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WEAK SOLUTIONS OF THE EXTERIOR BOUNDARY VALUE PROBLEMS OF PLANE COSSERAT ELASTICITY

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ABSTRACT. In this paper we formulate exterior Dirichlet and Neumann boundary value problems of plane Cosserat elasticity in Sobolev spaces, show that these problems are well-posed and find the corresponding weak solutions in terms of integral potentials.

1. Introduction. The theory of micropolar (Cosserat) elasticity [**4**] has been developed by Eringen to account for discrepancies between the classical theory and experiments when the effects of material microstructure were known to significantly affect the body's overall deformation, for example, in the case of granular bodies with large molecules (e.g. polymers) or human bones, see [**7**–**10**]. Significant progress has been achieved in this direction for the last 30 years (see [**12**] for a review of works in this area and an extensive bibliography), but investigations mainly have been confined to the case of boundary value problems for domains bounded by sufficiently smooth curves. For example, three-dimensional problems of Cosserat elasticity have been formulated in a rigorous setting and solved by means of methods of the potential theory by Kupradze in [**6**].

In [**5**, **13**–**15**], the corresponding boundary value problems for plane deformations of a micropolar homogeneous, linearly elastic solid were shown to be well posed and subsequently solved in a rigorous setting using the boundary integral equation method. Unfortunately, consideration of these problems in the space L^2 setting requires that we impose strict conditions on the curve which represents the boundary of the domain. To be precise, this curve must be expressed in terms of a twice differentiable function. If this condition is not satisfied, i.e., the boundary is not smooth enough or the domain contains cracks, the method presented in [**5**, **13**–**15**] fails to produce acceptable results. To overcome this difficulty it seems reasonable to formulate the corresponding

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boundary value problems of plane Cosserat elasticity in a Sobolev space and find weak solutions in terms of integral potentials using the boundary integral equation method.

Very recently in [**16**] we have formulated interior boundary value problems of Cosserat elasticity in the case of the reduced boundary smoothness in Sobolev spaces. Such an approach, which in a series of recent works by Chudinovich and Constanda [**1**–**3**] has been applied to the investigation of boundary value problems arising in plane classical elasticity and in the theory of plates, allows us to obtain solutions for domains with irregular boundaries and to facilitate the close monitoring of the performance of numerical schemes in domains with a relatively low degree of smoothness.

This work is the extension of our results obtained in [**16**] for the case of the exterior boundary value problems of plane micropolar elasticity. In this paper we formulate exterior Dirichlet and Neumann boundary value problems of plane Cosserat elasticity in Sobolev spaces and find the corresponding weak solutions in terms of integral potentials.

2. Preliminaries. In what follows Greek and Latin indices take the values 1, 2 and 1, 2, 3, respectively, the convention of summation over repeated indices is understood, $\mathcal{M}_{m\times n}$ is the space of $(m\times n)$-matrices, E_n is the identity element in $\mathcal{M}_{n\times n}$, the columns of a (3×3)-matrix P are denoted by $P^{(i)}$, a superscript T indicates matrix transposition, the generic symbol c denotes various strictly positive constants and $(\dots)_{,\alpha} \equiv \partial(\dots)/\partial x_\alpha$. Also, if X is a space of scalar functions and v a matrix, $v \in X$ means that every component of v belongs to X.

Let S^- be a region in $\mathbf{R}^2$ such that $\mathbf{R}^2\backslash S^-$ is a domain bounded by a closed curve ∂S. Let S^- be occupied by a homogeneous and isotropic linearly elastic micropolar material with elastic constants $\lambda, \mu, \alpha, \gamma$ and ε. The state of plane micropolar strain is characterized by a displacement field $u(x') = (u_1(x'), u_2(x'), u_3(x'))^T$ and a microrotation field $\phi(x') = (\phi_1(x'), \phi_2(x'), \phi_3(x'))^T$ of the form

$$\begin{aligned} &u_\alpha(x') = u_\alpha(x), \quad && u_3(x') = 0, \\ &\phi_\alpha(x') = 0, && \phi_3(x') = \phi_3(x), \end{aligned} \tag{2.1}$$

where $x' = (x_1, x_2, x_3)$ and $x = (x_1, x_2)$ are generic points in $\mathbf{R}^3$ and $\mathbf{R}^2$, respectively. The equilibrium equations of plane micropolar strain written in terms of displacements and microrotations are given by [**5, 15**]

$$L(\partial_x)u(x) + q(x) = 0, \quad x \in S^-, \tag{2.2}$$

in which now, denoting ϕ_3 by u_3, we have $u(x) = (u_1, u_2, u_3)^T$, the matrix partial differential operator $L(\partial x) = L(\partial/\partial x_\alpha)$ is defined by

$$L(\xi) = L(\xi_\alpha)$$
$$= \begin{pmatrix} (\mu+\alpha)\Delta + (\lambda+\mu-\alpha)\xi_1^2 & (\lambda+\mu-\alpha)\xi_1\xi_2 & 2\alpha\xi_2 \\ (\lambda+\mu-\alpha)\xi_1\xi_2 & (\mu+\alpha)\Delta + (\lambda+\mu-\alpha)\xi_2^2 & -2\alpha\xi_1 \\ -2\alpha\xi_2 & 2\alpha\xi_1 & (\gamma+\varepsilon)\Delta - 4\alpha \end{pmatrix},$$

where $\Delta = \xi_\alpha\xi_\alpha$, and vector $q = (q_1, q_2, q_3)^T$ represents body forces and body couples.

Together with L we consider the boundary stress operator $T(\partial x) = T(\partial/\partial x_\alpha)$ defined by

$$T(\xi) = T(\xi_\alpha)$$
$$= \begin{pmatrix} (\lambda+2\mu)\,\xi_1 n_1 + (\mu+\alpha)\,\xi_2 n_2 & (\mu-\alpha)\xi_1 n_2 + \lambda\xi_2 n_1 & 2\alpha n_2 \\ (\mu-\alpha)\xi_2 n_1 + \lambda\xi_1 n_2 & (\lambda+2\mu)\,\xi_2 n_2 + (\mu+\alpha)\,\xi_1 n_1 & -2\alpha n_1 \\ 0 & 0 & (\gamma+\varepsilon)\xi_\alpha n_\alpha \end{pmatrix},$$

where $n = (n_1, n_2)^T$ is the unit outward normal to ∂S. To guarantee the ellipticity of system (2.2), in what follows we assume that

$$\lambda + \mu > 0, \quad \gamma + \varepsilon > 0, \quad \mu > 0, \quad \alpha > 0.$$

The space of rigid displacements and microrotations $\mathcal{F}$ is spanned by the columns of the matrix

$$\mathcal{F} = \begin{pmatrix} 1 & 0 & -x_2 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{pmatrix}$$

from which it can be seen that $L\mathcal{F} = 0$ in $\mathbf{R}^2$, $T\mathcal{F} = 0$ on ∂S and a general rigid displacement can be written as $\mathcal{F}k$, where $k \in \mathcal{M}_{3\times 1}$ is constant and arbitrary.

