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ALGEBRAIC TYPE OF SOLUTIONS FOR SINGULAR INTEGRAL EQUATIONS OF THE FORM $(S+T)x = x_0$ IN BANACH SPACES

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ABSTRACT. The formulae of the algebraic type for the solutions of the singular integral equations with positive indices, by applying the analytic formulae of Buraczewski and the integral model of Verma for the determinant systems, are obtained.

1. Introduction. Buraczewski [**1**] generalized the Fredholm formulae of Sikorski [**10**] to the case of Fredholm operators with nonvanishing indices in Banach spaces. Sikorski [**10**], first of all, gave an integral model of the determinants in Banach spaces. Recently, Verma [**14–16**] generalized the integral model of Sikorski and applied the determinant formulae for the solutions of the singular integral equations in the spaces of functions, satisfying the Hölder condition with a fixed exponent.

The main aim of this paper is to apply the integral model [**16**, Theorem 1.2] and a theorem of Buraczewski [**1**, Theorem (x)] in obtaining the general solutions of the singular integral equations of the form $(S+T)x = x_0$ in Banach spaces. We first introduce the necessary definitions and notations. Let L be a closed curve in the complex plane. Suppose that L does not intersect itself and is rectifiable. By $H^\mu(L)$, we shall mean the space of all those functions satisfying the Hölder condition with exponent μ, $0 < \mu < 1$. If $0 < \alpha \leq \mu$, then $H^\alpha(L) \supset H^\mu(L)$. The conjugate space of $H^\alpha(L)$ is the same. The functions x, y, e, v, ξ, η (with indices if necessary) will always represent the elements of $H^\alpha(L)$. For more details, see [**8**].

$H^\mu(L)$ is a Banach space of all those functions x satisfying the Hölder condition with exponent μ on L under the *norm*

$$\| x \| = \max_{t\in L} |x(t)| + \sup_{t_1,t_2\in L} \frac{|x(t_2) - x(t_1)|}{|t_2 - t_1|^\mu}.$$

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For a small positive number ε, and for a fixed point $t \in L$, set $L_\varepsilon = \{\tau \in L : |\tau - t| \geq \varepsilon\}$. The *Cauchy principal value* (abbreviated by p.v.) of the integral is defined by

$$\text{p.v.} \int_L \frac{x(\tau)}{\tau - t}\, d\tau = \lim_{\varepsilon \to 0} \int_{L_\varepsilon} \frac{x(\tau)}{\tau - t}\, d\tau, \quad t \in L.$$

For the following section, we introduce the operator $\mathcal{J}$, defined by

$$(\mathcal{J}x)(t) = \frac{1}{\pi i}\, \text{p.v.} \int_L \frac{x(\tau)}{\tau - t}\, d\tau. \tag{1.1}$$

The operator $\mathcal{J}$ is defined in the space $H^\alpha(L)$ and has the property $\mathcal{J}^2 = I$.

Let $a, b \in H^\mu(L)$, and $a^2(t) - b^2(t) \neq 0$, $t \in L$. Then it is well known that the operator S, given by

$$(Sx)(t) = a(t)x(t) + \frac{b(t)}{\pi i}\, \text{p.v.} \int_L \frac{x(\tau)}{\tau - t}\, d\tau \tag{1.2}$$

is well defined on $H^\alpha(L)$ for $\alpha \leq \mu$.

The integer $r(A) = \min(v(A), v'(A))$ is called the *order* of an operator A, where $v(A) = \dim(N(A))$ and $v'(A) = \text{codim}(R(A))$. And the integer $d(A) = v(A) - v'(A)$ is called the *index* of operator A. An operator B is said to be a *quasi-inverse* of an operator A if

$$ABA = A \quad \text{and} \quad BAB = B.$$

Thus, the operator S can be written in compact form, $S = aI + bJ$, where I is the identity operator on $H^\alpha(L)$. The operator $S = aI + bJ$ is called a singular integral operator which is a Fredholm operator with a nonvanishing index.

