JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 4, Number 1, Winter 1992

NONLINEAR INTEGRAL EQUATIONS ON THE HALF LINE

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Dedicated to Professor John A. Nohel in appreciation
for his many important contributions to the study
of integral equations.

ABSTRACT. This paper treats the existence and approximation of solutions of nonlinear integral equations defined on the half line $[0,\infty)$. Integral equations on $[0,\infty)$ are approximated by finite-section approximations, which reduce to integral equations on bounded intervals $[0,\beta]$. In the case when solutions are unique, the solutions x_β to the finite-section approximations converge uniformly on compact sets to the solution x of the integral equation on $[0,\infty)$, under natural hypotheses on its kernel. When solutions are not unique, the solution sets of the finite-section approximations converge in an appropriate sense to the solution set of the given integral equation. Integral equations of the type treated here include certain nonlinear Wiener-Hopf equations and integral equations of Hammerstein type. There are implications pertaining to global existence questions for nonlinear initial and boundary value problems for ordinary differential equations, in particular for a semi-conductor problem.

1. Introduction. We shall consider nonlinear integral equations on the half line $R^+ = [0,\infty)$ of the form

$$x(s) - \int_0^\infty k(s,t,x(t))\,dt = y(s), \tag{1.1}$$

and more general nonlinear operator equations. By hypothesis, x and y are bounded, continuous functions on R^+. Assumptions on k will be imposed later.

Finite-section approximations for (1.1) are given by

$$x_\beta(s) - \int_0^\beta k(s,t,x_\beta(t))\,dt = y(s), \tag{1.2}$$

for $\beta \geq 0$. Since (1.2) determines $x_\beta(s)$ for $s > \beta$ in terms of $x_\beta(t)$ for $t \in [0,\beta]$, (1.2) reduces to an integral equation on $[0,\beta]$.

Various discretization and linearization procedures, such as numerical integration and Newton's method, are available for the approximate solution of (1.2). This leads to double or even triple approximation schemes for the approximate solution of (1.1).

The setting for the analysis is the Banach space X^+ of bounded, continuous, real or complex functions x on R^+ with $||x|| = \sup |x(t)|$. The integral equations (1.1) and (1.2) will be special cases of more general operator equations on X^+:

$$(1.3) \qquad (I-K)x = y, \qquad (I-K_\beta)x_\beta = y.$$

The main concern of this paper is the convergence of solutions x_β to solutions x in (1.3) and, more particularly, in (1.1) and (1.2). Convergence in the norm of X^+ is uniform convergence on R^+. However, it is not generally true that solutions x_β in (1.2) converge uniformly on R^+ to solutions x in (1.1). The most that can be expected in general is uniform convergence on finite intervals. Strict convergence, described in Section 2, embodies this feature. Strict convergence was introduced in a locally compact topological space [**6**]. It was first applied to integral equations in [**5**]. A current application of strict convergence to integral equations is the paper [**7**] by Eggermont in this journal.

We shall identify basic continuity, compactness, and convergence properties of the operators K and K_β that imply the strict convergence of solutions of $(I-K_\beta)x_\beta = y$ to solutions of $(I-K)x = y$. There are also implications concerning the existence and uniqueness of solutions. Hypotheses on k in (1.1) and (1.2) will enable us to apply the general results to integral equations. The convergence results obtained below (see Section 4) are of the following type. The existence of solutions x_β to $(I-K)x_\beta = y$ for all β in R^+ (or just for β in an unbounded subset of R^+) implies the existence of a solution x to $(I-K)x = y$. Moreover, the solution x is the strict limit of x_β with β in a subset of R^+. When $I-K$ is one-to-one, x_β converges strictly to x with β in R^+. In the case of nonuniqueness, we compare the solution sets of $(I-K)x_\beta = y$ and $(I-K)x = y$ and obtain strict convergence of the set of approximate solutions to the solution set of $(I-K)x = y$.

The operator-theoretic structure we use is adapted from [**1**] by Anselone and Ansorge, which is concerned with nonlinear operator approximation theory principally in a Banach space setting with norm convergence. The applications to the integral equations (1.1) and (1.2) are motivated by [**2, 3, 4**], which deal with linear integral equations on the half line.

2. Strict convergence. Let $\{x_\beta\} = \{x_\beta : \beta \in R^+\}$ be an ordered family of functions in X^+ with the natural order induced by R^+. We are particularly interested in the behavior of x_β as $\beta \to \infty$. The following definitions made for $\{x_\beta\}$ carry over directly to $\{x_\beta : \beta \in R'\}$ for any unbounded subset $R' \subset R^+$.