We introduce the class $\mathcal{A}$ of vectors $u \in \mathcal{M}_{3\times 1}$ whose components in terms of polar coordinates, as $r = |x| \to \infty$, are of the form

$$\begin{aligned} u_1(r,\theta) &= r^{-1}\left(\beta m_0 \sin\theta + m_1 \cos\theta + m_0 \sin 3\theta + m_2 \cos 3\theta\right) + O(r^{-2}), \\ u_2(r,\theta) &= r^{-1}\left(m_3 \sin\theta + \beta m_0 \cos\theta + m_4 \sin 3\theta - m_0 \cos 3\theta\right) + O(r^{-2}), \\ u_3(r,\theta) &= r^{-2}\left(m_5 \sin 2\theta + m_6 \cos 2\theta\right) + O(r^{-3}), \end{aligned}$$

where

$$\beta = \frac{3\mu + \lambda}{\lambda + \mu},$$

and $m_0, \dots, m_6$ are arbitrary constants. Also, let

$$\mathcal{A}^* = \left\{ u : u = \mathcal{F}c + \sigma^{\mathcal{A}} \right\},$$

where $c \in \mathcal{M}_{3\times 1}$ is constant and arbitrary and $\sigma^{\mathcal{A}} \in \mathcal{M}_{3\times 1} \cap \mathcal{A}$.

Using the same technique as in the derivation of the Betti formula [**15**], it is easy to show that if u is a solution of (2.2) in S^-, then for any $v \in C^2(S^-) \cap C^1(\overline{S}^-) \cap \mathcal{A}^*$

$$\int_{S^-} v^T q \, dx = -\int_{S^-} v^T L u \, dx = 2\int_{S^-} E(u,v)\, dx + \int_{\partial S} v^T T u \, ds, \tag{2.3}$$

where $E(u,v)$ is the internal energy density given by

$$\begin{aligned} 2E(u,v) = 2E_0(u,v) & \\ &+ (\mu+\alpha)\left((u_{1,2}+u_3)(v_{1,2}+v_3) + (u_{2,1}-u_3)(v_{2,1}-v_3)\right) \\ &+ (\mu-\alpha)\left((u_{1,2}+u_3)(v_{2,1}-v_3) + (v_{1,2}+v_3)(u_{2,1}-u_3)\right) \\ &+ (\gamma+\varepsilon)(u_{3,1}v_{3,1} + u_{3,2}v_{3,2}), \\ 2E_0(u,v) = (\lambda+2\mu)&\,(u_{1,1}v_{1,1} + u_{2,2}v_{2,2}) + \lambda(u_{1,1}v_{2,2} + u_{2,2}v_{1,1}). \end{aligned}$$

A Galerkin representation for the solution of (2.2) when $q(x) = -\delta(|x-y|)$, where δ is the Dirac delta distribution, yields the matrix of fundamental solutions [**15**]

$$D(x,y) = L^*(\partial x)t(x,y), \tag{2.4}$$

where L^* is the adjoint of L,

$$t(x,y) = \frac{a}{8\pi k^4}\left\{[k^2\,|x-y|^2 + 4]\ln|x-y| + 4K_0(k|x-y|)\right\}, \tag{2.5}$$

and the constants a, k^2 are defined by

$$a^{-1} = (\gamma + \varepsilon)(\lambda + 2\mu)(\mu + \alpha), \quad k^2 = \frac{4\mu\alpha}{(\gamma + \varepsilon)(\mu + \alpha)}.$$

In view of (2.4) and (2.5)

$$D(x, y) = D^T(x, y) = D(y, x).$$

Along with matrix $D(x, y)$ we consider the matrix of singular solutions

$$P(x, y) = (T(\partial y)D(y, x))^T. \tag{2.5}$$

It is easy to verify that $D^{(i)}(x, y)$ and $P^{(i)}(x, y)$ satisfy (2.2) with $q(x) = 0$ at all $x \in \mathbf{R}^2$, $x \neq y$.

Using classical techniques [**6**] the following theorem can be proved.

Theorem 1 (Somigliana formulae). *If* $u \in C^2(S^-) \cap C^1(\overline{S}^-) \cap \mathcal{A}$, *is a solution of* (2.2) *with* $q(x) = 0$ *in* S^-, *then*

$$-\int_{\partial S} [D(x, y)T(\partial y)u(y) - P(x, y)u(y)]\, ds(y) = \begin{cases} 0 & x \in S^+, \\ \dfrac{1}{2}\, u(x) & x \in \partial S, \\ u(x) & x \in S^-. \end{cases}$$

Further, we introduce the corresponding single and double layer potentials given respectively by

$$(V\varphi)(x) = \int_{\partial S} D(x, y)\varphi(y)\, ds(y),$$
$$(W\varphi)(x) = \int_{\partial S} P(x, y)\varphi(y)\, ds(y),$$

where $\varphi \in \mathcal{M}_{3\times 1}$ is an unknown density matrix.

The properties of the single and double layer integral potentials are well known and may be formulated in the following theorem, which can be proved using the technique described in [**5**, **13–15**].

Theorem 2. (i) *If* $\varphi \in C(\partial S)$, *then* $V\varphi$, $W\varphi$ *are analytic and satisfy* $L(V\varphi) = L(W\varphi) = 0$ *in* $S^+ \cup S^-$.

