

NONTRIVIAL SOLUTIONS FOR A CLASS
OF NONLINEAR VOLTERRA EQUATIONS
WITH CONVOLUTION KERNEL

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ABSTRACT. We consider the Volterra integral equation

$$u(x) = \int_0^x k(x-s)g(u(s)) ds, \quad x \geq 0,$$

where $k \geq 0$ is an integrable function and g is an increasing absolutely continuous function ($g(0) = 0$) which does not satisfy a Lipschitz condition. New necessary and sufficient conditions for the existence of positive nontrivial solutions are obtained.

1. Introduction. The nonlinear Volterra integral equation with convolution kernel

$$(1.1) \quad u(x) = \int_0^x k(x-s)g(u(s)) ds, \quad x \geq 0$$

has been studied recently in the modeling of problems in nonlinear diffusion and shock-wave propagation [4, 7]. In these problems the kernel function is nonnegative and g is an increasing continuous function such that $g(0) = 0$. Moreover, g does not satisfy a Lipschitz condition in the vicinity of the origin. A typical example of such a function g is $g(u) = u^p$, $p \in (0, 1)$. Obviously, $u \equiv 0$ is the trivial solution to (1.1). But the question of physical interest is the existence of nontrivial solutions to (1.1), i.e., continuous functions u such that $u(x) > 0$ for $x > 0$.

Some particular answers concerning the existence of nontrivial solutions can be found in Gripenberg's work [3]. Under restrictive assumptions concerning g , the author has presented there a condition which is necessary and sufficient for the existence of a nontrivial solution to (1.1)

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with the particular kernel $k(x) = x^{\alpha-1}$, $\alpha > 0$. But we must emphasize that under Gripenberg's assumptions the case $g(u) = u^p$, $p \in (0, 1)$, cannot be considered. In papers [1, 2] and [6] these restrictions are removed and Gripenberg's condition can be applied to a wider class of g including $g(u) = u^p$. One can ask if Gripenberg's condition can be generalized to other kernels. In fact, in the case of monotonic kernels we can give some integral sufficient and necessary conditions for the existence of nontrivial solutions to (1.1) (see [7]). For $k(x) = x^{\alpha-1}$, all these conditions are equivalent to that of Gripenberg's. The range of application of these generalized Gripenberg's conditions is sometimes limited. In the case of $g(u) = u^p$, $p \in (0, 1)$, these necessary conditions are fulfilled for all admissible kernels, and, because of this, they are useless. Applying the sufficient condition to (1.1) with $k(x) = \exp(1/x^\alpha)$ and $g(u) = u^p$, $p \in (0, 1)$, we can show the existence of nontrivial solutions for $\alpha \in (0, 1)$. But in [2] it has been shown that, for k and g mentioned above, a nontrivial solution exists for all $\alpha > 0$. These remarks suggest a search for different kinds of conditions.

The purpose of this paper is to present some new conditions for the existence of nontrivial solutions to (1.1) which will be widely applicable. Moreover, the new results can be applied to discontinuous kernels.

2. Statement of Results. We shall study equation (1.1) assuming that

(k) $k : \langle 0, \delta \rangle \rightarrow \langle 0, +\infty \rangle$, $\delta > 0$, is an integrable function such that $k > 0$ a.e.,

(g) $g : \langle 0, +\infty \rangle \rightarrow \langle 0, +\infty \rangle$ is a strictly increasing absolutely continuous function such that $g(0) = 0$ and $u/g(u) \rightarrow 0$ as $u \rightarrow 0^+$.

Let K^{-1} denote the inverse function to $K(x) \doteq \int_0^x k(s) ds$. For a given function f , we define the sequence of functions f^n , $n = 0, 1, \dots$, as follows: $f^0(x) = x$, $f^{n+1}(x) = (f^n \circ f)(x)$, $n = 0, 1, \dots$. We formulate the following sufficient condition.

Theorem 2.1. *Let (k) and (g) be satisfied. Let φ be a continuous function on $\langle 0, \delta_0 \rangle$, $\delta_0 > 0$, such that $x < \varphi(x) < g(x)$ for $x \in (0, \delta_0)$*

and $x/\varphi(x) \rightarrow 0$ as $x \rightarrow 0^+$. If the series

$$(2.1) \quad \sum_{n=0}^{\infty} K^{-1}((g^{-1} \circ \varphi)^n(x)/\varphi((g^{-1} \circ \varphi)^n(x)))$$

converges uniformly on $\langle 0, \delta_0 \rangle$, then equation (1.1) has a nontrivial solution on some interval.