Strict convergence is defined by:

$$x_\beta \xrightarrow{s} x \text{ as } \beta \to \infty$$
$$\text{if } \{x_\beta\} \text{ is bounded and } x_\beta \to x \text{ uniformly on finite intervals.}$$

Let $||x||_\alpha = \max |x(t)|$ for $t \in [0, \alpha]$. Then $x_\beta \xrightarrow{s} x$ if

$$||x_\beta|| \text{ is bounded uniformly}$$
$$\text{for } \beta \in R^+ \text{and } ||x_\beta - x||_\alpha \to 0 \text{ as } \beta \to \infty \quad \forall\, \alpha \in R^+.$$

If $x_\beta \xrightarrow{s} x$, then x is unique and $||x|| \leq \sup ||x_\beta||$.

Convergence in the norm of X^+ implies strict convergence but not conversely. For example, let

$$x(t) = 1, \qquad x_\beta(t) = e^{-t/\beta} \quad \text{for } \beta > 0.$$

Then $x_\beta \xrightarrow{s} x$ but $||x_\beta - x|| = 1$ for all β.

A *strict cluster point* of $\{x_\beta\}$ is a function $x \in X^+$ such that $x_\beta \xrightarrow{s} x$ with $\beta \in R'$ for some $R' \subset R^+$. The set of all strict cluster points of $\{x_\beta\}$ is denoted by $\{x_\beta\}^*$. We say that $\{x_\beta\}$ is *s-compact* if $\{x_\beta : \beta \in R'\}$ has a strict cluster point for any $R' \subset R^+$. This is analogous to the criterion for a sequence that every subsequence has a convergent subsequence. It is elementary that $\{x_\beta\}$ s-compact implies $\{x_\beta\}$ is bounded.

The following result is similar to a standard metric space proposition and it is proved in much the same way. Here $[x]$ denotes a singleton set.

Lemma 2.1. *$x_\beta \xrightarrow{s} x \Leftrightarrow \{x_\beta\}$ s-compact, $\{x_\beta\}^* = [x]$.*

Proof. The forward implication is immediate. For the converse, we shall prove that

$$\{x_\beta\}\ s\text{-compact}, \qquad x_\beta \overset{s}{\not\to} x \quad \Rightarrow \quad \{x_\beta\}^* \neq [x].$$

Since $x_\beta \overset{s}{\not\to} x$, there exist $\alpha \in R^+$, $\varepsilon > 0$, and $R' \subset R^+$ such that

$$||x_\beta - x||_\alpha > \varepsilon \quad \forall\, \beta \in R'.$$

Since $\{x_\beta\}$ is s-compact, $\{x_\beta : \beta \in R'\}$ has a strict cluster point $y \in X^+$. Then $y \in \{x_\beta\}^*$ and

$$||y - x||_\alpha \geq \varepsilon, \quad y \neq x, \quad \{x_\beta\}^* \neq [x]. \qquad \square$$

There is an analogue of the Arzélà-Ascoli theorem for strict convergence:

Lemma 2.2. *$\{x_\beta\}$ bounded, equicontinuous $\Rightarrow \{x_\beta\}$ s-compact.*

This is proved by applying the classical Arzélà-Ascoli theorem to successive intervals $[0, n]$, $n = 1, 2, 3, \dots$, and using a diagonal argument. See [**2, 5**].

A useful consequence of Lemmas 2.1 and 2.2 is

Lemma 2.3. *$\{x_\beta\}$ bounded equicontinuous, $x_\beta(t) \to x(t)$ for all $t \in R^+ \Rightarrow x_\beta \xrightarrow{s} x$.*

Definitions and results for strict convergence $x_\beta \xrightarrow{s} x$ carry over to sets $E, E_\beta \subset X^+$. In the comparison of $(I - K)x = y$ and $(I - K_\beta)x_\beta = y$, E and E_β will be sets of solutions in the absence of uniqueness. *Strict set convergence* is defined by:

$$E_\beta \xrightarrow{s} E \text{ as } \beta \to \infty \quad \text{if}$$

(1) $\bigcup_{\beta \in R^+} E_\beta$ is bounded,

(2) for all $\alpha \in R^+$ and for all $\varepsilon > 0$ there exists $\beta(\alpha, \varepsilon)$ such that $P_\alpha E_\beta$ lies in the ε-neighborhood of $P_\alpha E$ for $\beta \geq \beta(\alpha, \varepsilon)$,

where $P_\alpha : X^+ \to C[0,\alpha]$ is the restriction map. Strict set limits are not unique, for larger sets are also limits.