(ii) *If* $\varphi \in C^{0,\alpha}(\partial S)$, $\alpha \in (0,1)$, *then the direct values* $\widetilde{V}\varphi, \widetilde{W}\varphi$ *of* $V\varphi$, $W\varphi$ *on* ∂S *exist* (*the latter as principal value*), *the function*

$$V^-(\varphi) = (V\varphi)|_{\overline{S}^-},$$

is of class $C^{1,\alpha}(\overline{S}^-)$ *and*

$$TV^-(\varphi) = \left(\widetilde{W}^* - \frac{1}{2}\, I\right)\varphi,$$

where $\widetilde{W}^*$ *is the adjoint of* $\widetilde{W}$ *and* I *is the identity operator.*

(iii) *If* $\varphi \in C^{1,\alpha}(\partial S)$, $\alpha \in (0,1)$, *then the function*

$$W^-(\varphi) = \begin{cases} (W\varphi)|_{S^-} & \text{in } S^-, \\ \left(\widetilde{W} + \frac{1}{2}\, I\right)\varphi & \text{on } \partial S, \end{cases}$$

is of class $C^{1,\alpha}(\overline{S}^-)$.

3. Auxiliary estimates. We define the (2×1)-vector function $\breve{u} = (u_1, u_2)$ and, for a domain $\Omega \subseteq \mathbf{R}^2$, introduce the space $L^2_\omega(\Omega)$ of (3×1)-vector functions u such that

$$\|u\|^2_{0,\omega;\Omega} = \int_\Omega \frac{\left|\breve{u}(x)\right|^2}{(1+|x|)^2(1+\ln|x|)^2}\, dx + \int_\Omega \frac{|u_3(x)|^2}{(1+|x|)^4(1+\ln|x|)^2}\, dx < \infty.$$

Let $H_{1,\omega}(\mathbf{R}^2)$ be the space of distributions on $\mathbf{R}^2$ for which

$$\|u\|^2_{1,\omega} = \|u\|^2_{0,\omega;\mathbf{R}^2} + b_{\mathbf{R}^2}(u,u) < \infty,$$

where

$$b_\Omega(u,v) = \int_\Omega 2E(u,v)\, dx,$$

Also, let $H_{1,\omega}(\Omega)$ be the space of restrictions to Ω of the elements of $H_{1,\omega}(\mathbf{R}^2)$. The norm in this space can be defined in two equivalent ways, namely

$$\|u\|^2_{1,\omega;\Omega} = \|u\|^2_{0,\omega;\Omega} + b_\Omega(u,u) \quad \text{or} \quad \|u\|_{1,\omega;\Omega} = \inf_{\substack{v\in H_{1,\omega}(\mathbf{R}^2)\\ v|_\Omega = u}} \|v\|_{1,\omega},$$

and in what follows we make no distinction between the two. Clearly, if Ω is bounded, then the norm in $L^2_\omega(\Omega)$ is equivalent to that in $L^2(\Omega)$, and the norm in $H_{1,\omega}(\Omega)$ is equivalent to that in the Sobolev space $H_1(\Omega)$. Finally, $\overset{\circ}{H}_{1,\omega}(\Omega)$ is the subspace of $H_{1,\omega}(\mathbf{R}^2)$ of elements with support in $\overline{\Omega}$, equipped with the norm $\|\cdot\|_{1,\omega}$; $C_0^\infty(\Omega)$ is dense in $H_1(\Omega)$ and $C_0^\infty(\Omega)$ is dense in $\overset{\circ}{H}_{1,\omega}(\Omega)$.

If Ω has a compact boundary $\partial\Omega$, we denote by γ^- the trace operator defined first on $C_0^\infty(\Omega)$ and then extended by continuity to a surjection $\gamma^- : H_{1,\omega}(\Omega) \to H_{1/2}(\partial\Omega)$. This is possible because of the local equivalence of $H_{1,\omega}(\Omega)$ and $H_1(\Omega)$. We also consider a continuous extension operator $l^- : H_{1/2}(\partial\Omega) \to H_1(\Omega)$, which, since norm in $H_1(\Omega)$ is stronger than that in $H_{1,\omega}(\Omega)$, can also be regarded as a continuous operator from $H_{1/2}(\partial\Omega)$ into $H_{1,\omega}(\Omega)$.

Let $\overset{\circ}{H}_{-1,\omega}(\Omega)$ (with norm $\|\cdot\|_{-1,\omega}$) and $H_{-1,\omega}(\Omega)$ (with norm $\|\cdot\|_{-1,\omega;\Omega}$) be the duals of $H_{1,\omega}(\Omega)$ and $\overset{\circ}{H}_{1,\omega}(\Omega)$, respectively. It can be shown that if $u \in \overset{\circ}{H}_{-1}(\Omega)$ and has compact support in Ω, or if

$$\int_\Omega \left|\breve{u}(x)\right|^2 (1+|x|)^2(1+\ln|x|)^2\,dx + \int_\Omega |u_3(x)|^2\,(1+|x|)^4(1+\ln|x|)^2\,dx < \infty,$$

then $u \in \overset{\circ}{H}_{-1,\omega}(\Omega)$.

Let $\Omega = K_R^- = \{x\in\mathbf{R}^2 : |x| > R\}$, $R > 1$, and $\partial\Omega = \partial K_R = \{x\in\mathbf{R}^2 : |x| = R\}$.

Theorem 3. *There are* $c_i(R) = \text{const} > 0$ *such that*

$$\|u\|^2_{0,\omega;K_R^-} \leqslant c_1 b_{K_R^-}(u,u) + c_2\|\breve{u}\|^2_{1/2,\partial K_R} + c_3\|u_3\|^2_{0,\partial K_R} \quad \textit{for all}\quad u\in H_{1,\omega}(K_R^-), \tag{3.1}$$

where $\|\cdot\|_{0,\partial K_R}$ *and* $\|\cdot\|_{1/2,\partial K_R}$ *are the norms in* $L^2(\partial K_R)$ *and* $H_{1/2}(\partial K_R)$*, respectively.*

The proof of this theorem is similar to the procedure described in [**2**].

For simplicity we write $b_{S^-}(u,v) = b_-(u,v)$.

Theorem 4. *There is a* $c = c(S^-) = \text{const} > 0$ *such that any* $u \in H_{1,\omega}(S^-)$ *satisfies the estimates*

(3.2)
$$\|u\|^2_{1,\omega;S^-} \leqslant c\left[b_-(u,u) + \left|\int_{\Gamma_0} u\,\mathrm{d}s\right|^2\right],$$
$$\|u\|^2_{1,\omega;S^-} \leqslant c\left[b_-(u,u) + \sum_{i=1}^{3}\left\langle u, \gamma^-\mathcal{F}^{(i)}\right\rangle^2_{0,\partial S}\right], tag3.3$$

where $\Gamma_0 \subseteq \partial S$*, mes* $\Gamma_0 > 0$.

