{n+1} to be uniquely solvable. Therefore, in addition to $h^{(t)} \leq h_0^{(t)}$, a second stepsize restriction $h^{(t)} \leq h_1^{(t)}$ will usually apply.

4. Numerical results. We now solve the following VPIDEs via the MOL:

(4.1)
$$u_t(x,t) = g(x,t) + a_2(x,t)u_{xx}(u,t) + \int_0^t b(x,t,s,u(x,s))\,ds,$$

(4.2)
$$u_t(x,t) = g(x,t) + a_1(x,t)u_x(x,t) + \int_0^t b(x,t,s,u(x,s))\,ds.$$

The corresponding semi-discrete systems (2.6) are integrated numerically in time by the implicit Euler method.

In the following, we use the norm

$$||e||_2 = \sqrt{\frac{1}{M-1}\sum_{j=1}^{M-1} e_j^2}, \quad e \in \mathbf{R}^{M-1}.$$

If u_{xx} in (4.1) is discretized by the three-point finite difference scheme (2.2), the space truncation error $\alpha(t)$ satisfies

$$||\alpha(t)||_2 = \mathcal{O}(h^{(x)^2}), \quad t \in J. \tag{4.3}$$

By Theorem 3.1, the total error for (4.1) is

$$||E_n||_2 = \mathcal{O}(h^{(x)^2}) + \mathcal{O}(h^{(t)}), \quad n = 1, \dots, N. \tag{4.4}$$

If u_x in (4.2) is discretized by the symmetric scheme (2.1a), the space truncation error $\alpha(t)$ also satisfies (4.3). An estimate for the total error for (4.2) is also given by (4.4).

If u_x in (4.2) is discretized by the forward difference scheme (2.1b), the space truncation error $\alpha(t)$ is only of order 1, i.e.,

$$(4.5) \qquad ||\alpha(t)||_2 = \mathcal{O}(h^{(x)}), \quad t \in J.$$

The total error is (only)

$$(4.6) \qquad ||E_n||_2 = \mathcal{O}(h^{(x)}) + \mathcal{O}(h^{(t)}), \quad n = 1, \dots, N.$$

In general it is therefore preferable to use (2.1a) instead of (2.1b).

In the three cases listed above, it can be shown that $\mu_2[A(t)] = \lambda_{\max}[(A(t) + A(t)^\top)/2]$ is independent of $h^{(x)}$ if a_1 and a_2 are once, respectively, twice, continuously differentiable w.r.t. x. Here $\mu_2[\cdot]$ is the logarithmic norm associated to the norm $||\cdot||_2$. We note in passing that a_2 in (4.1) always has to be nonnegative as well as a_1 if forward differences are used for (4.2).

If we use the maximum norm in $\mathbf{R}^{M-1}$, the above estimates also hold, but we may no longer use the symmetrical scheme (2.1a) for the discretization of u_x in (4.2) since then $\mu_\infty[A(t)] = \mathcal{O}((h^{(x)})^{-1})$ which contradicts (3.4).

The above error bounds are illustrated in the following examples. Similar results were obtained for the maximum norm.

Example 1. Let $a_2(x,t) = 2 + \sin(x)$, $b(x,t,s,u) = (1-2x)\exp(s-t)u^2$ and $g(x,t)$ be chosen such that the exact solution of (4.1) is $u(x,t) = \sin(2\pi x)\exp(-xt)$. Here, as in the next two examples, T is equal to 1. Let E_N^M denote the error at $t = T$, after N integration steps in t-direction:

$$E_N^M := \sqrt{\frac{1}{M-1}\sum_{j=1}^{M-1}(E_N^{(j)})^2}, \qquad E_N = (E_N^{(1)}, \dots, E_N^{(M-1)})^\top.$$

For this particular equation, it turned out that the contribution $\mathcal{O}(h^{(t)})$ in (4.4) (with $h^{(t)} = T/N$ and $N = 80$) is small compared to $\mathcal{O}((h^{(x)})^2)$

and therefore the total error is dominated by the space truncation error. The results are listed in Table 4.1 and the observed convergence rate $\hat{p} := \ln(E_N^{2M}/E_N^M)/\ln(2)$ is nearly 2.

TABLE 4.1. Error for $N = 80$ integration steps in t-direction.

M	E_N^M	$\hat{p}$
5	7.291 e-2	
10	1.617 e-2	2.17
20	3.880 e-3	2.06
40	9.600 e-4	2.01
80	2.445 e-4	1.97

Example 2. Let $a_2(x,t) = x^2(1-x)/2$, $b(x,t,s,u) = (1-2x)\exp(s-t)u^2$ and $g(x,t)$ be chosen such that the exact solution of (4.1) is $u(x,t) = x(1-x)\exp(-xt)$. In this example, it is the time integration error that dominates the total error, i.e., $||E_N||_2 \approx \mathcal{O}(h^{(t)})$. The observed order $\hat{q} := \ln(E_{2N}^M/E_N^M)/\ln(2)$ is nearly 1 (see Table 4.2).

TABLE 4.2. Error for $M = 80$ subdivisions in x-direction.

N	E_N^M	$\hat{q}$
5	2.731 e-3	
10	1.401 e-3	0.96
20	7.093 e-4	0.98
40	3.565 e-4	0.99
80	1.782 e-4	1.00

Example 3. Let $a_1(x,t) = 2+\sin(x)$, $b(x,t,s,u) = (1-2x)\exp(s-t)u^2$ and $g(x,t)$ be chosen such that the exact solution of (4.2) is $u(x,t) = \sin(2\pi x)\exp(-xt)$. For the forward difference scheme (2.1b), the observed order $\hat{p}$ is nearly 1 (see Table 4.3), as predicted in (4.6). The central difference scheme (2.1a) provides better results and a higher order of convergence $\hat{p}$, namely 2 (see Table 4.3).

TABLE 4.3. Error for $N = 80$ integration steps in t-direction.

	Forward differences		Central differences	
M	E_N^M	$\hat{p}$	E_N^M	$\hat{p}$
5	4.887 e-1		4.178 e-1	
10	2.239 e-1	1.13	2.963 e-2	3.82
20	1.074 e-1	1.06	8.461 e-3	1.81
40	5.262 e-2	1.03	2.119 e-3	2.00
80	2.600 e-2	1.01	5.686 e-4	1.90

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