In our present work, we restrict ourselves to the case when order $r(S) = 0$ and index $d(S) = d > 0$.

From now on, the letter T shall represent two objects: the kernel and the operator given by the kernel. Let $T(t, \tau)$ be a kernel on $L \times L$ of the Hölder class in both variables of the form

$$T(t, \tau) = \frac{T(t, \tau) - T(t, t)}{\tau - t}$$

such that the integral operator T is defined by

$$(Tx)(t) = \frac{1}{\pi i}\int_L T(t,\tau)x(\tau)\,d\tau \tag{1.3}$$

in $H^\alpha(L)$ with $\alpha < \mu/2$. Then T is compact on the space $H^\alpha(L)$

Now we relate the following singular integral equation [**9**] to the singular integral operator $S+T$:

$$a(t)x(t) + \frac{b(t)}{\pi i}\,\text{p.v.}\int_L \frac{x(\tau)}{\tau - t}\,d\tau + \frac{1}{\pi i}\int_L T(t,\tau)x(\tau)\,d\tau = x_0(t), \tag{1.4}$$

or, more briefly,

$$(S+T)x = x_0. \tag{1.5}$$

The adjoint equation of (1.5) can be written as

$$v(S+T) = v_0 \quad \text{for } v, v_0 \in H^\alpha(L). \tag{1.6}$$

Next, we give an integral quasi-inverse of $S+T$ when its order, $r(S+T)$, is zero. The operator $S+T$ is a generalized Fredholm operator since the following conditions are satisfied:

$$(aI + bJ + T)(aI - Jb) = (I + T_1)(a^2 - b^2) \tag{1.7}$$

and

$$(aI - Jb)(aI + bJ + T) = (a^2 - b^2)(I + T_2), \tag{1.8}$$

where $aI + bJ = S$ and T_1 and T_2 are compact operators.

Thus,

(i) if $T_1 = 0$, then index $d(S+T) = d \geq 0$, and

(ii) if $T_2 = 0$, then index $d(S+T) = d \leq 0$.

It follows from this that the operator

$$\tilde{B} = \frac{1}{(a^2 - b^2)}(aI - Jb), \tag{1.9}$$

or

$$(1.10)\quad (\tilde{B}x)(t) = \frac{1}{a^2(t) - b^2(t)}\left\{a(t)x(t) - \frac{1}{\pi i}\,\text{p.v.}\int_L \frac{b(\tau)}{\tau - t}x(\tau)\,d\tau\right\}$$

is an *integral quasi-inverse* of $S + T$ when $r(S+T) = 0$.

We conclude this section by giving an example of a singular integral equation.

Example. We consider a real singular integral equation

$$(1.11)\quad a(s)x(s) + \frac{1}{2\pi}\,\text{p.v.}\int_0^{2\pi} T(s,\sigma)\cot\left(\frac{\sigma - s}{2}\right)x(\sigma)\,d\sigma = y(s)$$

with Hilbert kernel, where $a(s)$ and $T(s,\sigma)$ are real, 2π-periodic functions of s and σ, which satisfy Hölder condition with exponent μ.

If we set $t = e^{is}$, $\tau = e^{i\sigma}$, $\tilde{T}(t,\tau) = T(-i\ln t, -i\ln\tau)$, $\tilde{a}(t) = a(-i\ln t)$, $\tilde{y}(t) = y(-i\ln t)$, $\tilde{x}(t) = x(-i\ln t)$, then $\cot((\sigma - s)/2) = i((2\tau)/(\tau - t) - 1)$, $d\sigma = (1/i)(dt/\tau)$, and the equation (1.11) is transformed into the equation

$$(1.12)\quad \tilde{a}(t)\tilde{x}(t) + \frac{1}{\pi}\,\text{p.v.}\int_L \frac{\tilde{T}(t,\tau)}{\tau - t}\tilde{x}(\tau)\,d\tau - \frac{1}{2\pi}\int_L \tilde{T}(t,\tau)\tilde{x}(\tau)\frac{d\tau}{\tau} = \tilde{y}(t),$$

where L is the unit circle in the complex plane. The operator

$$-\frac{1}{2\pi}\int_L \tilde{T}(t,\tau)\tilde{x}(\tau)\frac{d\tau}{\tau}$$

is compact in space $H^\alpha(L)$ with $\alpha \le \mu$.