Proof. We claim that for any $u \in H_{1,\omega;S^-}$

$$\|u\|^2_{0,\omega;S^-} \leqslant c\left[b_-(u,u) + \left|\int_{\Gamma_0} u\mathrm{d}s\right|^2\right], \tag{3.4}$$

$$\|u\|^2_{0,\omega;S^-} \leqslant c\left[b_-(u,u) + \sum_{i=1}^{3}\left\langle u, \gamma^-\mathcal{F}^{(i)}\right\rangle^2_{0,\partial S}\right], \tag{3.5}$$

First suppose that the opposite of formula (3.4) is true. Then we can construct a sequence $\{u^{(n)}\} \subset H_{1,\omega}(S^-)$ such that

$$b_-(u^{(n)}, u^{(n)}) \longrightarrow 0, \qquad \int_{\Gamma_0} u^{(n)}\,ds \longrightarrow 0 \tag{3.6}$$

while

$$\|u\|^2_{0,\omega;S^-} = 1. \tag{3.7}$$

Let ∂K_R be a circle with the center at the origin and of radius $R > 1$ sufficiently large so that ∂S is contained inside ∂K_R. We write

$S_R = S^- \cap K_R^-$. Since S_R is bounded, we may repeat the proof [**3**, Theorem 4] to deduce that there is a $c_R = \text{const} > 0$ such that

$$\|u\|^2_{1;S_R} \leqslant c_R \left[b_{S_R}(u,u) + \left| \int_{\Gamma_0} u \; \mathrm{d}s \right|^2 \right] \quad \forall\, u \in H_1(S_R). \tag{3.8}$$

Then, by Theorem 3,

$$\begin{aligned}
\|u^{(n)}\|^2_{0,\omega;S^-} &= \|u^{(n)}\|^2_{0,\omega;S_R} + \|u^{(n)}\|^2_{0,\omega;K_R^-} \\
&\leqslant \|u^{(n)}\|^2_{0,S_R} + \|u^{(n)}\|^2_{0,\omega;K_R^-} \\
&\leqslant c_R \left[b_{S_R}(u^{(n)}, u^{(n)}) + \left| \int_{\Gamma_0} u^{(n)} \; \mathrm{d}s \right|^2 \right] + c_1 b_{K_R^-}(u^{(n)}, u^{(n)}) \\
&\quad + c_2 \|\widecheck{u}^{(n)}\|^2_{1/2,\partial K_R} + c_3 \|u_3^{(n)}\|^2_{0,\partial K_R}.
\end{aligned}$$

From (3.8) for $u^{(n)}$, we now conclude that $u^{(n)} \to 0$ in $H_1(S_R)$. Then $u^{(n)} \to 0$ in $H_{1/2}(\partial K_R)$, hence in $L^2(\partial K_R)$. Consequently, from the last inequality we find that $\lim_{n\to\infty} \|u^{(n)}\|^2_{0,\omega;S^-} = 0$, which contradicts (3.7). Formula (3.5) is proved similarly. □

4. Boundary value problems. We consider Dirichlet and Neumann exterior boundary value problems.

The (exterior) Dirichlet problem is formulated as follows:

(D$^-$) Find $u \in C^2(S^-) \cap C^1(\overline{S}^-) \cap \mathcal{A}^*$ satisfying (2.2) such that $u\,|_{\partial S} = f$,

where f is prescribed on ∂S.

Let (D_0^-) be the exterior Dirichlet problem with $f = 0$. From (2.3) we see that a solution u of (D_0^-) satisfies

$$b_-(u,v) = \langle q, v\rangle_{0,S^-} \quad \text{for all} \quad v \in C_0^\infty(S^-), \tag{4.1}$$

where $\langle \cdot,\cdot \rangle_{0,S^-}$ is the $L^2(S^-)$-inner product. Since $C_0^\infty(S^-)$ is dense in $\mathring{H}_{1,\omega}(S^-)$, it is clear that (4.1) holds for any $v \in \mathring{H}_{1,\omega}(S^-)$. Obviously, any $u \in C^2(S^-) \cap C^1(\overline{S}^-) \cap \mathcal{A}^*$ satisfying (4.1) for any $v \in \mathring{H}_{1,\omega}(S^-)$

and $u|_{\partial S} = 0$ is a classical (regular) solution of (D_0^-). It is easy to check that such a solution belongs to $\overset{\circ}{H}_{1,\omega}(S^-)$. Hence, the variational formulation of (D_0^-) is as follows.

Find $u \in \overset{\circ}{H}_{1,\omega}(S^-)$ such that

$$b_-(u,v) = \langle q, v\rangle_{0,S^-} \quad \text{for all} \quad v \in \overset{\circ}{H}_{1,\omega}(S^-). \tag{4.2}$$

Theorem 5. *The variational problem* (4.2) *has a unique solution* $u \in \overset{\circ}{H}_{1,\omega}(S^-)$ *for every* $q \in H_{-1,\omega}(S^-)$, *and this solution satisfies the estimate*

$$\|u\|_{1,\omega} \leqslant c\|q\|_{-1,\omega;S^-}.$$

Proof. By Theorem 4,

$$\|u\|_{1,\omega}^2 \leqslant cb_-(u,u) \quad \text{for all} \quad u \in \overset{\circ}{H}_{1,\omega}(S^-),$$

which means that $b_-(u,u)$ is coercive on $\overset{\circ}{H}_{1,\omega}(S^-)$. Since $b_-(u,u)$ is clearly continuous on $\overset{\circ}{H}_{1,\omega}(S^-)\times\overset{\circ}{H}_{1,\omega}(S^-)$, we apply the Lax-Milgram lemma [**11**] to complete the proof. □

The variational formulation of (D^-) is as follows.

