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ON A BOUNDARY VALUE PROBLEM IN SUBSONIC AEROELASTICITY AND THE COFINITE HILBERT TRANSFORM

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ABSTRACT. We study a boundary value problem in subsonic aeroelasticity and introduce the *cofinite Hilbert transform* as a tool for solving an auxiliary linear integral equation on the complement of a finite interval on the real line $\mathbf{R}$.

1. Introduction. We consider the linearized subsonic inviscid compressible flow equation in two dimensions [**2, 3**]

$$a^2\left(1-M^2\right)\frac{\partial^2\phi}{\partial x^2}+a^2\frac{\partial^2\phi}{\partial z^2}=\frac{\partial^2\phi}{\partial t^2}+2Ma\,\frac{\partial^2\phi}{\partial t\partial x}, \tag{1}$$

where a is the speed of sound, M is the Mach number ($M=U/a<1$, where U is the free stream velocity), and $\phi(x,z,t)$ is a small disturbance velocity potential, considered on

$$\mathbf{R}_+^2\times\overline{\mathbf{R}_+}=\left\{(x,z,t):-\infty<x<\infty,\ 0<z<\infty,\ 0\le t<\infty\right\}.$$

This velocity potential is assumed to satisfy the boundary conditions:

- flow tangency condition

$$\frac{\partial\phi}{\partial z}(x,0,t)=w(x,t),\quad |x|<b, \tag{2}$$

where b is the "half-chord," and w is the given normal velocity of the wing, without loss of generality we will assume in what follows that $b=1$,

- "*strong Kutta-Joukowski condition*" for the acceleration potential

$$\begin{gathered}\psi(x,z,t):=\frac{\partial\phi}{\partial t}+U\,\frac{\partial\phi}{\partial x},\\ \psi(x,0,t)=0\quad\text{for}\quad 1<|x|<A\quad\text{for some}\quad A>1,\end{gathered} \tag{3}$$

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• far field condition

$$\phi(x,z,t) \longrightarrow 0, \quad \text{as} \quad |x| \to \infty, \quad \text{or} \quad z \to \infty.$$

Boundary condition (3), though being motivated by one of the "auxiliary boundary conditions" from [**3**, p. 319], is weaker because it requires that $\psi(x,0,t) = 0$ not on the entire complement $\mathbf{R}\setminus[-1,1]$, but only on finite intervals adjacent to the interval $[-1,1]$. On the other hand, this change in the boundary condition allows application of some new mathematical tools different from tools in [**2, 3**]. We do not present here any physical motivation for the conditions above, referring instead to the book [**3**], where these conditions are given physical interpretations.

In order to formulate our main result we introduce the following notations. We denote by $\widehat{w}$ the Laplace transform of the function w with respect to time variable

$$\widehat{w}(x,z,\lambda) = \int_0^\infty e^{-\lambda t} w(x,z,t)\, dt$$

for $\mathrm{Re}\,\lambda > \sigma_0 > 0$. We also denote

$$r(\lambda) = \frac{\lambda M}{U\sqrt{1-M^2}},$$
$$d(\lambda) = \frac{\lambda M^2}{U(1-M^2)}.$$

In Sections 5 and 6 we construct a function $\mathcal{D}_N(\lambda)$, which is the Fredholm determinant, defined in (35), of the operator $\mathcal{N}_\lambda$, defined in (25). This function is analytic in the half-plane $\mathrm{Re}\,\lambda > \sigma_0 > 0$, and, as the function $R(x,\lambda)$ defined in (15), depends basically only on the function K_0–the modified Bessel function of the third kind.

The following theorem represents the main result of the paper.

Theorem 1. *Let the function $\mathcal{D}_N(\lambda)$ from equation (42), mentioned above, have no zeros in the strip $\{\mathrm{Re}\,\lambda \in [\sigma_1, \sigma_2]\}$, where $\sigma_1 > \sigma_0$. Let $I(1) = [-1,1]$, and let $w(\cdot,t) \in L^2(I(1))$ be such that for some $\varepsilon > 0$*

(4)

$$\|\widehat{w}(\cdot, \sigma + i\eta)\|_{L^2(I(1))} < \exp\left\{-e^{|\eta|} \cdot (1+|\eta|)^{2+\varepsilon}\right\} \quad \text{for} \quad \sigma \in [\sigma_1, \sigma_2].$$