2. Formulae of algebraic type. Let us define a bilinear function on $H^\alpha(L) \times H^\alpha(L)$ by the formula

$$(2.1)\quad v \cdot x = \int_L v(t) \cdot x(t)\,dt.$$

Let $T(s,t)$ be a function on $L \times L$ such that

$$(2.2)\quad F(I) = \int_L T(s,s)\,ds$$

exists, and such that

$$F(H) = \int_L \int_L T(t,s)H(s,t)\,ds\,dt, \tag{2.3}$$

for every integral operator H defined by

$$H(s,t) = x(s) \cdot v(t). \tag{2.4}$$

Then a quasi-nucleus F is said to be an *integral quasi-nucleus*, and the operator T is an integral operator determined by the kernel $T(s,t)$

$$vTx = \int_L \int_L v(s)T(s,t)x(t)\,ds\,dt, \tag{2.5}$$

$$(vT)(t) = \int_L v(s)T(s,t)\,ds, \tag{2.6}$$

and

$$(Tx)(s) = \int_L T(s,t)x(t)\,dt. \tag{2.7}$$

In order to write the integral formulae for the subdeterminants of an integral quasinucleus F, we introduce a formal expression $\delta(s,t)$, which is a substitute for the *Dirac delta distribution*, namely, we define

$$\int_L \delta(s,t)x(t)\,dt = x(s), \tag{2.8}$$

$$\int_L v(s)\delta(s,t)\,ds = v(t), \tag{2.9}$$

$$\int_L \int_L \delta(s,t)T(t,s)\,ds\,dt = \int_L T(s,s)\,ds = F(I), \tag{2.10}$$

and

$$\int_L \int_v v(s)\delta(s,t)x(t)\,ds\,dt = v \cdot x. \tag{2.11}$$

Now we recall the following result which is crucial to our solutions.

Lemma [**16**]. *Let $S = aI + bJ$ be a singular integral operator of order $r(S) = 0$ and index $d(S) = d > 0$, let Q be an integral quasi-inverse of S, and let $e_1, \ldots, e_d$ be all linearly independent solutions of $Sx = 0$. Then, for any integral quasi-nucleus F, we have the following determinant system, $\{D_n\}$ defined by the formula* (2.12) *for the operator $S + T$, which does not depend on the choice of an integral quasi-inverse Q*:

(2.12)

$$D_n(F)\begin{pmatrix} v_1, & \ldots, & v_{n+d} \\ x_1, & \ldots, & x_n \end{pmatrix}$$

$$= \int_L \cdots \int_L \mathcal{J}_{n,m}\begin{pmatrix} s_1, & \ldots, & s_{n+d} \\ t_1, & \ldots, & t_n \end{pmatrix}$$

$$\cdot\, v_1(s_1) \cdots v_{n+d}(s_{n+d}) \cdot x_1(t_1) \cdots x_n(t_n) \cdot e_1(u_1) \cdots e_d(u_d)$$