Find $u \in H_{1,\omega}(S^-)$ such that

$$b_-(u,v) = \langle q, v\rangle_{0,S^-} \quad \text{for all} \quad v \in \overset{\circ}{H}_{1,\omega}(S^-) \tag{4.3}$$

and

$$\gamma^- u = f. \tag{4.4}$$

Theorem 6. *Problem* (4.3)–(4.4) *has a unique solution* $u \in H_{1,\omega}(S^-)$ *for any* $q \in H_{-1,\omega}(S^-)$ *and any* $f \in H_{1/2}(\partial S)$, *and this solution satisfies the estimate*

$$\|u\|_{1,\omega;S^-} \leqslant c\left(\|q\|_{-1,\omega;S^-} + \|f\|_{1/2,\partial S}\right).$$

Proof. The substitution $u = u_0 + l^- f$ reduces (4.3)–(4.4) to a new variational problem, consisting in finding $u_0 \in \mathring{H}_{1,\omega}(S^-)$ such that

$$b_-(u_0, v) = \langle q, v\rangle_{0,S^-} - b_-(l^- f, v) \quad \text{for all} \quad v \in \mathring{H}_{1,\omega}(S^-). \tag{4.5}$$

Since, for any $v \in \mathring{H}_{1,\omega}(S^-)$,

$$\begin{aligned}
|\langle q, v\rangle_{0,S^-} &- b_-(l^- f, v)| \\
&\leqslant \|q\|_{-1,\omega;S^-}\|v\|_{1,\omega} + \left[b_-(l^- f, l^- f)\right]^{1/2}\left[b_-(v,v)\right]^{1/2} \\
&\leqslant \left(\|q\|_{-1,\omega;S^-} + \|l^- f\|_{1,\omega;S^-}\right)\|v\|_{1,\omega} \\
&\leqslant c\left(\|q\|_{-1,\omega;S^-} + \|f\|_{1/2,\partial S}\right)\|v\|_{1,\omega},
\end{aligned}$$

the linear form $\langle q, v\rangle_{0,S^-} - b_-(l^- f, v)$ is a continuous linear functional on $\mathring{H}_{1,\omega}(S^-)$. The statement of the theorem now follows from the Lax-Milgram lemma applied to the auxiliary problem (4.5) and the estimates

$$\begin{aligned}
\|u_0\|_{1,\omega} &\leqslant c\left(\|q\|_{-1,\omega;S^-} + \|f\|_{1/2,\partial S}\right) \\
\|u\|_{1,\omega;S^-} &\leqslant \|u_0\|_{-1,\omega;S^-} + \|l^- f\|_{1,\omega;S^-} \\
&\leqslant c\left(\|q\|_{-1,\omega;S^-} + \|f\|_{1/2,\partial S}\right). \qquad \square
\end{aligned}$$

The (exterior) Neumann problem is formulated as follows.

(N^-) Find $u \in C^2(S^-) \cap C^1(\overline{S}^-) \cap \mathcal{A}$ satisfying (2.2) and $Tu = g$ on ∂S,

where g is prescribed on ∂S.

In this case (2.3) leads to the following variational formulation.

Find $u \in H_{1,\omega}(S^-)$ such that

$$b_-(u, v) = \langle q, v\rangle_{0,S^-} - \langle g, \gamma^- v\rangle_{0,\partial S} \quad \text{for all} \quad v \in H_{1,\omega}(S^-). \tag{4.6}$$

It is easy to verify that $\mathcal{F} \subset H_{1,\omega}(S^-)$ and that, in view of the properties of rigid displacements,

$$\langle q, \mathcal{F}^{(i)}\rangle_{0,S^-} - \langle g, \gamma^- \mathcal{F}^{(i)}\rangle_{0,\partial S} = 0 \tag{4.7}$$

is a necessary solvability condition for (4.6). In what follows we assume (4.7) holds.

Theorem 7. *Problem* (4.6) *is solvable for any* $q \in \overset{\circ}{H}_{-1,\omega}(S^-)$ *and* $g \in H_{-1/2}(\partial S)$. *Any two solutions differ by a rigid displacement, and there is a solution* u_0 *that satisfies the estimate*

$$\|u_0\|_{1,\omega;S^-} \leqslant c\left(\|q\|_{-1,\omega} + \|g\|_{-1/2,\partial S}\right). \tag{4.8}$$

Proof. We introduce the factor space $\mathcal{H}_{1,\omega}(S^-) = H_{1,\omega}(S^-)/\mathcal{F}$, the bilinear form

$$\mathcal{B}_-(U,V) = b_-(u,v) \quad \text{on} \quad \mathcal{H}_{1,\omega}(S^-) \times \mathcal{H}_{1,\omega}(S^-),$$

and the linear functional

$$\mathcal{L}(V) = \langle q, v\rangle_{0,S^-} - \langle g, \gamma^- v\rangle_{0,\partial S} \quad \text{on} \quad \mathcal{H}_{1,\omega}(S^-),$$

where u and v are arbitrary representatives of the classes $U, V \in \mathcal{H}_{1,\omega}(S^-)$. We define the norm in $\mathcal{H}_{1,\omega}(S^-)$ by

$$\|U\|_{\mathcal{H}_{1,\omega}(S^-)} = \inf_{\substack{u \in H_{1,\omega}(S^-) \\ u \in U}} \|u\|_{1,\omega;S^-}.$$

Instead of (4.6) we now consider the new variational problem of finding $U \in \mathcal{H}_{1,\omega}(S^-)$ such that

$$\mathcal{B}_-(U,V) = \mathcal{L}(V) \quad \text{for all} \quad V \in \mathcal{H}_{1,\omega}(S^-). \tag{4.9}$$

In view of the definition of $\mathcal{B}_-(U,V)$, we see that for any U, $V \in \mathcal{H}_{1,\omega}(S^-)$ and any $u \in U$, $v \in V$,

$$|\mathcal{B}_-(U,V)| = |b_-(u,v)| \leqslant c\|u\|_{1,\omega;S^-}\|v\|_{1,\omega;S^-},$$

therefore

$$\begin{aligned}|\mathcal{B}_-(U,V)| &\leqslant c \inf_{\substack{u\in H_{1,\omega}(S^-)\\ u\in U}} \|u\|_{1,\omega;S^-} \inf_{\substack{v\in H_{1,\omega}(S^-)\\ v\in U}} \|v\|_{1,\omega;S^-}\\ &= c\|U\|_{\mathcal{H}_{1,\omega}(S^-)}\|V\|_{\mathcal{H}_{1,\omega}(S^-)},\end{aligned}$$

which shows that $\mathcal{B}_-(U,V)$ is continuous on $\mathcal{H}_{1,\omega}(S^-)\times\mathcal{H}_{1,\omega}(S^-)$.

Next, we can choose $\widetilde{u}\in U$ such that $\langle\gamma^-\widetilde{u},\gamma^-\mathcal{F}^{(i)}\rangle_{0,\partial S}=0$. Then, by (3.3),

$$\begin{aligned}\mathcal{B}_-(U,U) = b_-(\widetilde{u},\widetilde{u}) &\geqslant c\|\widetilde{u}\|^2_{1,\omega;S^-} \geqslant c \inf_{\substack{u\in H_{1,\omega}(S^-)\\ u\in U}} \|u\|^2_{1,\omega;S^-}\\ &= k\|U\|^2_{\mathcal{H}_{1,\omega}(S^-)},\end{aligned}$$

so $\mathcal{B}_-(U,U)$ is coercive on $\mathcal{H}_{1,\omega}(S^-)$.