Then equation (1) *has a solution of the form*

$$\phi(x,z,t) = -\frac{1}{2\pi\sqrt{1-M^2}} \times \int_{-\infty}^{\infty} e^{d(\sigma'+i\eta)x} \left[\int_{-\infty}^{\infty} K_0\left(r(\sigma'+i\eta) \left(\frac{(x-y)^2}{1-M^2} + z^2 \right)^{1/2} \right) \times h(y,\sigma'+i\eta)\, dy \right] e^{(\sigma'+i\eta)t}\, d\eta.$$

This solution is independent of $\sigma' \in [\sigma_1,\sigma_2]$, *satisfies boundary conditions above, and the function* h *satisfies the estimate*

$$\int_{-\infty}^{\infty} (1+|x|)^{p-2} \left|h(x,\sigma'+i\eta)\right|^p dx < \frac{C(m)}{(1+|\eta|)^m}$$

for arbitrary $m > 0$, $p < 4/3$, *and* $C(m) > 0$ *independent of* η.

In the course of the proof of Theorem 1 we deal with the question of invertibility or "almost invertibility" of a singular integral operator of the form: "Hilbert transform + integral operator with logarithmic kernel" on a finite interval. Almost invertibility of the pure Hilbert transform on a finite interval or of the so called *finite Hilbert transform* [**23**] in spaces L^p was investigated in [**19, 23**]. From the computational standpoint invertibility of singular integral operators mentioned above on weighted Lebesgue spaces on a finite interval was analyzed for example in [**15, 16**], where further references can be found. In the present paper we, among other things, give a rigorous discussion of invertibility of such operators on Lebesgue spaces on a finite interval. In a forthcoming paper we combine the technique developed here with the Possio construction to prove a criterion of solvability of the so-called Possio integral equation [**3**]. The *cofinite Hilbert transform*, considered in the paper, is just the Hilbert transform taken over the complement to a finite interval, and it is a special case of the Hilbert transform or of the Cauchy integral on a curve in $\mathbf{C} = \mathbf{R}^2$. Properties of this general transform are discussed in many papers and monographs, see for example, [**6, 8, 10, 14, 17, 20**] and the references therein.

2. "General" solution. We are seeking a solution of equation (1) of the form

$$\phi(x,z,t) = \int_{-\infty}^{\infty} \xi(x,z,\lambda) e^{(\sigma+i\eta)t}\, d\eta, \tag{6}$$

where $\lambda = \sigma + i\eta$, $\sigma > \sigma_0$ and $\xi(x,z,\lambda) \in L^1(\mathbf{R}_\eta)$. Then, substituting the expression above into equation (1), we find that the following auxiliary equation for ξ

$$a^2\left(1-M^2\right)\frac{\partial^2\xi}{\partial x^2} + a^2\frac{\partial^2\xi}{\partial z^2} - \lambda^2\xi - 2M\lambda a\frac{\partial\xi}{\partial x} = 0 \tag{7}$$

is sufficient to ensure that ϕ satisfies (1).

To describe the general solution of equation (7) satisfying the far field condition we consider, following [**2**]

$$D(\omega,\lambda) = M^2\left(\frac{\lambda}{U}\right)^2 + 2i\,\frac{\lambda}{U}\,M^2\omega + \left(1-M^2\right)\omega^2,$$

and prove two lemmas below.

Lemma 2.1. *There exists a function $\sqrt{D(\omega,\lambda)}$, analytic with respect to complex variable $\lambda/U + i\omega((\lambda/U) \in \mathbf{C},\ \omega \in \mathbf{R})$ in the half-plane $\operatorname{Re}\lambda > \sigma_0$, and such that $\operatorname{Re}\sqrt{D(\omega,\lambda)} > 0$.*

Proof. Representing $D(\omega,\lambda)$ as

$$\begin{aligned} D(\omega,\lambda) &= M^2\left(\frac{\lambda}{U}\right)^2 + 2i\,\frac{\lambda}{U}\,M^2\omega + \left(1-M^2\right)\omega^2 \\ &= M^2\left(\frac{\lambda}{U} + i\omega\right)^2 + \omega^2, \end{aligned}$$

we obtain that the image of the half-plane $\operatorname{Re}\lambda > \sigma_0$ under the map $D(\omega,\lambda)$ is contained in the domain $\mathbf{C}\setminus\mathbf{R}^-$. Then the branch of the function $\sqrt{\ }$ considered on the complex plane with the cut along the negative part of the real axis is well defined and analytic on the image of D, and its real part satisfies the condition of the lemma. Therefore, the composition $\sqrt{D}$ is also analytic and satisfies the same condition. □

