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ON NONCOMPACT HAMMERSTEIN INTEGRAL EQUATIONS AND A NONLINEAR BOUNDARY VALUE PROBLEM FOR THE HEAT EQUATION

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ABSTRACT. We discuss the solvability of noncompact Hammerstein integral equations, related to Volterra as well as Wiener-Hopf equations. Usually the solvability is well understood only in L^2 setting, e.g., if the integral operator is positive definite and the nonlinearity is monotone. We are interested in obtaining the L^∞ theory from the L^2 theory. This can be done by means of a result related to Hadamard's theorem, which allows us to consider the solvability of the linearizations of the Hammerstein equation; by means of a theorem in [**5**] concerning the spectra of convolution-like operators on Lebesgue spaces; and by means of a compactness argument involving the strict topology on L^∞. We apply this theory to the study of the solvability of the heat equation on a half space with (mildly) nonlinear heat radiation on the boundary.

1. Introduction. In this paper we study the solvability of noncompact Hammerstein integral equations, prototypical of which are nonlinear convolution equations on Lebesgue spaces. Usually, equations like these are well understood in L^2 setting, and it is desirable to obtain an L^∞ theory without additional conditions. Typically, a satisfactory L^∞ theory is helpful when we want to establish uniform error estimates of numerical methods for these equations, most notoriously for solutions obtained by Galerkin methods, but sometimes the intrinsic interest is in the L^∞ theory to begin with, such as problems related to the heat equation. We consider only *mild* nonlinearities, i.e., nonlinearities with a reasonable *Lipschitz constant.* This allows us to linearize the Hammerstein equation and leads to problems about the spectra of integral operators on Lebesgue spaces. An indispensable technical device turns out to be the *strict* topology on L^∞, see [**8**] and, particularly, [**2, 3, 4**]. For an application of some of these matters in a related context, see [**1**].

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We illustrate the above approach on an integral equation formulation for the heat equation on a half plane with nonlinear dissipation on the boundary.

To make matters more precise, we consider the Hammerstein equation on $\Omega \subset \mathbf{R}^M$,

$$u + \mathcal{K}\mathcal{N}(u) = v. \tag{1.1}$$

Here

$$\mathcal{K}u(x) = \int_\Omega k(x,y)u(y)\,dy, \quad x \in \Omega, \tag{1.2}$$

with the integral kernel k measurable, and there exists a function $b \in L^1(\mathbf{R}^M)$ such that

$$|k(x,y)| \le b(x-y), \quad \text{a.e. } x, y \in \Omega, \tag{1.3}$$

and

$$\sup_{x\in\Omega} \int_\Omega |k(x+h,y) - k(x,y)|\,dy \to 0, \quad |h| \to 0. \tag{1.4}$$

These two conditions cover the case $k(x,y) = b(x-y)a(y)$, where $a \in L^\infty(\Omega)$. The conditions (1.3)–(1.4) are of course well known, see [**4, 17**]. The nonlinearity $\mathcal{N}(u)$ is of Nemytskii type,

$$\mathcal{N}(u)(x) = N(x, u(x)), \quad \text{a.e. } x \in \Omega, \tag{1.5}$$

where N is a Carathéodory function, with $N(x,0) = 0$, for all $x \in \Omega$, and

$$\left|\frac{\partial N(x,z)}{\partial z}\right| \le D, \quad \text{a.e. } x \in \Omega,\ z \in \mathbf{R}. \tag{1.6}$$

To describe a typical situation in which the L^2 theory is well understood, we assume that $\mathcal{K}$ and $\mathcal{N}$ are monotone in the following sense.

$$\int_\Omega u(x)\mathcal{K}u(x)\,dx \ge 0, \quad \text{for all } u \in L^2(\Omega), \tag{1.7}$$

$$0 < d \le \frac{\partial N(x,z)}{\partial z} \le D < \infty, \quad \text{a.e. } x \in \Omega,\ z \in \mathbf{R}. \tag{1.8}$$

Note that the absolute sizes of d and D are not important; the only relevant quantity is the *condition number* D/d of the nonlinearity. Under these conditions (1.7)–(1.8) it is well known that equation (1.1) has a unique solution $u \in L^2$ for every $v \in L^2$ and that u depends Lipschitz continuously on v, [**16**]. In addition, it follows that the linearized equations

$$\varphi + \mathcal{K}\mathcal{N}'(u)\varphi = \psi \tag{1.9}$$

are uniformly solvable on L^2, in the sense that

$$\sup_{u \in L^2} ||[I + \mathcal{K}\mathcal{N}'(u)]^{-1}||_2 < \infty. \tag{1.10}$$

Here $\mathcal{N}'(u)$ is the operator of multiplication by $(\partial/\partial u)N(x, u(x))$,

$$[\mathcal{N}'(u)\varphi](x) = \frac{\partial}{\partial u}N(x, u(x))\varphi(x), \quad \text{a.e. } x \in \Omega. \tag{1.11}$$

The key observation is now that if (1.10) were to hold on L^p, i.e.,

$$\sup_{u \in L^p} ||[I + \mathcal{K}\mathcal{N}'(u)]^{-1}||_p < \infty, \tag{1.12}$$

then *Hadamard's theorem* would give us the L^p theory for equation (1.1), modulo the technical condition that $\mathcal{K}\mathcal{N}$ needs to be "C^1." One main theme of this paper is to prove that (1.10) implies (1.12), see Section 3. The other theme is that (1.12) is *all* that is needed to get the L^∞ theory, see Section 2. When the extra generality $1 \le p \le \infty$ seems to be more trouble that it is worth, we will restrict attention to $p = \infty$. In Section 4 we apply this theory to solve a boundary value problem for the heat equation in a half plane with nonlinear radiation on the boundary.

We finish this section by establishing some notations and conventions. When considering *nonlinear* operators, equations, etc., the Banach spaces in question are spaces of *real* functions. When we talk about the *spectra of linear* operators, then we replace the Banach spaces by their complexifications. So, for $\Omega \subset \mathbf{R}^M$, we let $L^p(\Omega)$ denote the Banach spaces of real measurable functions on Ω with $|u|^p$ integrable on Ω for $1 \le p < \infty$ and essentially bounded for $p = \infty$. We denote the norm on $L^p(\Omega)$ by $||\cdot||_{p,\Omega}$ or simply by $||\cdot||_p$ when the set Ω is clear from the context. We let

$$L_0^\infty(\Omega) = \{u \in L^\infty(\Omega) : \operatorname*{ess\,sup}_{|x|>n} |u(x)| \to 0, \text{ as } n \to \infty\}.$$

Let X be a Banach space over the complex numbers. For a bounded linear operator $\mathcal{L} : X \to X$ we define the *spectrum* $\sigma(\mathcal{L}, X)$ as

$$\sigma(\mathcal{L}, X) = \{\lambda \in \mathbf{C} : \mathcal{L} - \lambda\mathcal{I} : X \to X \text{ has no bounded inverse}\}.$$

We will use this for our integral operators $\mathcal{K}$ which are defined on all real L^p spaces ($1 \le p \le \infty$), and so also on all *complex* L^p.