Finally, since γ^- is continuous on $H_{1,\omega}(S^-)$, for any $V\in\mathcal{H}_{1,\omega}(S^-)$,

$$\begin{aligned}\mathcal{L}(V) &\leqslant \|q\|_{-1,\omega}\|v\|_{1,\omega;S^-} + \|g\|_{-1/2,\partial S}\|\gamma^- v\|_{1/2,\partial S}\\ &\leqslant c\left(\|q\|_{-1,\omega}+\|g\|_{-1/2,\partial S}\right)\|v\|_{1,\omega;S^-},\end{aligned}$$

which shows that $\mathcal{L}$ is continuous linear functional on $\mathcal{H}_{1,\omega}(S^-)$.

By the Lax-Milgram lemma, problem (4.9) has a unique solution $U\in\mathcal{H}_{1,\omega}(S^-)$, and this solution satisfies the estimate

$$\|U\|_{\mathcal{H}_{1,\omega}(S^-)} \leqslant c\left(\|q\|_{-1,\omega}+\|g\|_{-1/2,\partial S}\right).$$

Clearly, any $u\in U$ is a solution of (4.6), and $u_0\in U$ such that

$$\|u_0\|_{1,\omega;S^-} = \|U\|_{\mathcal{H}_{1,\omega}(S^-)}$$

satisfies (4.8). □

5. Poincaré-Steklov operator. Let $f\in H_{1/2}(\partial S)$, and let $u\in H_{1,\omega}(S^-)$ be the (unique) solution of the variational problem (D^-) (4.3)–(4.4) with $q=0$

$$b_-(u,v)=0 \quad\text{for all}\quad v\in \mathring{H}_{1,\omega}(S^-), \quad \gamma^- u = f.$$

We consider an arbitrary $\alpha \in H_{1/2}(\partial S)$ and write $w = l^- \alpha$. Using the Riesz representation theorem, we can define an operator $\mathcal{T}^-$ on $H_{1/2}(\partial S)$ by

$$\langle \mathcal{T}^- f, \alpha \rangle_{0;\partial S} = -b_-(u, w). \tag{5.1}$$

The definition is consistent, for if $\breve{w} \in H_{1,\omega}(S^-)$ is another extension of α, then $w - \breve{w} \in \mathring{H}_{1,\omega}(S^-)$ and $b_-(w - \breve{w}, u) = 0$, for all $\alpha \in H_{1/2}(\partial S)$.

$\mathcal{T}^-$ is known as the Poincaré-Steklov operator corresponding to (2.2).

Denoting the space of the rigid displacements on ∂S by $\mathcal{Z}(\partial S)$, let $\mathcal{H}_{1/2}(\partial S)$ be the subspace of $H_{1/2}(\partial S)$ of all u such that

$$\langle u, z \rangle_{0;\partial S} = 0 \quad \text{for all} \quad z \in \mathcal{Z}(\partial S),$$

and let $\mathcal{H}_{-1/2}(\partial S)$ be the subspace of $H_{-1/2}(\partial S)$ of all g such that

$$\langle g, z \rangle_{0;\partial S} = 0 \quad \text{for all} \quad z \in \mathcal{Z}(\partial S).$$

Theorem 8. (i) *$\mathcal{T}^-$ is a continuous operator from $H_{1/2}(\partial S)$ to $H_{-1/2}(\partial S)$.*

(ii) *$\mathcal{T}^-$ is self-adjoint in the sense that*

$$\langle \mathcal{T}^- f, v \rangle_{0;\partial S} = \langle f, \mathcal{T}^- v \rangle_{0;\partial S} \quad \textit{for all} \quad f, v \in H_{1/2}(\partial S). \tag{5.2}$$

(iii) *The kernel of $\mathcal{T}^-$ coincides with $\mathcal{Z}(\partial S)$.*

(iv) *The range of $\mathcal{T}^-$ coincides with $\mathcal{H}_{-1/2}(\partial S)$.*

Proof. (i) By the definition of $\mathcal{T}^-$, for $f, v \in H_{1/2}(\partial S)$

$$\langle \mathcal{T}^- f, v \rangle^2_{0;\partial S} = b_-(u, l^- v)^2 \le b_-(u, u) b_-(l^- v, l^- v).$$

Since

$$b_-(l^- v, l^- v) \le c \|l^- v\|^2_{1,\omega;S^-} \le c \|v\|^2_{1/2;\partial S},$$

it follows that

$$\langle \mathcal{T}^- f, v \rangle^2_{0;\partial S} \le c b_-(u, u) \|v\|^2_{1/2;\partial S};$$

therefore, $\mathcal{T}^- f \in H_{-1/2}(\partial S)$ and

$$\begin{aligned}\|\mathcal{T}^- f\|^2_{-1/2;\partial S} &\le c b_-(u,u) = -c\langle \mathcal{T}^- f, f\rangle_{0;\partial S}\\ &\le c\|\mathcal{T}^- f\|_{-1/2;\partial S}\|f\|_{1/2;\partial S},\end{aligned}$$

from which

$$\|\mathcal{T}^- f\|_{-1/2;\partial S} \le c\|f\|_{1/2;\partial S}.$$

(ii) We take l^- to be the operator that associates with $v \in H_{1/2}(\partial S)$ the solution of corresponding problem (D^-), and (5.2) follows from (5.1).

(iii) If $z \in \mathcal{Z}(\partial S)$, then z is the solution of (D^-) and we have

$$\langle \mathcal{T}^- z, v\rangle_{0;\partial S} = -b_-(z, l^- v) = 0 \quad \text{for all} \quad z \in \mathcal{Z}(\partial S);$$

hence,

$$\mathcal{T}^- z = 0.$$

Conversely, if $\mathcal{T}^- f = 0$, then

$$\langle \mathcal{T}^- f, f\rangle_{0;\partial S} = -b_-(u,u) = 0;$$

therefore, $u \in \mathcal{F}$ and $f \in \mathcal{Z}(\partial S)$.