Lemma 2.2. *The following equality holds*

$$\frac{e^{d(\lambda)x}}{\sqrt{1-M^2}}\,K_0\biggl(r(\lambda)\Bigl(\frac{x^2}{1-M^2}+z^2\Bigr)^{1/2}\biggr)=\mathcal{F}\Bigl[\frac{e^{-z\sqrt{D(\omega,\lambda)}}}{2\sqrt{D(\omega,\lambda)}}\Bigr],$$

where $\mathcal{F}$ denotes the Fourier transform, that is,

$$\frac{e^{d(\lambda)x}}{\sqrt{1-M^2}}\,K_0\biggl(r(\lambda)\Bigl(\frac{x^2}{1-M^2}+z^2\Bigr)^{1/2}\biggr)$$
$$=\int_{-\infty}^{\infty}e^{ix\omega}\,\frac{e^{-z\left((1-M^2)(\omega+id(\lambda))^2+r^2(\lambda)\right)^{1/2}}}{2\sqrt{(1-M^2)(\omega+id(\lambda))^2+r^2(\lambda)}}\,d\omega.$$

Proof. First, we represent $D\,(\omega,\lambda)$ as

$$\begin{aligned}D\,(\omega,\lambda)&=\left(1-M^2\right)\omega^2+2i\,\frac{\lambda}{U}\,M^2\omega+M^2\left(\frac{\lambda}{U}\right)^2\\&=\left(\omega\sqrt{1-M^2}+i\,\frac{\lambda M^2}{U\sqrt{1-M^2}}\right)^2+\left(\frac{\lambda M^2}{U\sqrt{1-M^2}}\right)^2+M^2\left(\frac{\lambda}{U}\right)^2\\&=(1-M^2)(\omega+id(\lambda))^2+r^2(\lambda).\end{aligned}$$

Changing variables in equality [**9**]

$$K_0\left(r(x^2+z^2)^{1/2}\right)=\int_{-\infty}^{\infty}e^{ixu}\,\frac{e^{-z\left(u^2+r^2\right)^{1/2}}}{2\sqrt{u^2+r^2}}\,du,$$

we obtain

$$K_0\left(r(x^2+z^2)^{1/2}\right)$$
$$=\int_{-\infty}^{\infty}e^{ix\sqrt{1-M^2}\omega}\;\frac{e^{-z\left((1-M^2)\omega^2+r^2\right)^{1/2}}}{2\sqrt{(1-M^2)\omega^2+r^2}}\,d\left(\sqrt{1-M^2}\,\omega\right),$$

and then

$$\frac{1}{\sqrt{1-M^2}}\,K_0\left(r\left(\frac{x^2}{1-M^2}+z^2\right)^{1/2}\right) = \int_{-\infty}^{\infty} e^{ix\omega}\,\frac{e^{-z\left((1-M^2)\omega^2+r^2\right)^{1/2}}}{2\sqrt{(1-M^2)\omega^2+r^2}}\,d\omega.$$

We transform the equality above by integrating the function

$$g(x,w) = e^{ixw}\,\frac{e^{-z\left((1-M^2)w^2+r^2\right)^{1/2}}}{\sqrt{(1-M^2)w^2+r^2}}, \quad w \in \mathbf{C},$$

which is analytic with respect to w, over the closed rectangular contour

$$[-C, C, C+id, -C+id] \in \mathbf{C}, \quad \text{with} \quad C \in \mathbf{R}, \quad C > 0, \quad d \in \mathbf{C}, \quad \operatorname{Re} d > 0.$$

Then we obtain

$$\int_{-C}^{C} g(x,w)\,dw + \int_{C}^{C+id} g(x,w)\,dw + \int_{C+id}^{-C+id} g(x,w)\,dw + \int_{-C+id}^{-C} g(x,w)\,dw = 0. \tag{8}$$

For C large enough we have the following estimates for $w = u+iv \in [-C, -C+id]$, and $w \in [C, C+id]$

$$\left|e^{ix(u+iv)}\right| < e^{|x|\cdot\operatorname{Re} d}, \qquad \left|\sqrt{(1-M^2)w^2+r^2}\right| > \sqrt{1-M^2}\,\frac{C}{2},$$
$$\left|e^{-z\left((1-M^2)w^2+r^2\right)^{1/2}}\right| < e^{-z\sqrt{1-M^2}(C/2)},$$

and therefore for $z > 0$

$$\left|\int_{C}^{C+id} g(x,w)\,dw\right|, \ \left|\int_{-C}^{-C+id} g(x,w)\,dw\right| \longrightarrow 0 \quad \text{as} \quad C \to \infty.$$