2. Hadamard's theorem, and how to bypass it. We begin by quoting the version of Hadamard's theorem that seems to be the most appropriate for our purposes. For a reference, see [**6**].

Let X be a Banach space, with norm $||\cdot||_X$. The map $F : X \to X$ is Fréchet differentiable if for every $u \in X$ there exists a bounded linear operator $F'(u) : X \to X$ such that

$$\frac{||F(u+h) - F(u) - F'(u)h||_X}{||h||_X} \to 0, \quad \text{as } ||h||_X \to 0.$$

If this map $F'(u)$ is continuous in $u \in X$, then we say that $F \in C^1(X)$.

2.1 Hadamard's theorem. *Let X be a Banach space, and $F \in C^1(X)$. If*

$$\sup_{u \in X} ||[F'(u)]^{-1}||_X < \infty,$$

then $F : X \to X$ is onto, and F has a Lipschitz continuous inverse.

The condition $F \in C^1(X)$ is quite strong, but fortunately the nonlinear map $F(u) = u + \mathcal{K}\mathcal{N}(u)$ has some additional properties we may put to good use. To be precise,

(2.2) the nonlinearity $\mathcal{N}$ satisfies the mean value property

$$\mathcal{N}(u) - \mathcal{N}(v) = \mathcal{N}'(w)(u - v), \quad \text{for all } u, v \in X,$$

and so does $\mathcal{KN}$. Here w is some element of $X = L^p(\Omega)$ between u and v. (Recall that we take L^p to be real here.)

(2.3) the nonlinear map $F(u) = u + \mathcal{KN}(u) : Y \to Y$ is onto for a Banach space Y, with $Y \cap X$ dense in X. (At least for $Y = L^2(\Omega)$, and $X = L^p(\Omega)$, with $1 \le p < \infty$, and for $X = L_0^\infty(\Omega)$.)

We thus get the following theorem.

2.4 Theorem. *Let X and Y be Banach spaces, with $Y \cap X$ dense in X. Assume that F maps $X \cap Y$ into itself, and that this map extends to a continuous map from X into X, as well as from Y into Y in the respective topologies. Assume that $F(0) = 0$ both in X and Y, and that*

(2.5) *F satisfies the mean value property on X as well as Y,*

(2.6) *$F : Y \to Y$ is onto,*

(2.7) *$F : X \to X$ is Lipschitz continuous and Gateaux differentiable, and likewise for $F : Y \to Y$. Moreover, for all $u \in Y$, the Gateaux derivative $F'(u) : Y \to Y$ extends to a bounded linear operator from X to X, and*

$$||[F'(u)]^{-1}||_X \le C, \quad \textit{for all } u \in X \cup Y.$$

Then $F : X \to X$ is onto and has a Lipschitz continuous inverse.

Proof. Let $w \in X$, and consider the equation $F(u) = w$. Since $X \cap Y$ is dense in X, choose $\{w_n\}_n \subset X \cap Y$ such that $w_1 = 0$ and $||w - w_n||_X \to 0$ as $n \to \infty$. By (2.6), the equation $F(u_n) = w_n$ has a solution $u_n \in Y$. Then we have $F(u_n) - F(u_m) = w_n - w_m$, and by the mean value property (2.5), for each n, m there exists a $u_{nm} \in Y$ such that $F(u_n) - F(u_m) = F'(u_{nm})(u_n - u_m)$, so that

$$F'(u_{nm})(u_n - u_m) = w_n - w_m. \tag{2.8}$$

This equation holds in Y, but we may also think of it as an equation in X. Now (2.7) lets us conclude from (2.8) that

$$||u_n - u_m||_X \le C||w_n - w_m||_X. \tag{2.9}$$

Now observe that $w_1 = 0$ corresponds to $u_1 = 0$. Then (2.9) implies that $u_n \in X$ for all n. Moreover, since $\{w_n\}_n$ is a Cauchy sequence in X, it follows that $\{u_n\}_n$ is Cauchy as well. Thus, $\{u_n\}_n$ converges to some $u \in X$, and then $F(u) = \lim F(u_n) = \lim w_n = w$. Thus, $F : X \to X$ is onto. The Lipschitz continuity of the inverse of F follows likewise. □

We may apply this theorem to our situation, with $F(u) = u + \mathcal{K}\mathcal{N}(u)$, and $X = L^p(\Omega)$ for $1 \le p < \infty$, and $X = L_0^\infty(\Omega)$, and $Y = L^2(\Omega)$.

2.10 Theorem. *Let X be one of $L^p(\Omega)$, $1 \le p < \infty$, or $L_0^\infty(\Omega)$. Assume that $F(u) = u + \mathcal{K}\mathcal{N}(u)$ maps X into X, is onto on $L^2(\Omega)$, and assume that*

$$\sup_{u \in X} \|[I + \mathcal{K}\mathcal{N}'(u)]^{-1}\|_X < \infty. \tag{2.11}$$

Then $F : X \to X$ is onto, with a Lipschitz continuous inverse.

Finally, we must work a little bit to get from $L_0^\infty(\Omega)$ to the whole space $L^\infty(\Omega)$. The crucial notion is that of the *strict* topology on $\mathbf{R}^M$, see [**2, 4, 8**].

Let $\{u_n\} \subset L^\infty(\Omega)$ and $u \in L^\infty(\Omega)$. We say that u_n converges to u in the *strict* topology on $L^\infty(\Omega)$ if for each compact subset S of Ω we have

$$\operatorname*{ess\,sup}_{x \in S} |u_n(x) - u(x)| \to 0, \quad \text{as } n \to \infty. \tag{2.12}$$

A crucial property is that we have an analogue of the Arzelà-Ascoli theorem.

2.13 Arzelà-Ascoli Theorem [**4**]. *If $\{u_n\}_n$ is a bounded, equi-uniformly-continuous sequence in $L^\infty(\Omega)$, then $\{u_n\}_n$ has a subsequence which converges to some element of $L^\infty(\Omega)$ in the strict topology.*

With these notions in hand, we can extend Theorem 2.10 to L^∞.