$$\cdot\, ds_1 \ldots ds_{n+d} \cdot dt_1 \ldots dt_n \cdot du_1 \ldots du_d,$$

where

$$
\begin{aligned}
&\mathcal{J}_{n,m}\begin{pmatrix} s_1, & \ldots, & s_{n+d} \\ t_1, & \ldots, & t_n \end{pmatrix} = \sum_{m=0}^{\infty} (1/m!) \int_L \cdots \int_L \\
&\quad \cdot \begin{bmatrix}
Q(s_1,t_1) & \cdots & Q(s_1,t_n) & QT(s_1,r_1) & \cdots & QT(s_1,r_m) & \delta(s_1,u_1) & \cdots & \delta(s_1,u_d) \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
Q(s_{n+d},t_1) & \cdots & Q(s_{n+d},t_n) & QT(s_{n+d},r_1) & \cdots & QT(s_{n+d},r_m) & \delta(s_{n+d},u_1) & \cdots & \delta(s_{n+d},u_d) \\
Q(r_1,t_1) & \cdots & Q(r_1,t_n) & QT(r_1,r_1) & \cdots & QT(r_1,r_m) & \delta(r_1,u_1) & \cdots & \delta(r_1,u_d) \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
Q(r_m,t_1) & \cdots & Q(r_m,t_n) & QT(r_m,r_1) & \cdots & QT(r_m,r_m) & \delta(r_m,u_1) & \cdots & \delta(r_m,u_d)
\end{bmatrix} \\
&\qquad \cdot dr_1 \ldots dr_m,
\end{aligned}
$$

with $Q(s,t), QT(s,t), T(s,t)$ and $\delta(s,t)$, respectively, the kernels of the operators Q, QT, T and δ.

Let us define the series

$$\sum_{m=0}^{\infty} D_{n,m}(F) = \int_L \cdots \int_L \mathcal{J}_{n,m}\begin{pmatrix} s_1, & \cdots, & s_{n+d} \\ t_1, & \cdots, & t_n \end{pmatrix} \cdot v_1(s_1)\cdots v_{n+d}(s_{n+d}) \cdot x_1(t_1)\ldots x_n(t_n) \cdot e_1(u_1)\ldots e_d(u_d) \cdot ds_1 \ldots ds_{n+d} \cdot dt_1 \ldots dt_n \cdot du_1 \ldots du_d.$$

The series $\sum_{m=0}^{\infty} D_{n,m}(F)$ converges to $D_n(F)$ in the space $\aleph$ of all bounded bi-skew symmetric $(2n+d)$-linear functionals with respect to norm topology in $\aleph$ given by

$$\| D \| = \left(\sup \left| D_n \begin{pmatrix} v_1, & \cdots, & v_{n+d} \\ x_1, & \cdots, & x_n \end{pmatrix} \right| : ||v_i|| = 1, ||x_i|| = 1 \right).$$

We are just about ready to write the solutions of the singular integral equations $(S+T)x = x_0$ and $v(S+T) = v_0$.

Theorem 2.1. *Let the singular integral operator $S+T$ be of order $r(S+T) = r$ and index $d(S+T) = d > 0$. Let $\{D_n\}$, defined by (2.12), be a determinant system for the operator $S+T$, and let $\eta_1, \ldots, \eta_{r+d}$ and $y_1, \ldots, y_r$ be fixed elements such that*

$$\tilde{D}_r = D_r \begin{pmatrix} \eta_1, & \cdots, & \eta_{r+d} \\ y_r, & \cdots, & y_r \end{pmatrix} \neq 0.$$

(i) *Then there exist elements $z_i \in H^\alpha(L)$, $i = 1, \ldots, r+d$, and $\xi_i \in H^\alpha(L)$, $i = 1, \ldots, r$, such that*

(2.13)
$$v z_i = [1/\tilde{D}_r] D_r \begin{pmatrix} \eta_1, & \cdots, & \eta_{i-1}, & v, & \eta_{i+1}, & \cdots, & \eta_{r+d} \\ y_1, & \cdots & \cdots & \cdots & \cdots & \cdots, & y_r \end{pmatrix},$$

for every $v \in H^\alpha(L)$, and

(2.14)
$$\xi_i x = [1/\tilde{D}_r] D_r \begin{pmatrix} \eta_1, & \cdots & \cdots & \cdots & \cdots & \cdots, & \eta_{r+d} \\ y_1, & \cdots, & y_{i-1}, & x, & y_{i+1}, & \cdots, & y_r \end{pmatrix}$$

for every $x \in H^\alpha(L)$.