A *strict cluster point* of $\{E_\beta\}$ is an element $x \in X^+$ such that $x_\beta \xrightarrow{s} x$ for some $x_\beta \in E_\beta$ with $\beta \in R'$, and some $R' \subset R^+$. The set of strict cluster points of $\{E_\beta\}$ is denoted by $\{E_\beta\}^*$. We say that $\{E_\beta\}$ is *s-compact* if $\{E_\beta : \beta \in R'\}$ has a strict cluster point for any $R' \subset R^+$, in which case $\{E_\beta\}^* \neq \varnothing$.

Lemma 2.4. *$\{E_\beta\}$ s-compact, $\{E_\beta\}^* \subset E \Rightarrow E_\beta \xrightarrow{s} E \neq \varnothing$.*

The proof is almost the same as for the converse in Lemma 2.1.

3. Nonlinear operators on X^+. Let $K, K_\beta : X^+ \to X^+$ for $\beta \in R'$. The operator K is *s-continuous* if

$$x_\beta \xrightarrow{s} x \quad \Rightarrow \quad Kx_\beta \xrightarrow{s} Kx,$$

and K is *s-compact* if

$$\{x_\beta\} \quad \text{bounded} \quad \Rightarrow \quad \{Kx_\beta\} \ s\text{-compact}.$$

Similarly, $\{K_\beta\}$ is *asymptotically s-compact* if

$$\{x_\beta\} \quad \text{bounded} \quad \Rightarrow \quad \{K_\beta x_\beta\} \ s\text{-compact}.$$

Strict convergence $K_\beta \xrightarrow{s} K$ is defined by

$$K_\beta x \xrightarrow{s} Kx \quad \forall\, x \in X^+.$$

A stronger property, *continuous strict convergence* $K_\beta \xrightarrow{cs} K$, is defined by

$$x_\beta \xrightarrow{s} x \quad \Rightarrow \quad K_\beta x_\beta \xrightarrow{s} Kx.$$

Now, let K and K_β, $\beta \in R^+$, be the integral operators on X^+ defined by

$$Kx(s) = \int_0^\infty k(s,t,x(t))\,dt, \tag{3.1}$$

$$K_\beta x(s) = \int_0^\beta k(s,t,x(t))\,dt, \tag{3.2}$$

where the kernel $k(s,t,u)$ satisfies the following hypotheses.

H1. $k(s,t,u)$ is continuous in u.

H2. $k(s,t,u)$ is measurable in t.

H3. $\Phi_b = \sup_{s\in R^+} \int_0^\infty \sup_{|u|\le b} |k(s,t,u)|\,dt < \infty$ for each $b > 0$.

H4. $\Gamma_b(s',s) = \int_0^\infty \sup_{|u|\le b} |k(s',t,u) - k(s,t,u)|\,dt \to 0$ as $s' \to s$ for each $s \in R^+$.

We will show shortly that the integral operators K and K_β in (3.1) and (3.2) have the strict continuity, compactness, and convergence properties described in the previous paragraph. First, we consider some examples.

Hammerstein integral operators provide important special cases of the general nonlinear integral operators above. In the Hammerstein case $k(s,t,u) = l(s,t)f(t,u)$ with $l(s,t) = l(t,s)$, so that

$$\begin{aligned} Kx(s) &= \int_0^\infty l(s,t)f(t,x(t))\,dt, \\ K_\beta x(s) &= \int_0^\beta l(s,t)f(t,x(t))\,dt. \end{aligned}$$

The symmetry condition $l(s,t) = l(t,s)$ is not required in what follows. It is readily verified that the kernel $k(s,t,u) = l(s,t)f(t,u)$ satisfies H1–H4 when the following conditions hold:

A. $l(s,t)$ is measurable in t.

B. $\sup_{s\in R^+} \int_0^\infty |l(s,t)|\,dt < \infty$.

C. $\int_0^\infty |l(s',t) - l(s,t)|\,dt \to 0$ as $s' \to s$ for all $s \in R^+$.

D. $f(t,u)$ is measurable in t for each u, continuous in u for each t, and bounded for $t \in R^+$ uniformly for u in any bounded set.