Moreover, the following necessary condition is true.

Theorem 2.2. *Let (k) and (g) be satisfied. Let ψ be a continuous function such that $\psi(x) > 0$ for $x > 0$ and $\lim_{x \rightarrow 0^+} g(x)/\psi(x) < 1$. If equation (1.1) has a nontrivial solution on an interval, then the series*

$$(2.2) \quad \sum_{n=0}^{\infty} K^{-1}((g^{-1})^n(x)/\psi((g^{-1})^n(x)))$$

is convergent on $\langle 0, \delta_0 \rangle$, $\delta_0 > 0$.

In the next section we present corollaries and examples based on the above theorems.

3. Some consequences and comments. We can formulate two corollaries:

Corollary 3.1. *If functions $\varphi(x)$ and $x/\varphi(x)$ are increasing, then the convergence of (2.1) at one of the points of $\langle 0, \delta_0 \rangle$ is sufficient for the existence of nontrivial solutions to (1.1).*

Corollary 3.2. *Assume additionally that k and $u/g(u)$ are nondecreasing. If the series*

$$(3.1) \quad \sum_{n=0}^{\infty} K^{-1}((x/2^n)/g(x/2^n))$$

is convergent at one point $x_0 > 0$, then equation (1.1) has a nontrivial solution.

In fact, if we put $\varphi(x) \equiv g(x/2)$, then the series (2.1) reduces to

$$\sum_{n=1}^{\infty} K^{-1}((x/2^{n-1})/g(x/2^n)).$$

Since K^{-1} is concave, then the last series is dominated by the series

$$2 \sum_{n=1}^{\infty} K^{-1}((x/2^n)/g(x/2^n)).$$

By (3.1) and Corollary 3.1 we obtain Corollary 3.2.

Remark 3.1. Suppose assumptions of Corollary 3.2 are satisfied. In this case the Gripenberg's generalized sufficient condition has the form

$$(3.2) \quad \int_0^{\delta} K^{-1}(s/g(s))/s \, ds < \infty \quad [7].$$

Since the function $K^{-1}(x/g(x))/x$ is decreasing, we obtain the estimate

$$\sum_{n=1}^{\infty} K^{-1}((x/2^n)/g(x/2^n)) \leq 2 \int_0^x K^{-1}(s/g(s))/s \, ds.$$

Hence, by Corollary 3.2, we infer that Theorem 2.1 is a generalization of condition (3.2).

Now we give some corollaries concerning the necessary condition.

Corollary 3.3. *The function $\psi(x) \equiv 1$ satisfies assumptions of Theorem 2.2. In this case the series (2.2) is equal to*

$$(3.3) \quad \sum_{n=0}^{\infty} K^{-1}((g^{-1})^n(x)).$$

The convergence of this last series is a necessary condition for the existence of nontrivial solutions to (1.1).

Corollary 3.4. *Assume additionally that k is nondecreasing. If equation (1.1) has a nontrivial solution, then the series*

$$(3.4) \quad \sum_{n=0}^{\infty} K^{-1}((g^{-1})^n(x)/(g^{-1})^{n-1}(x))$$

is convergent on an interval $\langle 0, \delta_0 \rangle$, $\delta_0 > 0$.

In fact, we can put $\psi(x) \equiv g(x)/(1 - \varepsilon)$. In this case the series (2.2) is

$$\sum_{n=0}^{\infty} K^{-1}((1 - \varepsilon)(g^{-1})^n(x)/(g^{-1})^{n-1}(x)).$$

Since K^{-1} is concave, then $K^{-1}((1 - \varepsilon)x) \geq (1 - \varepsilon)K^{-1}(x)$, and Corollary 3.4 is proved.

Example. We consider the equation

$$(3.5) \quad u(x) = \int_0^x k(x-s)\{u(s)\}^p ds, \quad p \in (0, 1),$$

with $k(x) = K'(x)$, where $K(x) = \exp(-\exp(1/x^\alpha))$, $\alpha > 0$. In this case $K^{-1}(x) = 1/(\ln \ln(1/x))^{1/\alpha}$ and $g^{-1}(x) = x^{1/p}$. We calculate that the series (3.3) becomes

$$\sum_{n=0}^{\infty} 1/(\ln \ln(1/x) + n \ln(1/p))^{1/\alpha}.$$

If $\alpha \geq 1$, then this series is divergent on an interval. This means that equation (3.5) has no nontrivial solutions for $\alpha \geq 1$. Now we shall try to apply Theorem 2.1. Let $\varphi(x) = x^q$, where $p < q < 1$. In this case the series (2.1) becomes

$$\sum_{n=0}^{\infty} 1/(\ln \ln(1/x) + \ln(1-q) + n \ln(q/p))^{1/\alpha}.$$

If $\alpha \in (0, 1)$, then this last series is convergent for small x . This implies that equation (3.5) has nontrivial solutions for $\alpha \in (0, 1)$.