(iv) By (5.2), for any $f \in H_{1/2}(\partial S)$ and $z \in \mathcal{Z}(\partial S)$

$$\langle \mathcal{T}^- f, z\rangle_{0;\partial S} = -b_-(u,z) = 0,$$

so $\mathcal{T}^- f \in \mathcal{H}_{-1/2}(\partial S)$ for any $f \in H_{1/2}(\partial S)$. We define operator $\overset{\curlywedge}{\mathcal{T}}{}^-$ on the factor space $H_{1/2}(\partial S)\backslash\mathcal{Z}(\partial S)$ by

$$\overset{\curlywedge}{\mathcal{T}}{}^- F = \mathcal{T}^- f, \quad F \in H_{1/2}(\partial S)\backslash\mathcal{Z}(\partial S).$$

Clearly, $\overset{\curlywedge}{\mathcal{T}}{}^-$ is injective and its range coincides with the range of $\mathcal{T}^-$. We now show that inverse operator $(\overset{\curlywedge}{\mathcal{T}}{}^-)^{-1}$ is continuous. Let $F \in H_{1/2}(\partial S)\backslash\mathcal{Z}(\partial S)$, let f be representative of F such that

$$\langle f, z\rangle_{0;\partial S} = 0 \quad \text{for all} \quad z \in \mathcal{Z}(\partial S),$$

and let u be the solution of (D^-) with boundary data f. By Theorem 4, we have

$$\begin{aligned}\|F\|^2_{H_{1/2}(\partial S)\setminus\mathcal{Z}(\partial S)} &\le \|f\|^2_{1/2;\partial S} \le c\|u\|^2_{1,S^+}\\ &\le cb_-(u,u) = -c\langle \mathcal{T}^- f, f\rangle_{0;\partial S}\\ &\le c\|\mathcal{T}^- f\|_{-1/2;\partial S}\|f\|_{1/2;\partial S},\end{aligned}$$

from which it follows that

$$\|F\|_{H_{1/2}(\partial S)\setminus\mathcal{Z}(\partial S)} \le \|f\|_{1/2;\partial S} \le c\|\mathcal{T}^- f\|_{-1/2;\partial S} = c\|\overset{\curlywedge}{\mathcal{T}}{}^- f\|_{-1/2;\partial S}.$$

To prove that the range of $\mathcal{T}^-$ coincides with $\mathcal{H}_{-1/2}(\partial S)$, it suffices to establish that this range is dense in $\mathcal{H}_{-1/2}(\partial S)$. Suppose that this is not true. Then we can find nonzero Φ in the dual $H_{1/2}(\partial S)\setminus\mathcal{Z}(\partial S)$ of $\mathcal{H}_{-1/2}(\partial S)$ such that

$$\langle \mathcal{T}^- f, \varphi\rangle_{0;\partial S} = 0 \quad \text{for all} \quad f \in H_{1/2}(\partial S),$$

where φ is any representative of Φ. Taking $f = \varphi$, we arrive at

$$\langle \mathcal{T}^- \varphi, \varphi\rangle_{0;\partial S} = 0;$$

therefore, $\varphi \in \mathcal{Z}(\partial S)$ and $\Phi = 0$. This contradiction completes the proof. □

Let $\mathcal{N}^-$ be the restriction of $\mathcal{T}^-$ to $\mathcal{H}_{1/2}(\partial S)$.

Theorem 9. *The operator $\mathcal{N}^-$ is a homeomorphism from $\mathcal{H}_{1/2}(\partial S)$ to $\mathcal{H}_{-1/2}(\partial S)$.*

Proof. The bijectivity and continuity of $\mathcal{N}^-$ were established in the previous theorem, while the continuity of the inverse operator $(\mathcal{N}^-)^{-1}$ follows from Banach's theorem. □

6. Boundary equations. We define the modified single and double layer potentials $\mathcal{V}^-\varphi$ and $\mathcal{W}^-\psi$ and their corresponding boundary operators $\mathcal{V}_0\varphi$ and $\mathcal{W}_0^-\psi$ by

$$\begin{aligned}
\mathcal{V}^-\varphi &= V\varphi - \langle V_0\varphi, \widetilde{z}^{(i)}\rangle_{0;\partial S}\widetilde{z}^{(i)}, && \varphi \in \mathcal{H}_{-1/2}(\partial S),\\
\mathcal{V}_0\varphi &= V_0\varphi - \langle V_0\varphi, \widetilde{z}^{(i)}\rangle_{0;\partial S}\widetilde{z}^{(i)}, && \varphi \in \mathcal{H}_{-1/2}(\partial S),\\
\mathcal{W}^-\psi &= W\psi - \langle W_0^-\psi, \widetilde{z}^{(i)}\rangle_{0;\partial S}\widetilde{z}^{(i)}, && \psi \in \mathcal{H}_{1/2}(\partial S),\\
\mathcal{W}_0^-\psi &= W_0^-\psi - \langle W_0^-\psi, \widetilde{z}^{(i)}\rangle_{0;\partial S}\widetilde{z}^{(i)}, && \psi \in \mathcal{H}_{1/2}(\partial S),
\end{aligned}$$

where $V\varphi$ and $W\psi$ are the single and double layer potentials, $\{\widetilde{z}^{(i)}\}$ is the set obtained from $\{F^{(i)}\}$ by orthonormalization in $L^2(\partial S)$, and V_0 and W_0^- are the boundary operators defined by $V_0\varphi = \gamma^- V\varphi$ and $W_0^-\psi = \gamma^- W\psi$.

Theorem 10. *The operator $\mathcal{V}_0$, extended by continuity from the space $C^{0,\alpha}(\partial S)\cap\mathcal{H}_{-1/2}(\partial S)$ to $\mathcal{H}_{-1/2}(\partial S)$, is a homeomorphism from $\mathcal{H}_{-1/2}(\partial S)$ to $\mathcal{H}_{1/2}(\partial S)$.*

This theorem can be proved following procedure similar to the proof of Theorem 3.5 in [**2**].

Theorem 11. *The operator $\mathcal{W}_0^-$, extended by continuity from the space $C^{1,\alpha}(\partial S)\cap\mathcal{H}_{1/2}(\partial S)$ to $\mathcal{H}_{1/2}(\partial S)$, is a homeomorphism from $\mathcal{H}_{-1/2}(\partial S)$ to $\mathcal{H}_{1/2}(\partial S)$.*

The proof of this assertion can be obtained following procedure shown in the proof of the Theorem 3.9 in [**2**].