2.14 Theorem. *Under the same conditions as Theorem 2.10, the map $F(u) = u + \mathcal{K}\mathcal{N}(u)$ maps $L^\infty(\Omega)$ onto itself and has a Lipschitz continuous inverse.*

Proof. Let $v \in L^\infty(\Omega)$. For $n \in \mathbf{N}$, define $v_n \in L_0^\infty(\Omega)$ by truncation.

$$v_n(x) = \begin{cases} v(x), & |x| \le n, \\ 0, & \text{otherwise.} \end{cases}$$

Observe that $v_n \to v$ in the *strict* topology on $L^\infty(\Omega)$. By Theorem 2.10 the equation $F(u_n) = v_n$ has a solution $u_n \in L_0^\infty(\Omega)$, and

$$||u_n||_\infty \le C||v_n||_\infty \le C||v||_\infty.$$

It follows that $\{\mathcal{N}(u_n)\}_n$ is a bounded sequence in $L^\infty(\Omega)$, and so

$$|\mathcal{K}\mathcal{N}(u_n)(x) - \mathcal{K}\mathcal{N}(u_n)(y)| \le ||\mathcal{N}(u_n)||_\infty \int_\Omega |k(x,z) - k(y,z)|\, dz,$$

from which it follows by (1.4) that $\{\mathcal{K}\mathcal{N}(u_n)\}_n$ is equi-uniformly-continuous on Ω. It then follows from the Arzel̀-Ascoli theorem that $\{\mathcal{K}\mathcal{N}(u_n)\}_n$ has a subsequence which converges to some $w \in L^\infty(\Omega)$ in the *strict* topology. Moreover, w itself is uniformly continuous on Ω. From the equation

$$u_n + \mathcal{K}\mathcal{N}(u_n) = v_n$$

it then follows without loss of generality that $u_n \to v - w$ (*strictly*). Finally, since $\mathcal{K}\mathcal{N}$ is continuous in the *strict* topology [**1, 2**], it follows that for $u = v - w$ we have

$$u + \mathcal{K}\mathcal{N}(u) = v.$$

This shows that $F(u) = u + \mathcal{K}\mathcal{N}(u)$ as a map from $L^\infty(\Omega)$ into itself is onto. The Lipschitz continuity of the inverse follows from assumption (2.11) in Theorem 2.10. □

For later reference, we quote a slightly different version of the Arzelà-Ascoli theorem. A *sliding Arzelà-Ascoli technique* already appears in [**2**] and in improved form in [**3**].

2.15 Sliding Arzelà-Ascoli Theorem [**2, 3**]. *Let* $\Omega = \mathbf{R}_+^M$ $(= (\mathbf{R}^+)^M)$ *and let* $\{x_n\}_n \subset \Omega$, *with* $|x_n| \to \infty$. *Define the translations-extensions* $\mathcal{T}_n : L^2(\Omega) \to L^2(\mathbf{R}^M)$ *for* $u \in L^2(\Omega)$ *by*

$$\mathcal{T}_n u(x) = \begin{cases} u(x + x_n), & x + x_n \in \Omega, \\ 0, & \textit{otherwise,} \end{cases}$$

If $\{u_n\}_n$ is a bounded, equi-uniformly-continuous sequence in $L^\infty(\Omega)$, then $\{\mathcal{T}_n u_n\}_n$ has a subsequence which converges to some element u of $L^\infty(\mathbf{R}^M)$ in the strict topology. Moreover, u is uniformly continuous on $\mathbf{R}^M$.

Proof. We only consider the case where every component of x_n tends to ∞ as $n \to \infty$. (If some component $x_n(i)$ remains bounded, then for our purposes we should *redefine* the x_n by setting $x_n(i)$ equal to 0 and make the appropriate changes in the proof.) Then the *maximal supports* (to give them a name) of $\mathcal{T}_n u_n$, given by $-x_n + \Omega$, are strictly nested, and their union covers $\mathbf{R}^M$. The only catch in the proof is that the $\mathcal{T}_n u_n$ are not continuous on $\mathbf{R}^M$, but $\{\mathcal{T}_n u_n\}_{n>N}$ is equi-uniformly-continuous on $\Omega_N \stackrel{\text{def}}{=} -x_N + \Omega$. So, for each ball $B_m \stackrel{\text{def}}{=} \{x \in \mathbf{R}^M : |x| \le m\}$, there exists an Ω_N containing it, and so we can extract a subsequence of $\{\mathcal{T}_n u_n\}_{n>N}$ which converges uniformly on B_m to some element of $L^\infty(B_m)$. A diagonal argument then shows that there exists a subsequence of $\{T_n u_n\}_n$ which converges to some element $u \in L^\infty(\mathbf{R}^M)$, uniformly on every compact subset of $\mathbf{R}^M$. It also follows that u is uniformly continuous on $\mathbf{R}^M$. □

3. Uniform invertibility of integral operators. The previous section tells us that all we need to get a satisfactory L^∞ theory for the equation $u + \mathcal{KN}(u) = v$ is to establish the theorem which says that

$$\sup_{u \in L^2(\Omega)} ||[I + \mathcal{K}\mathcal{N}'(u)]^{-1}||_{2,\Omega} < \infty$$

implies that

$$\sup_{u \in L^\infty(\Omega)} ||[I + \mathcal{K}\mathcal{N}'(u)]^{-1}||_{\infty,\Omega} < \infty.$$

In [**11**] this is proved (for $M = 1$), in the following form. (See also [**10**], where an extra condition on b was needed.)

3.1 Theorem. *Let $b \in L^1(\mathbf{R}^M)$ and $e \in C(\mathbf{R}^M)$, with $e(0) = 0$. Consider a set $\mathbf{K}$ of integral operators with integral kernels $k(x, y)$ satisfying*

$$|k(x, y)| \le b(x - y), \quad a.e.\ x, y \in \Omega,$$

and

$$\int_{\Omega} |k(x,z) - k(y,z)|\, dz \le e(x-y), \quad \textit{for all } x, y \in \Omega.$$

If $-1 \notin \sigma(\mathcal{K}, L^2(\Omega))$ *for each* $\mathcal{K} \in \mathbf{K}$ *and*

$$\sup_{\mathcal{K} \in \mathbf{K}} ||[I + \mathcal{K}]^{-1}||_{2,\Omega} < \infty,$$

then $-1 \notin \sigma(\mathcal{K}, L^\infty(\Omega))$ *for each* $\mathcal{K} \in \mathbf{K}$ *and*

$$\sup_{\mathcal{K} \in \mathbf{K}} ||[I + \mathcal{K}]^{-1}||_{\infty,\Omega} < \infty.$$

The proof for $M > 1$ goes through with minor changes. Here we give the more natural proof, in that we first establish the theorem for a single operator and then use compactness arguments to get results which hold uniformly on $\mathbf{K}$. Unfortunately, the first step seems to require *stronger* conditions.