The elements $z_1, \dots, z_{r+d}$ *and* $\xi_1, \dots, \xi_r$ *are linearly independent solutions of the equations* $(S+T)x = 0$ *and* $v(S+T) = 0$, *respectively.*

(ii) *Then the operator* B, *defined by*

$$vBx = [1/\tilde{D}_r]D_{r+1}\begin{pmatrix} v, & \eta_1, & \cdots, & \eta_{r+d} \\ x, & y_1, & \cdots, & y_r \end{pmatrix}, \tag{2.15}$$

is a quasi-inverse of the operator $S+T$.

(iii) *Then the equation* $(S+T)x = x_0$ *has a solution* x *if and only if* $\xi_i x_0 = 0$ *for* $i = 1, \dots, r$, *and the adjoint equation* $v(S+T) = v_0$ *has a solution* v *if and only if* $v_0 z_i = 0$, *for* $i = 1, \dots, r+d$.

(iv) *Then the general solution of the equation* $(S+T)x = x_0$ *is given by*

$$x = Bx_0 + \alpha_1 z_1 + \cdots + \alpha_{r+d} z_{r+d}, \tag{2.16}$$

and the general solution of the equation $v(S+T) = v_0$ *is given by*

$$v = v_0 B + \beta_1 \xi_1 + \cdots + \beta_r \xi_r, \tag{2.17}$$

where $\alpha_1, \dots, \alpha_{r+d}$, *and* $\beta_1, \dots, \beta_r$ *are arbitrary constants.*

Proof. The formulae for the solutions of the equations $(S+T)x = x_0$ and $v(S+T) = v_0$ follow from an application of the identities [**16**, Theorem 2.2]:

$$D_{n+1}\begin{pmatrix} v_0(S+T), & & v_1, & \cdots, & v_{n+d} \\ x_0, & x_1, & \cdots & \cdots, & x_n \end{pmatrix}$$
$$= \sum_{i=0}^{n}(-1)^i v_0 \cdot x_i \cdot D_n\begin{pmatrix} v_1, & \cdots & \cdots & \cdots & \cdots & \cdots, & v_{n+d} \\ x_0, & x_1, & \cdots & x_{i-1}, & x_{i+1}, & \cdots, & x_n \end{pmatrix},$$

and

$$D_{n+1}\begin{pmatrix} v_0, & v_1, & \cdots & \cdots, & v_{n+d} \\ (S+T)x_0, & & x_1, & \cdots, & x_n \end{pmatrix}$$
$$= \sum_{i=0}^{n+d}(-1)^i v_i \cdot x_0 \cdot D_n\begin{pmatrix} v_0, & v_1, & \cdots, & v_{i-1}, & v_{i+1}, & \cdots, & v_{n+d} \\ x_1, & \cdots & \cdots & \cdots & \cdots & \cdots, & x_n \end{pmatrix}$$

for $n = r$, and replacement of x and v, respectively, by $(S+T)x$ and $v(S+T)$ in (2.15). As a consequence of this, we get the following expressions:

$$vB(S+T)x = vx - \sum_{i=1}^{r+d} vz_i \cdot v_i x,$$

and

$$v(S+T)Bx = vx - \sum_{i=1}^{r} vx_i \cdot \xi_i x,$$

or, equivalently,

$$B(S+T) = I - \sum_{i=1}^{r+d} z_i v_i, \tag{2.18}$$

and

$$(S+T)B = I - \sum_{i=1}^{r} x_i \xi_i. \tag{2.19}$$

Multiplying the equation (2.18) on the left by v_0 and the equation (2.19) on the right by x_0, and assuming that $v_0 z_i = 0$ and $\xi_j x_0 = 0$ for $i = 1, \dots, r+d$, $j = 1, \dots, r$, we get