We formulate two remarks concerning kernels which do not satisfy (k).

Remark 3.2. Let $k \geq 0$ be an integrable function such that $K(x) > 0$ for $x > 0$. Let $\overline{K}^{-1}(y) \doteq \max\{x \in \langle 0, 1 \rangle : K(x) = y\}$. If we substitute \overline{K}^{-1} instead of K^{-1} in (2.1), then Theorem 2.1 will still be true.

Remark 3.3. Let $k \geq 0$ be a locally bounded measurable function. Let $k_1 = \sum_{n=1}^{\infty} k^{*n}$, where $k^{*1} = k$ and $k^{*(n+1)} = k^{*n} * k$, $n = 1, 2, \dots$; here $*$ denotes the convolution. It can be shown (see [2]) that (1.1) has a nontrivial solution if and only if

$$(3.6) \quad u(x) = \int_0^x k_1(x-s)g_1(u(s)) ds,$$

where $g_1(u) \doteq g(u) - u$, has a nontrivial solution. Suppose assumption (k) is not fulfilled for k but is satisfied for k_1 (see the example in [2]). Let g_1 satisfy (g). In such cases, one can try to apply results of this paper to (3.6).

4. Proofs of the theorems. Throughout this part, we assume that (k) and (g) are satisfied. On the basis of results presented in [5], we can formulate the following remarks:

Remark 4.1. If there exists a nontrivial solution to (1.1), then it is a unique nontrivial solution on an interval $\langle 0, \delta_1 \rangle$, $\delta_1 > 0$. Moreover, it is a strictly increasing absolutely continuous function. We shall denote this solution by u_0 .

Remark 4.2. For every $\varepsilon \in (0, 1)$ the equation

$$(4.1) \quad u_\varepsilon(x) = \varepsilon x + \int_0^x k(x-s)g(u_\varepsilon(s)) ds$$

has a unique strictly increasing, absolutely continuous solution u_ε on an interval $\langle 0, \delta_1 \rangle$, where $\delta_1 > 0$ is independent of ε . Moreover, $u_{\varepsilon_1} \leq u_{\varepsilon_2}$ for $\varepsilon_1 \leq \varepsilon_2$.

We formulate the following lemma.

Lemma 4.1. *Let $\varepsilon \in \langle 0, 1 \rangle$. If u_ε is the nontrivial solution to (4.1), then the inverse function u_ε^{-1} satisfies the equation*

$$(4.2) \quad x = \varepsilon u_\varepsilon^{-1}(x) + \int_0^{g(x)} K(u_\varepsilon^{-1}(x) - u_\varepsilon^{-1}(g^{-1}(s))) ds$$

for $x \in \langle 0, u_\varepsilon(\delta_1) \rangle$.

Proof. Let $\varepsilon \geq 0$ and u_ε be the solution to (4.1) mentioned in Remarks 4.1 and 4.2. Since u_ε is absolutely continuous, then

$$(4.3) \quad u'_\varepsilon(x) = \varepsilon + \int_0^x k(x-s)g'(u_\varepsilon(s))u'_\varepsilon(s) ds \quad \text{a.e.}$$

on $\langle 0, \delta_1 \rangle$. By the assumptions and (4.3), we have $u_\varepsilon > 0$ a.e. We infer u_ε^{-1} is an absolutely continuous function. From (4.3) we get

$$(4.4) \quad u'_\varepsilon(x) = \varepsilon + \int_0^{u_\varepsilon(x)} k(x - u_\varepsilon^{-1}(s))g'(s) ds \quad \text{a.e.}$$

Substituting $u_\varepsilon^{-1}(x)$ for x gives

$$(4.5) \quad u'_\varepsilon(u_\varepsilon^{-1}(x)) = \varepsilon + \int_0^x k(u_\varepsilon^{-1}(x) - u_\varepsilon^{-1}(s))g'(s) ds \quad \text{a.e.}$$