From now on, we assume throughout that $\Omega = \mathbf{R}_+^M$. The results below apply also to the case $\Omega = \mathbf{R}_+^{M-N} \times \mathbf{R}^N$.

3.2 Theorem. *Let* $e \in C(\mathbf{R}^M)$, *with* $e(0) = 0$, *and let* ℓ *be measurable and nonnegative on* $\Omega \times \Omega$, *and for some* $\alpha > 0$,

$$\operatorname*{ess\,sup}_{x \in \Omega} \int_{|x-y|>n} \{|l(x,y)| + |l(y,x)|\}(1 + |x-y|)^\alpha \, dy \tag{3.3}$$

is bounded for $n = 0$ *and tends to* 0 *as* $n \to \infty$. *Consider a set* $\mathbf{L}$ *of integral operators with* (*integral*) *kernels* $k(x,y)$ *satisfying*

$$\int_{\Omega} |k(x,z) - k(y,z)|\, dz \le e(x-y), \quad \textit{for a.e. } x, y \in \Omega \tag{3.4}$$

as well as

$$|k(x,y)| \le l(x,y), \quad \textit{for a.e. } x, y \in \Omega. \tag{3.5}$$

If $-1 \notin \sigma(\mathcal{K}, L^2(\Omega))$ *for each* $\mathcal{K} \in \mathbf{L}$ *and*

$$\sup_{\mathcal{K} \in \mathbf{L}} ||[I + \mathcal{K}]^{-1}||_{2,\Omega} < \infty, \tag{3.6}$$

then $-1 \notin \sigma(\mathcal{K}, L^\infty(\Omega))$ *for each* $\mathcal{K} \in \mathbf{L}$ *and*

$$\sup_{\mathcal{K}\in\mathbf{L}} \|[I+\mathcal{K}]^{-1}\|_{\infty,\Omega} < \infty. \tag{3.7}$$

From a practical point of view, the difference between the condition (3.3) and the condition "$|l(x,y)| \le b(x-y)$ for some $b \in L^1(\mathbf{R}^M)$" is hardly going to make a difference.

A first natural step in the proof of the above theorem is to consider the case where $\mathbf{L}$ is a singleton. Then it is just a question about spectra of integral operators on L^p spaces. The following result follows from [**5**, Theorem 4.8(2)].

3.8 Theorem [**5**]. *For every* $\mathcal{K} \in \mathbf{L}$,

$$\sigma(\mathcal{K}, L^\infty(\Omega)) \subset \sigma(\mathcal{K}, L^2(\Omega)).$$

It is here that the condition (3.3) is needed, since this is a requirement of [**5**].

In order to prove Theorem 3.2, we need the following construction. This construction is implicit for Wiener-Hopf equations, for which the solvability is quite naturally associated with certain equations on the *whole* real line, [**17**], and essentially appears in [**2**]. Let $\mathbf{L}$ be as in Theorem 3.2, and let $\mathcal{T}\mathbf{L}$ (translates of operators in $\mathbf{L}$) be defined as follows. An operator $\mathcal{L}$ is an element of $\mathcal{T}\mathbf{L}$ if and only if there exist

(3.9) a sequence $\{\mathcal{K}_n\}_n \subset \mathbf{L}$,

(3.10) a sequence of translations/extensions $\mathcal{T}_n : L^2(\Omega) \to L^2(\mathbf{R}^M)$ defined for suitable $\{x_n\}_n \subset \Omega$ as

$$\mathcal{T}_n u(x) = \begin{cases} u(x+x_n), & x+x_n \in \Omega, \\ 0, & \text{otherwise}, \end{cases}$$

for all $u \in L^2(\Omega)$, whose adjoints $\mathcal{T}_n^* : L^2(\mathbf{R}^M) \to L^2(\Omega)$ are given by

$$\mathcal{T}_n^* v(x) = v(x - x_n), \quad x \in \Omega,$$

such that

(3.11) $\{\mathcal{T}_n\mathcal{K}_n\mathcal{T}_n^*\}_n$ converges strictly to $\mathcal{L}$, i.e., for all $v \in L^\infty(\mathbf{R}^M)$,

$$\mathcal{T}_n\mathcal{K}_n\mathcal{T}_n^* v \to \mathcal{L}v, \text{ in the strict topology on } L^\infty(\mathbf{R}^M).$$

We need the following theorem, which enunciates a very useful notion of compactness.

3.12 Theorem. *Let $\{\mathcal{K}_n\}_n \subset \mathbf{L}$, and let $\{\mathcal{T}_n\}_n$ be a sequence of translations-extensions, as in* (3.10)*. Then the sequence $\{\mathcal{T}_n\mathcal{K}_n\mathcal{T}_n^*\}_n$ has a subsequence which converges to an operator $\mathcal{L} \in \mathcal{T}\mathbf{L}$ in the sense of* (3.11)*.*

We first prove a weaker version of the above theorem.

3.13 Lemma. *The sequence $\{\mathcal{T}_n\mathcal{K}_n\mathcal{T}_n^*\}_n$ has a subsequence which converges to an integral operator $\mathcal{L}$ on $L_0^\infty(\mathbf{R}^M)$ in the sense that there exists an infinite subset $\mathbf{N}_1$ of $\mathbf{N}$ such that for every $u \in L_0^\infty(\mathbf{R}^M)$,*

$$\mathcal{T}_n\mathcal{K}_n\mathcal{T}_n^* u \to \mathcal{L}u,$$

as $n \to \infty$, $n \in \mathbf{N}_1$, in the strict topology on $L^\infty(\mathbf{R}^M)$. Moreover, the integral kernel of $\mathcal{L}$ satisfies the integrability conditions (3.3)–(3.5)*.*