Specializing further, if $g \in L_1(R)$, then the translation kernel $l(s,t) = g(s-t)$ satisfies A–C. Consequently, the kernel $k(s,t,u) = g(s-t)f(t,u)$ will satisfy H1–H4 provided f satisfies D. The special choice $g(z) = e^{-a|z|}$ with $a > 0$ yields the Picard kernel for $l(s,t)$. Further choices for $l(s,t)$ which satisfy A–C include

$$l(s,t) = \frac{1}{s^2+t^2+1},$$
$$l(s,t) = \frac{\sin(s+t)}{s^2+t^2+1},$$
$$l(s,t) = e^{-t}\frac{s}{s+t+1}.$$

Functions which satisfy D include $f(t,u) = u^2$ and $f(t,u) = e^{cu}$, with c any constant. Thus, our results apply to integral operators of the form

$$Kx(s) = \int_0^\infty l(s,t)x(t)^2\,dt,$$
$$Kx(s) = \int_0^\infty l(s,t)e^{cx(t)}\,dt,$$

where $l(s,t)$ satisfies A, B and C.

Let K and K_β be given by (3.1) and (3.2). Assume $k(s,t,u)$ satisfies H1–H4. The following lemmas establish key relations among K and the K_β.

Lemma 3.1. (a) $K : X^+ \to X^+$.

(b) $\{Kx : ||x|| \le b\}$ *is bounded, equicontinuous for all* $b > 0$.

(c) K *is s-compact.*

Proof. From H3 and H4,

$$|Kx(s)| \le \Phi_b \quad \text{for } ||x|| \le b,$$
$$|Kx(s') - Kx(s)| \le \Gamma_b(s', s) \quad \text{for } ||x|| \le b,$$

which imply (a) and (b). Then Lemma 2.2 yields (c). □

Virtually the same reasoning proves

Lemma 3.2. (a) $K_\beta : X^+ \to X^+$ *for all* $\beta \in R^+$.

(b) $\{K_\beta x : ||x|| \le b, \beta \in R^+\}$ *is bounded, equicontinuous for all* $b > 0$.

(c) $\{K_\beta : \beta \in R^+\}$ *is asymptotically s-compact.*

Lemma 3.3. $K_\beta \xrightarrow{s} K$. *Thus,*

$$K_\beta x \xrightarrow{s} Kx \quad \text{as } \beta \to \infty \quad \forall\, x \in X^+.$$

Moreover, for any $\gamma \in R^+$ *and any* $b > 0$,

$$||K_\beta x - Kx||_\gamma \to 0 \quad \text{as } \beta \to \infty, \quad \text{uniformly for } ||x|| \leq b.$$

Proof. For any $x \in X^+$,

$$Kx(s) - K_\beta x(s) = \int_\beta^\infty k(s,t,x(t))\,dt,$$

$$|Kx(s) - K_\beta x(s)| \leq \int_\beta^\infty \sup_{|u|\leq b} |k(s,t,u)|\,dt \text{ for } ||x|| \leq b.$$

In view of H3,

$$K_\beta x(s) \to Kx(s) \quad \text{as } \beta \to \infty, \quad \text{uniformly for } ||x|| \leq b.$$

By Lemma 3.2, $\{K_\beta x : ||x|| \leq b, \beta \in R^+\}$ is bounded and equicontinuous. By Lemma 2.3, the conclusions of the lemma follow. □

Lemma 3.4. K_β *is s-continuous for each* $\beta \in R^+$. *Thus,*

$$x_\alpha \xrightarrow{s} x \quad \Rightarrow \quad K_\beta x_\alpha \xrightarrow{s} K_\beta x \quad \text{as } \alpha \to \infty.$$

Proof. Assume $x_\alpha \xrightarrow{s} x$. Then $||x_\alpha|| \leq b$ for some $b < \infty$ and all α. For each $s \in R^+$,

$$K_\beta x_\alpha(s) - K_\beta x(s) = \int_0^\beta [k(s,t,x_\alpha(t)) - k(s,t,x(t))]\,dt.$$

By H1 and H2, the integrand is measurable in t and pointwise convergent to 0 as $\alpha \to \infty$. It is also bounded by

$$2 \sup_{|u|\leq b} |k(s,t,u)|.$$

Now H3 and the Lebesgue dominated convergence theorem yield

$$K_\beta x_\alpha(s) \to K_\beta x(s) \text{ as } \alpha \to \infty \quad \text{for each } s \in R^+.$$