But $(u_\varepsilon^{-1})' = 1/u'_\varepsilon \circ u_\varepsilon^{-1}$ a.e. Hence, by (4.5),

$$(4.6) \quad (u_\varepsilon^{-1})'(x) \left[\varepsilon + \int_0^x k(u_\varepsilon^{-1}(x) - u_\varepsilon^{-1}(s))g'(s) ds \right] = 1 \quad \text{a.e.}$$

Integrating (4.6) with respect to x obtains

$$(4.7) \quad \varepsilon u_\varepsilon^{-1}(x) + \int_0^x K(u_\varepsilon^{-1}(x) - u_\varepsilon^{-1}(s))g'(s) ds = x$$

on $\langle 0, u_\varepsilon(\delta_1) \rangle$, and (4.7) yields (4.2). \square

Remark 4.3. The function $G_\varepsilon(x, s) \doteq K(u_\varepsilon^{-1}(x) - u_\varepsilon^{-1}(g^{-1}(s)))$ is decreasing with respect to s . Moreover, $G_\varepsilon(x, 0) = K(u_\varepsilon^{-1}(x))$ and $G_\varepsilon(x, g(x)) = 0$.

Now we prove

Lemma 4.2. *Let φ be a continuous function on $\langle 0, \delta_0 \rangle$, $\delta_0 > 0$ such that $x < \varphi(x) < g(x)$ for $x \in \langle 0, \delta_0 \rangle$ and $x/\varphi(x) \rightarrow 0$ as $x \rightarrow 0^+$. Let $\varepsilon > 0$. If u_ε is the solution to (4.1), then*

$$(4.8) \quad u_\varepsilon^{-1}(x) \leq K^{-1}(x/\varphi(x)) + u_\varepsilon^{-1}((g^{-1} \circ \varphi)(x))$$

for $x \in \langle 0, u_\varepsilon(\delta_1) \rangle$.

Remark 4.4. Without loss of generality, we can assume $u_\varepsilon(\delta_1) < \delta_0$ for any $\varepsilon \in (0, 1)$.

Proof of Lemma 4.2. On the basis of (4.2) we have

$$(4.9) \quad x \geq \int_0^{g(x)} G_\varepsilon(x, s) ds,$$

where G_ε is defined in Remark 4.3. By Remark 4.3 and the assumptions concerning φ ,

$$(4.10) \quad \int_0^{g(x)} G_\varepsilon(x, s) ds \geq \varphi(x) G_\varepsilon(x, \varphi(x)).$$

Using (4.9) and (4.10) gives

$$(4.11) \quad x/\varphi(x) \geq G_\varepsilon(x, \varphi(x)),$$

and from (4.11) we obtain (4.8). \square

Proof of Theorem 2.1. Let φ be given on $\langle 0, \delta_0 \rangle$. Let $\{u_\varepsilon\}$, $\varepsilon \in (0, 1)$, denote the family of solutions to (4.1) on $\langle 0, \delta_1 \rangle$ mentioned in Remark 4.2. Fixing ε , we can iterate the inequality (4.8). After n iterations we get

$$(4.12) \quad u_\varepsilon^{-1}(x) \leq \sum_{i=0}^n K^{-1}((g^{-1} \circ \varphi)^i(x)/\varphi((g^{-1} \circ \varphi)^i(x))) + u_\varepsilon^{-1}((g^{-1} \circ \varphi)^{n+1}(x))$$

on $\langle 0, u_\varepsilon(\delta_1) \rangle$. Since $(g^{-1} \circ \varphi)(x) < x$ for $x > 0$, then $u_\varepsilon^{-1}((g^{-1} \circ \varphi)^{n+1}(x))$ tends uniformly to 0 on $\langle 0, u_\varepsilon(\delta_1) \rangle$ as $n \rightarrow \infty$. This means that the right side of (4.12) tends uniformly to

$$F(x) \doteq \sum_{i=0}^{\infty} K^{-1}((g^{-1} \circ \varphi)^i(x) / \varphi((g^{-1} \circ \varphi)^i(x))) \quad \text{on } \langle 0, u_\varepsilon(\delta_1) \rangle.$$