Proof. Let $\mathcal{L}_n = \mathcal{T}_n\mathcal{K}_n\mathcal{T}_n^*$. Let $\{u_k\}_k \subset L_0^\infty(\mathbf{R}^M)$ be dense in $L_0^\infty(\mathbf{R}^M)$, and set $w_k = u_k/||u_k||_{L^\infty(\mathbf{R}^M)}$. Then $\{\mathcal{L}_n w_k\}_n$ is equi-uniformly-continuous on $\mathbf{R}^M$, hence by the Sliding Arzelà-Ascoli Theorem 2.15 we may extract a subsequence $\{\mathcal{L}_n w_k\}_{n\in\mathbf{N}_k}$ which converges in the strict topology on $L^\infty(\mathbf{R}^M)$ to some $v_k \in L^\infty(\mathbf{R}^M)$, and $\mathbf{N}_1 \supset \mathbf{N}_2 \supset \ldots \mathbf{N}_k \supset \mathbf{N}_{k+1} \supset \ldots$. A diagonalization process yields an infinite subset $\mathbf{N}_\infty \subset \mathbf{N}$ such that $\{\mathcal{L}_n w_k\}_{n\in\mathbf{N}_\infty}$ converges strictly for every k. Without loss of generality, we assume that $\mathbf{N}_\infty = \mathbf{N}$. It follows that $\{\mathcal{L}_n w\}_n$ converges strictly for every $w \in L_0^\infty(\mathbf{R}^M)$. Denote the limit by $\mathcal{L}w$. Then, obviously, $\mathcal{L} : L_0^\infty(\mathbf{R}^M) \to L^\infty(\mathbf{R}^M)$ is a linear operator. Moreover, since $\mathcal{L}w$ is the uniform limit of $\{\mathcal{L}_n w\}_n$ (on every compact set), it follows that $\mathcal{L}$ is *bounded*.

Next we show that $\mathcal{L}$ is an integral operator, whose integral kernel satisfies (3.3)–(3.5). From the Sliding Arzelà-Ascoli Theorem 2.15 we have that $\mathcal{L}w$ is (uniformly) continuous. Thus, for each $x \in \mathbf{R}^M$, we have that $\mathcal{L}w(x)$ is a bounded linear functional on $L_0^\infty(\mathbf{R}^M)$. Therefore, we may represent this functional as

$$\mathcal{L}w(x) = \int_{\mathbf{R}^M} w(y)\, d\lambda_x(y), \quad x \in \mathbf{R}^M,$$

where λ_x is a Radon measure on $\mathbf{R}^M$, see, e.g., [**9**]. Since the integral kernels $k_n(x,y)$ of the operators $\mathcal{L}_n$ satisfy

$$|k_n(x,y)| \le l(x + x_n, y + y_n), \quad x, y \in \mathbf{R}^M,$$

(with $l(x,y) \equiv 0$ for $(x,y) \notin \Omega \times \Omega$), we thus get that λ_x is absolutely continuous with respect to Lebesgue measure. So $d\lambda_x(y) = \lambda(x,y)\, dy$, with $\lambda(x,\cdot) \in L^1(\mathbf{R}^M)$ for all $x \in \mathbf{R}^M$, and we may write

$$\mathcal{L}w(x) = \int_{\mathbf{R}^M} \lambda(x,y) w(y)\, dy, \quad x \in \mathbf{R}^M.$$

The final conclusion that

$$\operatorname{ess\,sup}_{x\in\Omega} \int_{|x-y|>n} \{|\lambda(x,y)| + |\lambda(y,x)|\}(1 + |x-y|)^\alpha\, dy$$

is bounded for $n = 0$ and tends to 0 as $n \to \infty$, now follows easily. □

The following corollary requires no proof now.

3.14 Corollary. *If $\mathcal{L} \in \mathcal{T}\mathbf{L}$, then $\mathcal{L}$ is an integral operator whose integral kernel satisfies* (3.3)–(3.5).

3.15 Proof of theorem 3.12. The sequence $\{\mathcal{L}_n\}_n \stackrel{\text{def}}{=} \{\mathcal{T}_n \mathcal{K}_n \mathcal{T}_n^*\}_n$ has a subsequence which converges to some integral operator $\mathcal{L}$ in the sense of Lemma 3.13, and the integral kernel of $\mathcal{L}$ satisfies (3.3)–(3.5). For convenience, we assume that the whole sequence converges. We must now strengthen the convergence to the sense of (3.11).

We denote the integral kernels of $\mathcal{L}_n$ and $\mathcal{L}$ by $k_n(x,y)$ and $k(x,y)$, respectively. Let $u \in L^\infty(\mathbf{R}^M)$. Let $A \subset \mathbf{R}^M$ be compact, and let

$\varepsilon > 0$. Since all $k_n(x,y)$ are dominated by $l(x,y)$, and both $l(x,y)$ and $k(x,y)$ satisfy (3.3)–(3.5), we may choose $m \in \mathbf{N}$ such that for all $n \in \mathbf{N}$,

$$\operatorname{ess\,sup}_{x\in A}\left|\int_{|y|>m} k_n(x,y)u(y)\,dy\right| < \varepsilon, \tag{3.16}$$

and likewise for k. Now define v_m by truncation:

$$v_m(x) = \begin{cases} u(x), & |x| \le m, \\ 0, & \text{otherwise.} \end{cases}$$

Then $v_m \in L_0^\infty(\mathbf{R}^M)$ and so by Lemma 3.13,

$$\operatorname{ess\,sup}_{x\in A}|\mathcal{L}_n v_m - \mathcal{L} v_m| < \varepsilon,$$

for all n large enough. Combined with (3.16), it follows that

$$\operatorname{ess\,sup}_{x\in A}|\mathcal{L}_n u - \mathcal{L} u| < \varepsilon,$$

and the theorem is proved. □

Before we can prove Theorem 3.2, we need a connection between the strict convergence of (3.11) with convergence in L^2 and associated spectral properties of the classes $\mathbf{L}$ and $\mathcal{T}\mathbf{L}$.

3.17 Lemma. *If $\{\mathcal{T}_n\mathcal{K}_n\mathcal{T}_n^*\}_n$ converges to $\mathcal{L} \in \mathcal{T}\mathbf{L}$ in the sense of* (3.11)*, then the convergence is in effect strong convergence on $L^2(\mathbf{R}^M)$.*

Proof. Let $\mathcal{L}_n \overset{\text{def}}{=} \mathcal{T}_n\mathcal{K}_n\mathcal{T}_n^*$. By (3.3)–(3.5), we have that for $w \in L_0^\infty(\mathbf{R}^M)$,

$$||\mathcal{L}_n w|_{|x|>m}||_\infty \to 0, \quad m \to \infty, \tag{3.18}$$

uniformly in n, and by Corollary 3.14, also

$$||\mathcal{L} w|_{|x|>m}||_\infty \to 0, \quad m \to \infty. \tag{3.19}$$

Since $\mathcal{L}_n w$ converges to $\mathcal{L}w$, uniformly on compacta, then

$$||\mathcal{L}_n w - \mathcal{L}w||_{\infty,\mathbf{R}^M} \to 0, \quad n \to \infty, \tag{3.20}$$

still for all $w \in L_0^\infty(\mathbf{R}^M)$. Since (3.18)–(3.19) also hold for the L^2-norm, it is then an easy exercise to show that (3.20) implies that