Lemma 3.2 implies that $\{K_\beta x_\alpha : \alpha \in R^+\}$ is bounded and equicontinuous. By Lemma 2.3, $K_\beta x_\alpha \xrightarrow{s} K_\beta x$ as $\alpha \to \infty$, so that K_β is s-continuous. □

Lemma 3.5. *K is s-continuous. Thus,*

$$x_\alpha \xrightarrow{s} x \quad \Rightarrow \quad Kx_\alpha \xrightarrow{s} Kx \quad \text{as } \alpha \to \infty.$$

Proof. This can be proved by the same argument used for Lemma 3.4. The following proof is based on different ideas. Similar reasoning will be used to establish Lemma 3.6. Assume $x_\alpha \xrightarrow{s} x$. For any $\alpha, \beta \in R^+$,

$$Kx_\alpha - Kx = (Kx_\alpha - K_\beta x_\alpha) + (K_\beta x_\alpha - K_\beta x) + (K_\beta x - Kx).$$

Fix $\gamma \in R^+$ and $\varepsilon > 0$. By Lemma 3.3, there exists β such that

$$||K_\beta x - Kx||_\gamma < \varepsilon, \qquad ||K_\beta x_\alpha - Kx_\alpha||_\gamma < \varepsilon \quad \text{for } \alpha \in R^+.$$

Now β is fixed. By Lemma 3.4, there exists $\alpha_0 \in R^+$ such that

$$||K_\beta x_\alpha - K_\beta x||_\gamma < \varepsilon \quad \text{for } \alpha \geq \alpha_0.$$

It follows that

$$||Kx_\alpha - Kx||_\gamma < 3\varepsilon \quad \text{for } \alpha \geq \alpha_0,$$

so that $Kx_\alpha \xrightarrow{s} Kx$ as $\alpha \to \infty$ and K is s-continuous. □

Lemma 3.6. *$K_\beta \xrightarrow{cs} K$. Thus,*

$$x_\beta \xrightarrow{s} x \quad \Rightarrow \quad K_\beta x_\beta \xrightarrow{s} Kx.$$

Proof. Assume $x_\beta \xrightarrow{s} x$. For any $\alpha, \beta \in R^+$,

$$\begin{aligned} K_\beta x_\beta - Kx &= (K_\beta x_\beta - Kx_\beta) + (Kx_\beta - K_\alpha x_\beta) \\ &\quad + (K_\alpha x_\beta - K_\alpha x) + (K_\alpha x - Kx). \end{aligned}$$

Fix $\gamma \in R^+$ and $\varepsilon > 0$. By Lemma 3.3, there exists $\alpha \in R^+$ such that

$$\begin{aligned} ||K_\alpha x - Kx||_\gamma &< \varepsilon, \\ ||K_\beta x_\beta - Kx_\beta||_\gamma &< \varepsilon \quad \text{for } \beta \geq \alpha, \\ ||Kx_\beta - K_\alpha x_\beta||_\gamma &< \varepsilon \quad \text{for } \beta \in R^+. \end{aligned}$$

Now α is fixed. By Lemma 3.4, there exits $\beta_0 \geq \alpha$ such that

$$||K_\alpha x_\beta - K_\alpha x||_\gamma < \varepsilon \quad \text{for } \beta \geq \beta_0.$$

It follows that

$$||K_\beta x_\beta - Kx||_\gamma < 4\varepsilon \quad \text{for } \beta \geq \beta_0.$$

Therefore, $K_\beta x_\beta \xrightarrow{s} Kx$ and $K_\beta \xrightarrow{cs} K$. □

The principal results of the preceding lemmas are summarized as follows.

Theorem 3.7. *Let K and K_β, $\beta \in R^+$ be the nonlinear integral operators in* (3.1) *and* (3.2), *where the kernel $k(s,t,u)$ satisfies* H1–H4. *Then*

(a) *K is s-compact.*

(b) *$\{K_\beta\}$ is asymptotically s-compact.*

(c) *$K_\beta \xrightarrow{cs} K$.*

4. Convergence of approximate solutions. Let K, $K_\beta : X^+ \to X^+$ for $\beta \in R^+$. We shall compare solutions of equations

$$(I - K)x = y, \qquad (I - K_\beta)x_\beta = y,$$

where $\{K_\beta\}$ is asymptotically s-compact and $K_\beta \xrightarrow{cs} K$. Special cases are the integral operators K and K_β in (3.1) and (3.2) with the hypotheses H1–H4 on $k(s,t,u)$.