Let us note that F is a continuous function. Inequality (4.12) implies

$$(4.13) \quad u_\varepsilon^{-1}(x) \leq F(x).$$

Let $\bar{F}(x) \doteq \sup_{s \in \langle 0, x \rangle} F(s) + x$. It is a strictly increasing continuous function such that

$$(4.14) \quad u_\varepsilon^{-1}(x) \leq \bar{F}(x)$$

on $\langle 0, u_\varepsilon(\delta_1) \rangle$. From (4.14), we get

$$(4.15) \quad u_\varepsilon(x) \geq \bar{F}^{-1}(x)$$

for $x \in \langle 0, \delta_1 \rangle$, where \bar{F}^{-1} is the inverse function to \bar{F} . Let $\varepsilon \searrow 0^+$. Since the sequence u_ε is decreasing with respect to ε , we infer $u(x) \doteq \lim_{\varepsilon \searrow 0^+} u_\varepsilon(x)$, $x \in \langle 0, \delta_1 \rangle$, is a continuous solution to (1.1). But with respect to (4.15), $u(x) \geq \bar{F}^{-1}(x)$, and we have constructed a nontrivial solution to (1.1).

To prove the necessary condition, we need the following lemma

Lemma 4.3. *Let ψ be a continuous function such that $\psi(x) > 0$ for $x > 0$ and $\overline{\lim}_{x \rightarrow 0^+} g(x)/\psi(x) < 1$. If equation (1.1) has the nontrivial solution u_0 , then*

$$(4.16) \quad u_0^{-1}(x) \geq K^{-1}(x/\psi(x)) + u_0^{-1}(g^{-1}(x))$$

on an interval $\langle 0, \delta_0 \rangle$, $\delta_0 > 0$.

Proof. Let us note that (4.16) is equivalent to

$$(4.17) \quad G_0(x, x) \geq x/\psi(x)$$

for $x \in \langle 0, \delta_0 \rangle$, where G_0 is defined in Remark 4.3 for $\varepsilon = 0$. Suppose (4.17) does not hold. This means that there exists a sequence $x_n \rightarrow 0$ as $n \rightarrow \infty$ such that

$$(4.18) \quad G_0(x_n, x_n) < x_n/\psi(x_n).$$

Lemma 4.1 gives

$$(4.19) \quad x_n = \int_0^{x_n} G_0(x_n, s) ds + \int_{x_n}^{g(x_n)} G_0(x_n, s) ds.$$

Since G_0 is decreasing with respect to s , we get

$$(4.20) \quad x_n \leq K^{-1}(u_0^{-1}(x_n))x_n + g(x_n)G_0(x_n, x_n).$$

By (4.18) and (4.20),

$$x_n < K^{-1}(u_0^{-1}(x_n))x_n + x_n g(x_n)/\psi(x_n),$$

and, hence,

$$(4.21) \quad 1 < K^{-1}(u_0^{-1}(x_n)) + g(x_n)/\psi(x_n).$$

Since $\overline{\lim}_{x \rightarrow 0^+} g(x)/\psi(x) < 1$, then $g(x)/\psi(x) < 1 - \eta$, $\eta \in (0, 1)$, on $\langle 0, \delta_\eta \rangle$. Since $x_n \in \langle 0, \delta_\eta \rangle$ for $n \geq n_0$, then (4.21) implies

$$(4.22) \quad 1 < K^{-1}(u_0^{-1}(x_n)) + 1 - \eta$$

for $n \geq n_0$. If $n \rightarrow \infty$, then $K^{-1}(u_0^{-1}(x_n)) \rightarrow 0$, and (4.22) yields the contradiction. \square

Proof of Theorem 2.2. Let ψ satisfy the assumptions of the theorem. If u_0 is the nontrivial solution to (1.1), inequality (4.16) holds. We can iterate (4.16). After n iterations, we get

$$(4.23) \quad u_0^{-1}(x) \geq \sum_{i=0}^n K^{-1}((g^{-1})^i(x)/\psi((g^{-1})^i(x))) + u_0^{-1}((g^{-1})^{n+1}(x))$$

on $\langle 0, \delta_0 \rangle$. Without loss of generality, we can assume that $g^{-1}(x) < x$ for $x \in \langle 0, \delta_0 \rangle$. Hence, $u_0^{-1}((g^{-1})^{n+1}(x))$ tends uniformly to 0 on $\langle 0, \delta_0 \rangle$ as $n \rightarrow \infty$. Let $n \rightarrow \infty$. From (4.23) we obtain

$$(4.24) \quad u_0^{-1}(x) \geq \sum_{i=0}^{\infty} K^{-1}((g^{-1})^i(x)/\psi((g^{-1})^i(x))),$$

and the theorem is proved. \square

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