$$||\mathcal{L}_n v - \mathcal{L}v||_{2,\mathbf{R}^M} \to 0, \quad n \to \infty,$$

for each $v \in L^2(\mathbf{R}^M)$. □

3.21 Lemma. *If* $-1 \notin \sigma(\mathcal{K}, L^2(\Omega))$ *for all* $\mathcal{K} \in \mathbf{L}$*, and*

$$\sup_{\mathcal{K}\in\mathbf{L}} ||[\mathcal{I} + \mathcal{K}]^{-1}||_{2,\Omega} < \infty,$$

then there exists a constant $c > 0$ *such that for every* $\mathcal{L} \in \mathcal{T}\mathbf{L}$*, and for every* $u \in L^2(\mathbf{R}^M)$*,*

$$||u + \mathcal{L}u||_{2,\mathbf{R}^M} \geq c||u||_{2,\mathbf{R}^M}.$$

Proof. Let $\mathcal{L} \in \mathcal{T}\mathbf{L}$, and let $\mathcal{L}_n = \mathcal{T}_n\mathcal{K}_n\mathcal{T}_n^*$ converge to $\mathcal{L}$ on $L^2(\mathbf{R}^M)$ in the sense of (3.11). By Lemma 3.20, we may assume that the convergence is in effect strong on $L^2(\mathbf{R}^M)$. Since the equation $u + \mathcal{L}_n u = v$ on $\mathbf{R}^M$ is equivalent to $w + \mathcal{K}_n w = \mathcal{T}_n^* v$ on Ω, it follows that $||[\mathcal{I} + \mathcal{L}_n]^{-1}||_{2,\mathbf{R}^M} \leq c||[\mathcal{I} + \mathcal{K}_n]^{-1}||_{2,\Omega}$, and we have for a suitable constant C,

$$\sup_n ||[\mathcal{I} + \mathcal{L}_n]^{-1}||_{2,\mathbf{R}^M} \leq C.$$

The lemma now follows from, e.g., [**14**, Chapter 3, Lemma 1.1]. □

Finally, we need a simple fact about the (uniform) continuity in the strict topology of operators in $\mathbf{L}$. See [**1,3**]. We omit the proof.

3.22 Lemma. *Suppose* $\{u_n\}_n$ *is a bounded sequence in* $L^\infty(\mathbf{R}^M)$*, which converges to some* $u \in L^\infty(\mathbf{R}^M)$ *in the strict topology. If* $\{\mathcal{K}_n\}_n \subset \mathbf{L}$*, with* $\mathbf{L}$ *as in Theorem* 3.2*, then* $\mathcal{L}_n(u_n - u) \to 0$ *in the strict topology on* $L^\infty(\mathbf{R}^M)$*. Here* $\mathcal{L}_n = \mathcal{T}_n\mathcal{K}_n\mathcal{T}_n^*$*.*

We are now ready to prove Theorem 3.2.

3.23 Proof of Theorem 3.2. We suppose that $-1 \notin \sigma(\mathcal{K}, L^2(\Omega))$ for all $\mathcal{K} \in \mathbf{L}$, but that (3.7) does not hold. We then prove that (3.6) does not hold either.

By [**5**, Theorem 3.8] we know that $-1 \notin \sigma(\mathcal{K}, L^\infty(\Omega))$; thus, if (3.7) does not hold then by the uniform boundedness principle there must exist a sequence $\{\mathcal{K}_n\}_n \subset \mathbf{L}$, and a $v \in L^\infty(\Omega)$ such that $||[\mathcal{I} + \mathcal{K}_n]^{-1} v||_{\infty,\Omega} \to \infty$. It follows that there exists $\{u_n\}_n \subset L^\infty(\Omega)$, with $||u_n||_{\infty,\Omega} = 1$, and

$$||u_n + \mathcal{K}_n u_n||_{\infty,\Omega} \to 0. \tag{3.24}$$

Now $\{\mathcal{K}_n u_n\}_n$ is equi-uniformly-continuous on Ω, by assumption (3.4), and then without loss of generality, so is $\{u_n\}_n$. Let $\{x_n\}_n \subset \Omega$ be such that $|u_n(x_n)| > 1 - n^{-1}$. Let $\mathcal{T}_n$ be the translation-extension operator defined in (3.10). Then from (3.24)

$$||\mathcal{T}_n u_n + \mathcal{T}_n \mathcal{K}_n \mathcal{T}_n^* \mathcal{T}_n u_n||_{\infty,\mathbf{R}^M} \to 0. \tag{3.25}$$

By applying the Sliding Arzelà-Ascoli Theorem 2.15 we see that $\{\mathcal{T}_n u_n\}_n$ contains a subsequence which converges to some $u \in L^\infty(\mathbf{R}^M)$ in the strict topology and u is uniformly continuous. For notational convenience, assume that the whole sequence converges. Since $|u(0)| = \lim_n |u_n(x_n)| = 1$, it follows that $u \neq 0$. Now, by Lemma 3.22 and (3.25) we get that $\{u + \mathcal{T}_n \mathcal{K}_n \mathcal{T}_n^* u\}_n$ converges to 0 in the strict topology on $L^\infty(\mathbf{R}^M)$. By Theorem 3.12, the sequence $\{\mathcal{T}_n \mathcal{K}_n \mathcal{T}_n^*\}_n$ contains a subsequence which converges strictly on $L^\infty(\mathbf{R}^M)$ to some $\mathcal{L} \in \mathcal{T}\mathbf{L}$, so that $u + \mathcal{L}u = 0$, with $u \neq 0$. Then $-1 \in \sigma(\mathcal{L}, L^\infty(\mathbf{R}^M))$, and so by [**5**, Theorem 3.8] also $-1 \in \sigma(\mathcal{L}, L^2(\mathbf{R}^M))$. By Lemma 3.21, it follows that (3.6) does not hold. □

4. A nonlinear boundary value problem for the heat equation. In this section we consider an application of the theory developed in the previous sections. We consider the heat equation in a half plane, with nonlinear heat exchange on the boundary. The one-dimensonal case has been treated exhaustively [**18**], with different methods under weaker assumptions on the nonlinearity. A different treatment of the two-dimensional problem is given in [**13**].