Theorem 4.1. *Assume $\{K_\beta\}$ asymptotically s-compact and $K_\beta \xrightarrow{cs} K$. Fix $y \in X^+$. Assume there exists $x_\beta \in X^+$ for $\beta \in R'$ such that*

$$(I - K_\beta)x_\beta = y \quad \textit{and} \quad \{x_\beta : \beta \in R'\} \textit{ is bounded.}$$

Then there exist $R'' \subset R'$ and $x \in X^+$ such that

$$x_\beta \xrightarrow{s} x \quad \text{with } \beta \in R'', \qquad (I-K)x = y.$$

If x is the unique solution of $(I-K)x = y$, then

$$x_\beta \xrightarrow{s} x \quad \text{with } \beta \in R'.$$

Proof. Since $\{K_\beta\}$ is asymptotically s-compact and $x_\beta = K_\beta x_\beta + y$, $\{x_\beta\}$ is s-compact. So there exist $x \in X^+$ and $R'' \subset R'$ such that

$$x_\beta \xrightarrow{s} x \quad \text{with } \beta \in R''.$$

Now $K_\beta \xrightarrow{cs} K$ implies that $K_\beta x_\beta \xrightarrow{s} Kx$ with $\beta \in R''$. Hence,

$$y = x_\beta - K_\beta x_\beta \xrightarrow{s} x - Kx \quad \text{with } \beta \in R'', \qquad (I-K)x = y.$$

Finally, if x is the unique solution of $(I-K)x = y$, then Lemma 2.1 gives

$$x_\beta \xrightarrow{s} x \quad \text{with } \beta \in R'. \qquad \square$$

The next theorem extends Theorem 4.1 to sets of solutions of $(I-K)x = y$ and $(I-K_\beta)x_\beta = y$ in the absence of uniqueness. The proof involves the same arguments.

Theorem 4.2. *Assume $\{K_\beta\}$ asymptotically s-compact and $K_\beta \xrightarrow{cs} K$. Fix $y \in X^+$. Let*

$$E = \{x \in X^+ : (I-K)x = y,\ ||x|| \le b\},$$
$$E_\beta = \{x_\beta \in X^+ : (I-K_\beta)x_\beta = y,\ ||x_\beta|| \le b\}.$$

Assume $E_\beta \neq \varnothing$ for $\beta \in R'$. Then $E \neq \varnothing$. Moreover,

$$\{E_\beta\} \text{ is } s\text{-compact}, \ \{E_\beta\}^* \subset E, \text{ and } E_\beta \xrightarrow{s} E.$$

Proof. Let $x_\beta \in E_\beta$ for $\beta \in R''$. Then $x_\beta = K_\beta x_\beta + y$. Since $\{K_\beta\}$ is asymptotically s-compact, $\{x_\beta\}$ is s-compact. Therefore, $\{E_\beta\}$ is

s-compact. Let $x \in \{E_\beta\}^*$. Then there exist $R'' \subset R'$ and $x_\beta \in E_\beta$ for $\beta \in R''$ such that $x_\beta \xrightarrow{s} x$ with $\beta \in R''$. Hence,

$$y = x_\beta - K_\beta x_\beta \to x - Kx \quad \text{with } \beta \in R'', \qquad (I-K)x = y.$$

Thus, $x \in E$ and $\{E_\beta\}^* \subset E$. Finally, Lemma 2.4 gives $E_\beta \xrightarrow{s} E \neq \varnothing$. □

5. A semiconductor example. Integral equations of the type treated above arise in a variety of physical applications and are closely related to the global solvability of initial and/or boundary value problems for ordinary differential equations. For example, the analysis of semiconductor devices leads to a problem in which Poisson's equation must be solved in two adjacent domains, one of which is unbounded, subject to suitable continuity and jump relations along the common boundary. When specialized to one spatial dimension [**8**], a typical problem can be reduced to