$$v_0 B(S+T) = v_0 \tag{2.20}$$

and

$$(S+T)Bx_0 = x_0. \tag{2.21}$$

This completes the proof. □

Corollary 2.2. *Let the singular integral operator $S+T$ be of order $r(S+T) = 0$ and index $(S+T) = d > 0$. Let $z_i \in H^\alpha(L)$, $i = 1, \dots, d$, be linearly independent solutions of $(S+T)x = 0$. Let $\tilde{B}$ be an integral quasi-inverse of $S+T$. Then the solution of $(S+T)x = x_0$ is given by*

$$x = \tilde{B}x_0 + \alpha_1 z_1 + \cdots + \alpha_d z_d, \tag{2.22}$$

and the general solution of $v(S+T)=v_0$ *is given by*

$$v = v_0\tilde{B}, \tag{2.23}$$

where $\tilde{B}=\{1/(a^2-b^2)\}(aI-Jb)$ *and* $\alpha_1,\dots,\alpha_d$ *are arbitrary constants.*

Corollary 2.3. *For* $S=I$ *and* $d=0$*, the general solutions of the equations* $(I+T)x=x_0$ *and* $v(I+T)=v_0$ *are, respectively, given by*

$$x = B_1x_0+\alpha_1z_1+\cdots+\alpha_rz_r \tag{2.24}$$

and

$$v = v_0B_1+\beta_1\xi_1+\cdots+\beta_r\xi_r \tag{2.25}$$

for $vB_1x=\{1/\tilde{D}_r\}D_{r+1}\begin{pmatrix} v, & \eta_1, & \cdots, & \eta_r \\ x, & y_1, & \cdots, & y_r\end{pmatrix}$*, where*

$$\tilde{D}_r = D_r\begin{pmatrix}\eta_1, & \cdots, & \eta_r \\ y_1, & \cdots, & y_r\end{pmatrix}\neq 0.$$

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REFERENCES

1. A. Buraczewski, *Determinant theory of generalized Fredholm operators*, Studia Math. **22** (1963), 265–307.

2. ———, *On certain property of determinant systems*, Coll. Mat. **10** (1963), 325–330.

3. ———, *Determinant system for composite of generalized Fredholm operators*, Studia Math. **34** (1970), 197–207.

4. ———, *Generalization of formulae of Fredholm type*, Coll. Mat. **31** (1974), 107–113.

5. ———, *The general theory of determinants in Banach spaces*, Studia Math., preprint.

6. A. Grothendieck, *La theorie de Fredholm*, Bull. Soc. Math. France **84** (1956), 319–384.

7. T. Lezanski, *The Fredholm theory of linear equations in Banach spaces*, Studia Math. **13** (1953), 244–276.

8. S. Mikhlin and S. Prössdorf, *Singular integral operators*, Springer-Verlag, Berlin, 1986.

9. N. Muskhelishvili, *Singular integral equations*, Groningen, Holland, 1953.

10. R. Sikorski, *Remarks on Lezanski's determinants*, Studia Math. **20** (1961), 145–161.

11. ———, *Determinant systems*, Studia Math. **18** (1959), 161–186.

12. ———, *The determinant theory in Banach spaces*, Coll. Mat. **8** (1961), 141–198.

13. ———, *On multiplication of determinants in Banach spaces*, Bull. Acad. Polon. Sci., Classe III, **1** (1953), 219–221.

14. R.U. Verma, *Determinant theory of singular integral equations*, Demonstratio Math. **13** (1980), 65–75.

15. ———, *Formulae of Fredholm type for singular integral operators with nonvanishing indices*, Demonstratio Math. **16** (1983), 211–228.

16. ———, *The generalized Fredholm operators with finite indices*, Nonlinear Anal. **8** (1984), 1443–1453.

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