Let $\Pi = \{(x,y) : -\infty < x < \infty, 0 < y < \infty\}$ with boundary $\partial\Pi$ (the x=axis, and consider the temperature distribution $u(x,y;t)$ in the half plane at time t. We assume that u satisfies the initial boundary value problem

$$\begin{aligned} &u_t - \Delta u = 0, \quad \text{in } \Pi \times \mathbf{R}^+, \\ &u(x,y;0) = f(x,y), \quad \text{on } \Pi, \\ &u_y(x,0;t) = g(x,t,u(x,0;t)), \quad \text{on } \partial\Pi \times \mathbf{R}^+. \end{aligned} \tag{4.1}$$

We assume that f is bounded on Π. The nonlinear heat exchange may vary over time and may also vary with the position on the boundary. If it is independent of the position on the boundary, then it reduces to a problem with one spatial dimension. We assume that g is a Carathéodory function, $g(x,t,0) \equiv 0$ for all x,t and satisfies

$$0 < d \le \frac{\partial g}{\partial u} \le D < \infty, \quad \text{for all } (x,t,u) \in \partial\Pi \times \mathbf{R}^+ \times \mathbf{R}. \tag{4.2}$$

In the sequel we will show that the initial boundary value problem (4.1) has a unique solution $u \in L^\infty(\Pi \times \mathbf{R}^+)$ for every $f \in L^\infty(\Pi \times \mathbf{R}^+)$, and that u, restricted to $\partial\Pi$, depends Lipschitz continuous on f in the L^∞-topology, i.e., there exists a constant C such that if u_i corresponds to f_i, then

$$||u_1\Big|_{\partial\Pi} - u_2\Big|_{\partial\Pi}||_{\infty,\partial\Pi\times\mathbf{R}^+} \le C\,||f_1 - f_2||_{\infty,\Pi\times\mathbf{R}^+}. \tag{4.3}$$

Similar estimates with respect to g, e.g., the dependence of u with respect to u_0 in case $g(x,t,u) = \gamma(x,t,u-u_0(x,t))$ will not be pursued but are equally important.

It is, of course, well known that to solve problem (4.1) it suffices to find $u(x,0;t)$ for all x,t, see [**13**]. Using Green's formula, and assuming that $|u(x,y;t)|$ and $|\nabla u(x,y;t)|$ are bounded as $|x|+|y| \to \infty$ (for fixed t), it can be shown that $\varphi(x,t) = u(x,0;t)$ satisfies the integral equation

$$\frac{1}{2}\varphi + \mathcal{K}G(\varphi) = \psi, \tag{4.4}$$

where

$$\mathcal{G}(\varphi)(x,t) = g(x,t,\varphi(x,t)), \quad (x,t) \in \partial\Pi \times \mathbf{R}^+, \tag{4.5}$$

and $\mathcal{K}$ is given by

$$\mathcal{K}u(x,t) = \int_0^t \int_{-\infty}^{\infty} k(x-\xi, t-\tau) u(\xi,\tau)\, d\xi\, d\tau, \tag{4.6}$$

for $(x,t) \in \partial\Pi \times \mathbf{R}^+$, with $k(x,t) = K((x,0);t)$, where K is the causal Green's function for the heat equation in the plane

$$K(\mathbf{x};t) = \frac{\exp(-|\mathbf{x}|^2/4t)}{4\pi t}, \quad (\mathbf{x},t) \in \Pi \times \mathbf{R}^+, \tag{4.7}$$

and

$$\psi(x,t) = \int\int_{\Pi} K((x-\xi;\eta);t) f(\xi,\eta)\, d\xi\, d\eta, \tag{4.8}$$

for $(x,t) \in \partial\Pi \times \mathbf{R}^+$. Then the solution of the boundary value problem is (tentatively) given by Green's formula for $\mathbf{x} \in \Pi$, $t > 0$

$$\begin{aligned} u(\mathbf{x};t) &= \int_{\Pi} K(\mathbf{x}-\mathbf{y};t) f(\mathbf{y})\, d\mathbf{y} \\ &+ \int_0^t \int_{-\infty}^{\infty} [K(\mathbf{x}-(\xi,0);t-\tau) g(\xi,\tau,\varphi) - \varphi K_{\mathbf{n}}(\mathbf{x}-(\xi,0);t-\tau)]\, d\xi\, d\tau, \end{aligned} \tag{4.9}$$

where $K_{\mathbf{n}}$ denotes the normal derivative of $K(\mathbf{x}-\mathbf{y};t)$ with respect to $\mathbf{y}$. Assuming that φ is bounded on $\Pi \times \mathbf{R}^+$, it is a well-known exercise to show that $|u(\mathbf{x};t)|$ and $|\nabla u(\mathbf{x};t)|$ are bounded by const $\sqrt{t}$, uniformly in $\mathbf{x}$, and then that (4.9) gives the solution of the boundary problem (4.1). Finally, from the Phragmèn-Lindelöf principle [**19**], it follows that for all $\mathbf{x} \in \Pi$, $t > 0$,

$$|u(\mathbf{x};t)| \le \max\{||f||_{\infty,\Pi}, ||\varphi||_{\infty,\partial\Pi\times\mathbf{R}^+}\}, \tag{4.10}$$

and thus u is bounded on $\Pi \times \mathbf{R}^+$.

From the above, in order to solve (4.1), it suffices to show that the integral equation (4.4) has a bounded solution φ. We want to apply the theory from Sections 2–3, but the equation (4.4) is not yet in the required form, since $k \notin L^1(\partial\Pi \times \mathbf{R}^+)$. The situation is even worse than this, since $\mathcal{K}$ is not even a bounded operator on $L^2(\partial\Pi \times \mathbf{R}^+)$. However, just as for the one-dimensional case, we may transform it into an equation which has the required form, see [**10**]. First note that the integral operator (4.6) is densely defined and is monotone on its domain:

$$\langle u, \mathcal{K}u \rangle \stackrel{\text{def}}{=} \int_0^\infty \int_{-\infty}^\infty u(x,t)[\mathcal{K}u](x,t)\,dx\,dt > 0,$$

for all $u \in \text{domain}\,(\mathcal{K})$, $u \neq 0$. As a matter of fact, by the Plancherel formula,

$$\langle u, \mathcal{K}u \rangle = \int_{\mathbf{R}^2} \hat{k}(\omega,\sigma) |\hat{u}(\omega,\sigma)|^2 \,d\omega\,d\sigma, \tag{4.11}$$

with $\hat{u}(\omega,\sigma)$ the Fourier transform of u, and likewise for $\hat{k}$. A simple calculation shows that

$$\begin{aligned}\hat{k}(\omega,\sigma) &\stackrel{\text{def}}{=} \int_0^\infty \int_{-\infty}^\infty k(x,t) e^{-2\pi i x\omega - 2\pi i t\sigma}\,dx\,dt \\ &= [4\pi(i\sigma + 8\pi\omega^2)]^{-1/2},\end{aligned}$$

with the principal value of the square root, so that the real part of $\hat{k}$ is positive. Then the right-hand side of (4.11) must be positive as well (the imaginary part vanishes). It follows, [**7**], that $\mathcal{I} + \lambda\mathcal{K}$ has a bounded inverse on $L^2(\partial\Pi \times \mathbf{R}^+)$ for all $\lambda > 0$,

$$||[\mathcal{I} + \lambda\mathcal{K}]^{-1}||_{2,\partial\Pi\times\mathbf{R}^+} \le 1, \tag{4.12}$$

and from $\lambda(\mathcal{I} + \lambda\mathcal{K})^{-1}\mathcal{K} = \mathcal{I} - (\mathcal{I} + \lambda\mathcal{K})^{-1}$, we also get

$$||\lambda[\mathcal{I} + \lambda\mathcal{K}]^{-1}\mathcal{K}||_{2,\partial\Pi\times\mathbf{R}^+} \le 1. \tag{4.13}$$