$$z''(t) = g(t, z(t)), \quad 0 \le t < \infty, \tag{5.1}$$

$$z'(0) - \alpha z(0) = r, \quad \alpha > 0,\ r \in R, \tag{5.2}$$

$$\exists \lim_{t\to\infty} z(t), \tag{5.3}$$

where, on physical grounds, the functions $g(t,z)$ used in practical models satisfies regularity conditions more restrictive than D of Section 3. Here, $z(t)$ is an electrical potential and the principal mathematical questions concern the existence of a solution and its numerical evaluation. In realistic, physical models, $g(t,z)$ is such that *a priori* any bounded solution $z(t)$ to (5.1) and (5.2) automatically has a limit at infinity. Thus, we are led to the problem

$$\begin{aligned} &z''(t) = g(t,z), \quad 0 \le t < \infty, \\ &z'(0) - \alpha z(0) = r, \quad \alpha > 0,\ r \in R, \\ &z(t) \text{ bounded on } R^+. \end{aligned}$$

The change of dependent variable $z(t) = x(t) + ae^{-t}$ with $a = -r/(1+\alpha)$ reduces this problem to the more convenient form

$$\left\{ \begin{aligned} &-x''(t) + x(t) = f(t, x(t)) + h(t), \quad 0 \le t < \infty, \\ &x'(0) - \alpha x(0) = 0, \\ &x \in X^+, \end{aligned} \right\} \tag{5.4}$$

where $h(t) = ae^{-t}$ and $f(t,u) = u - g(t, ae^{-t} + u)$ satisfies D because g does. An elementary calculation confirms that the linear differential operator defined by $Lx = -x'' + x$ and the boundary conditions $x'(0) - \alpha x(0) = 0$, $x \in X^+$ has the Green's function

$$l(s,t) = \begin{cases} \frac{1}{2}[e^{t-s} + \gamma e^{-t-s}], & 0 \le t \le s < \infty, \\ \frac{1}{2}[e^{s-t} + \gamma e^{-s-t}], & 0 \le s \le t < \infty, \end{cases}$$

where $\gamma = (1-\alpha)/(1+\alpha)$. Thus, the boundary value problem (5.4) is equivalent to the Hammerstein integral equation

$$x(s) - \int_0^\infty l(s,t) f(t, x(t))\, dt = y(s), \tag{5.5}$$

where $y(s) = \int_0^\infty l(s,t)h(t)\, dt$. It is routine to check that $l(s,t)$ satisfies A, B, and C of Section 3. Consequently, the results in Sections 3 and 4 apply to (5.5) and its finite section approximations

$$x_\beta(s) - \int_0^\beta l(s,t) f(t, x_\beta(t))\, dt = y(s). \tag{5.6}$$

Existence results for Hammerstein equations on bounded domains [**9, 10**] yield solutions $x_\beta(s)$ to (5.6). Then Theorems 4.1 and 4.2 imply that (5.5) has a solution $x(t)$ and, in the case of uniqueness, the strict convergence of x_β to x. We shall not formulate more precise results here. It is clear, however, that the results in Sections 3 and 4 have fruitful applications to the *global* existence of solutions of nonlinear differential equations, in areas other than semiconductor devices.

REFERENCES

1. P.M. Anselone and R. Ansorge, *Compactness principles in nonlinear operator approximation theory*, Num. Func. Anal. Optim. **1** (1979), 598–618.

2. P.M. Anselone and I.H. Sloan, *Integral equations on the half line*, J. Integral Equations **9** (1986), 3–23.

3. ———, *Numerical solutions of integral equations on the half line* II. *The Wiener-Hopf case*, J. Integral Equations Appl. **1** (1988), 203–223.

4. ———, *Spectral approximations for Wiener-Hopf operators*, J. Integral Equations Appl. **2** (1990), 239–264.

5. K.E. Atkinson, *The numerical solution of integral equations on the half-line*, SIAM J. Numer. Anal. **6** (1969), 375–397.

6. R.C. Buck, *Bounded continuous functions on a locally compact space*, Michigan Math. J. **5** (1985), 95–104.

7. P.P.B. Eggermont, *On noncompact Hammerstein integral equations and a nonlinear boundary value problem for the heat equation*, J. Integral Equations Appl. **4** (1992), 47–68.

8. A. Granas, R.B. Guenther, J.W. Lee and D. O'Regan, *Boundary value problems on infinite intervals and semiconductor devices*, J. Math. Anal. Appl. **116** (1986), 335–348.

9. T.L. Saaty, *Modern nonlinear equations*, McGraw-Hill Book Co., New York, 1967.

10. F.G. Tricomi, *Integral equations*, Interscience Publishers, Inc., New York, 1957.

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