We write

$$\mathcal{G}^- = \mathcal{N}^-\mathcal{W}_0^-.$$

Theorem 12. *$\mathcal{G}^-$ is a homeomorphism from $\mathcal{H}_{1/2}(\partial S)$ to $\mathcal{H}_{-1/2}(\partial S)$.*

Proof. The theorem follows from properties of the operators $\mathcal{N}^-$ and $\mathcal{W}_0^-$. $\square$

We start with (D$^-$). As it was shown in [**2**], we may consider (2.2) with $q = 0$ without loss of generality. Thus, we arrive at variational formulation of (D$^-$) for the homogeneous equilibrium equation of finding $u \in H_{1,\omega}(S^-)$ such that

$$\begin{aligned} b_-(u, v) &= 0 \quad \text{for all} \quad v \in \overset{\circ}{H}_{1,\omega}(S^-) \\ \gamma^- u &= f. \end{aligned}$$

We represent the solution of (D$^-$) with $q = 0$ in the form

$$u = \mathcal{V}^- \varphi + z, \tag{6.1}$$

where the density $\varphi \in \mathcal{H}_{-1/2}(\partial S)$ and $z \in \mathcal{F}$ are unknown. Representation (6.1) leads to the system of boundary equations

$$\mathcal{V}_0 \varphi + z = f. \tag{6.2}$$

Representing the weak solution of (D$^-$) with $q = 0$ in the form

$$u = \mathcal{W}^- \psi + z, \tag{6.3}$$

where $\psi \in \mathcal{H}_{1/2}(\partial S)$ and $z \in \mathcal{F}$ are unknown, we obtain the following system of boundary equations

$$\mathcal{W}_0^- \psi + z = f. \tag{6.4}$$

Theorem 13. *Systems* (6.2) *and* (6.4) *have unique solutions*

$$\begin{aligned} \{\varphi, z\} &\in \mathcal{H}_{-1/2}(\partial S) \times \mathcal{Z}(\partial S), \\ \{\psi, z\} &\in \mathcal{H}_{1/2}(\partial S) \times \mathcal{Z}(\partial S), \end{aligned}$$

respectively, for any $f \in H_{1/2}(\partial S)$, *and*

$$\begin{aligned} \|\varphi\|_{-1/2;\partial S} &\leq c\|f\|_{1/2;\partial S}, \\ \|\psi\|_{1/2;\partial S} &\leq c\|f\|_{1/2;\partial S}. \end{aligned} \tag{6.5}$$

In this case, (6.1) *and* (6.3) *are the solutions of problem* (D^-) *with* $q = 0$*, and they satisfy the estimate*

$$\|u\|_{1,\omega;S^-} \leq c\|f\|_{1/2;\partial S}. \tag{6.6}$$

Proof. In both cases we choose $z \in \mathcal{Z}(\partial S)$ defined by

$$z = -\langle f, \mathcal{F}^{(i)}\rangle_{0;\partial S}\mathcal{F}^{(i)}.$$

Then it is obvious that $f - z \in \mathcal{H}_{1/2}(\partial S)$ and

$$\|f - z\|_{1/2;\partial S} \leq c\|f\|_{1/2;\partial S}. \tag{6.7}$$

The solvability of systems (6.2), (6.4) and estimates (6.5) now follow from Theorems 10, 11 and (6.7). To prove uniqueness, let $\{\varphi_1, z_1\}$ and $\{\psi_1, z_1\}$ be other solutions of (6.2) and (6.4), respectively. Then, writing

$$\breve{\varphi} = \varphi - \varphi_1, \quad \breve{\psi} = \psi - \psi_1, \quad \breve{z} = z - z_1,$$

we see that

$$\mathcal{V}_0\breve{\varphi} + \breve{z} = 0,$$
$$\mathcal{W}_0^-\breve{\psi} + \breve{z} = 0.$$

Since $\mathcal{V}_0\breve{\varphi}$ and $\breve{z}$ belong to L^2-orthogonal subspaces of $\mathcal{H}_{1/2}(\partial S)$, it follows that

$$\mathcal{V}_0\breve{\varphi} = 0, \quad \breve{z} = 0;$$

therefore, we also have $\breve{\varphi} = 0$.

The proof that $\breve{\psi} = 0$ and $\breve{z} = 0$ in the second case is similar.

Estimate (6.6) follows from (6.7) and Theorem 4. □

Instead of (N^-) we may consider without loss of generality its version for the homogeneous equilibrium equations.

In the problem (N^-) with $q = 0$ we seek $u \in H_{1,\omega}(S^-)$ such that

$$b_-(u, v) = -\langle g, v\rangle_{0;\partial S} \quad \text{for all} \quad v \in H_{1,\omega}(S^-).$$

We write

$$\mathcal{K}^- = \mathcal{N}^- \mathcal{V}_0$$

and represent the solution of problem (N$^-$) with $q = 0$ in the form

$$u = \mathcal{V}^- \varphi + z, \tag{6.8}$$

where $z \in \mathcal{F}$. Representation (6.8) leads to the systems of boundary equations

$$\mathcal{K}^- \varphi = g. \tag{6.9}$$

If we represent the solution as

$$u = \mathcal{W}^- \psi + z, \tag{6.10}$$

then we arrive at the system of boundary equations

$$\mathcal{G}^- \psi = g. \tag{6.11}$$

Theorem 14. *Systems* (6.9) *and* (6.11) *have unique solutions* $\varphi \in \mathcal{H}_{-1/2}(\partial S)$ *and* $\psi \in \mathcal{H}_{1/2}(\partial S)$ *for any* $g \in H_{-1/2}(\partial S)$, *and*

$$\|\varphi\|_{-1/2;\partial S} \leq c\|g\|_{-1/2;\partial S}, \tag{6.12}$$

$$\|\psi\|_{1/2;\partial S} \leq c\|g\|_{-1/2;\partial S}. \tag{6.13}$$

In this case, (6.8) *and* (6.10) *are the solutions of problem* (N^-) *with* $q = 0$, *and they satisfy the estimate*

$$\|u\|_{1,\omega;S^-} \leq c\|f\|_{-1/2;\partial S}. \tag{6.14}$$

Proof. The unique solvability of (6.9) and (6.12) follows from the properties of the operators $\mathcal{N}^-$ and $\mathcal{V}_0$ established in Theorems 9 and 10. The unique solvability of (6.11) and (6.13) follows from Theorem 12. Finally, estimate (6.14) is obtained from Theorem 4. □

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