We now rewrite (4.4) with $\lambda = (d+D)/2$ as $(1/2)\varphi + \lambda\mathcal{K}\varphi + \lambda\mathcal{K}\mathcal{G}_\lambda(\varphi) = \psi$, where $\mathcal{G}_\lambda(\varphi) = (\mathcal{G}(\varphi) - \lambda\varphi)/\lambda$, and so

$$\varphi + \mathcal{L}_\lambda \mathcal{G}_\lambda(\varphi) = \Psi, \tag{4.14}$$

where $\mathcal{L}_\lambda = 2\lambda(\mathcal{I} + 2\lambda\mathcal{K})^{-1}\mathcal{K}$, and $\Psi = 2(\mathcal{I} + 2\lambda\mathcal{K})^{-1}\psi$. One easily verifies that $\mathcal{G}_\lambda$ is a strong contraction, with contraction constant $(D-d)/(D+d)$. It follows that $\mathcal{L}_\lambda G_\lambda$ is a strong contraction. By the Banach contraction principle, whenever Ψ is in $L^2(\partial\Pi \times \mathbf{R}^+)$, we have that equation (4.14), and so also (4.4), has a unique solution φ, which depends Lipschitz continuous on Ψ in the $L^2(\partial\Pi\times\mathbf{R}^+)$-topology. More importantly, the same reasoning gives us the crucial property

$$(4.15)\quad ||[\mathcal{I}+\mathcal{L}_\lambda\mathcal{G}'_\lambda(\varphi)]^{-1}||_{2,\partial\Pi\times\mathbf{R}^+}\le \text{const},\qquad \text{for all measurable } \varphi$$

with const $=(d+D)/2d$. We now assume that $\mathcal{L}_\lambda$ is an integral operator with integral kernel $l(x-\xi,t-\tau)$, and $l\in L^1(\partial\Pi\times\mathbf{R}^+)$, and that

$$(4.16)\qquad \int_0^\infty\int_{-\infty}^\infty |l(x,t)|(1+|x|+t)^{1/4}\,dx\,dt<\infty.$$

Then we are all set up for the application of Theorem 3.2 and Theorems 2.10–2.14. Note that we have from (4.7)–(4.8),

$$(4.17)\qquad ||\psi||_{\infty,\partial\Pi\times\mathbf{R}^+}\le ||f||_{\infty,\Pi},$$

and so, if $l\in L^1(\partial\Pi\times\mathbf{R}^+)$,

$$(4.18)\qquad ||\Psi||_{\infty,\partial\Pi\times\mathbf{R}^+}\le ||l||_{1,\partial\Pi\times\mathbf{R}^+}||f||_{\infty,\Pi}.$$

4.19 Theorem. *Let g satisfy* (4.2). *Then the integral equation* (4.4) *has a unique solution $\varphi\in L^\infty(\partial\Pi\times\mathbf{R}^+)$ for every $f\in L^\infty(\Pi)$, and φ depends Lipschitz continuously on f in the L^∞-topology.*

Finally, the unique solvability of the initial boundary value problem (4.1) for bounded f now follows, and the solution is bounded, and

$$(4.20)\qquad ||u\Big|_{\partial\Pi\times\mathbf{R}^+}||_\infty\le C||f||_{\infty,\Pi}.$$

The Lipschitz continuity (4.3) follows likewise.

We finish by proving (4.16). Observing that

$$\int_\sigma^t\int_{-\infty}^\infty k(x-\xi,t-\tau)k(\xi-\eta,\tau-\sigma)\,d\xi\,d\tau = \frac{1}{2}\sqrt{\pi(t-\sigma)}k(x-\eta,t-\sigma),$$

we note that the integral kernel $k_n(x-\xi, t-\tau)$ of $\mathcal{K}^n$ satisfies

$$k_n(x,t) = \kappa_n(t)\exp(-x^2/4t) \tag{4.21}$$

for some function κ_n. From the Neumann series for $(\mathcal{I}+2\lambda\mathcal{K})^{-1}$, we then conclude that the kernel $l(x-\xi, t-\tau)$ of the operator $\mathcal{L}_\lambda$ satisfies

$$l(x,t) = m(t)\exp(-x^2/4t), \tag{4.22}$$

for some function m. Then, from the equation $\mathcal{L}_\lambda + 2\lambda\mathcal{K}\mathcal{L}_\lambda = \mathcal{K}$, we get that $m(t)$ satisfies the integral equation

$$m(t) + \frac{\lambda}{\sqrt{t}}\int_0^t \frac{\sqrt{\tau}m(\tau)}{\sqrt{\pi(t-\tau)}}\,d\tau = \frac{1}{4\pi t}, \quad t>0. \tag{4.23}$$

It follows that $\mu(t) = \sqrt{t}m(t)$ satisfies the equation

$$\mu(t) + \frac{\lambda}{\sqrt{\pi}}\int_0^t \frac{\mu(\tau)}{\sqrt{t-\tau}}\,d\tau = \frac{1}{4\pi\sqrt{t}}, \quad t>0. \tag{4.24}$$

As shown in [**15**], this shows that μ is locally integrable on $\mathbf{R}^+$, and that

$$\mu(t) = -\frac{1}{4\lambda\sqrt{\pi}}\frac{d}{dt}E_{\frac{1}{2}}(-\lambda t^{\frac{1}{2}}), \quad t>0, \tag{4.25}$$

where $E_{\frac{1}{2}}$ is the Mittag-Leffler function, see [**12**]. From the known asymptotic properties of $E_{\frac{1}{2}}$, it follows that $\mu(t) = \mathcal{O}(t^{-3/2})$ for $t\to\infty$, and so we get that

$$m(t) = \mathcal{O}(t^{-2}), \quad t\to\infty. \tag{4.26}$$

It is now an easy exercise to show that the above estimate combined with (4.22) implies that (4.16) holds.

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