Using the last estimate in (8) we obtain the equation

$$\int_{-\infty}^{\infty} e^{ix\omega}\,\frac{e^{-z\left((1-M^2)\omega^2+r^2(\lambda)\right)^{1/2}}}{\sqrt{(1-M^2)\omega^2+r^2(\lambda)}}\,d\omega$$
$$=\int_{-\infty}^{\infty} e^{ix(\omega+id(\lambda))}\,\frac{e^{-z\left((1-M^2)(\omega+id(\lambda))^2+r^2(\lambda)\right)^{1/2}}}{\sqrt{(1-M^2)(\omega+id(\lambda))^2+r^2(\lambda)}}\,d(\omega+id(\lambda)),$$

and finally

$$\frac{e^{d(\lambda)x}}{\sqrt{1-M^2}}\,K_0\left(r(\lambda)\left(\frac{x^2}{1-M^2}+z^2\right)^{1/2}\right) = \int_{-\infty}^{\infty} e^{ix\omega}\,\frac{e^{-z\left((1-M^2)(\omega+id(\lambda))^2+r^2(\lambda)\right)^{1/2}}}{2\sqrt{(1-M^2)(\omega+id(\lambda))^2+r^2(\lambda)}}\,d\omega. \quad \square \tag{9}$$

Using Lemmas 2.1 and 2.2, we now consider a special representation of the general solution of (7). Namely, using notations of Lemma 2.2, and denoting

$$S(x,z,\lambda) = -\frac{e^{d(\lambda)x}}{\sqrt{1-M^2}}\,K_0\left(r(\lambda)\left(\frac{x^2}{1-M^2}+z^2\right)^{1/2}\right),$$

we consider

$$\begin{aligned}\xi(x,z,\lambda) &= \int_{-\infty}^{\infty} S(x-y,z,\lambda)v(y,\lambda)\,dy \\ &= -\frac{e^{d(\lambda)x}}{\sqrt{1-M^2}}\int_{-\infty}^{\infty} e^{-d(\lambda)y}K_0\left(r(\lambda)\left(\frac{(x-y)^2}{1-M^2}+z^2\right)^{1/2}\right)v(y,\lambda)\,dy,\end{aligned} \tag{10}$$

where v is an arbitrary function such that

$$e^{-d(\lambda)y}v(y,\lambda)\in L^p(\mathbf{R})$$

for some $p>1$ and λ, with $\operatorname{Re}\lambda>\sigma_0$ in order for the expression above to be well defined.

Proposition 2.3. *Function ξ defined by formula* (10) *satisfies equation* (7). *If*

$$\begin{aligned} &\xi(x,z,\lambda), \quad \frac{\partial^2\xi}{\partial x^2}(x,z,\lambda), \quad \frac{\partial^2\xi}{\partial z^2}(x,z,\lambda),\\ &|\eta|^2\xi(x,z,\lambda), \quad |\eta|\,\frac{\partial\xi(x,z,\lambda)}{\partial x} \in L^1(\mathbf{R}_\eta), \end{aligned}$$

where $\lambda = \sigma + i\eta$, then the inverse Laplace transform of ξ, defined by the formula [**5**]

$$\phi(x,z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(\sigma+i\eta)t}\,\xi(x,z,\sigma+i\eta)\,d\eta \tag{11}$$

satisfies equation (1).

Proof. To prove that ξ defined above satisfies equation (7), it suffices to prove that the function S satisfies the same equation. For S we have

$$\begin{aligned} &a^2\left(1-M^2\right)\frac{\partial^2 S}{\partial x^2} + a^2\frac{\partial^2 S}{\partial z^2} - \lambda^2 S - 2M\lambda a\,\frac{\partial S}{\partial x}\\ &\quad = a^2\left[\frac{\partial^2 S}{\partial z^2} + \left(1-M^2\right)\frac{\partial^2 S}{\partial x^2} - M^2\left(\frac{\lambda}{U}\right)^2 S - 2M^2\,\frac{\lambda}{U}\,\frac{\partial S}{\partial x}\right]. \end{aligned}$$

Using formula (9), we then obtain

$$\begin{aligned} &\frac{\partial^2 S}{\partial z^2} + \left(1-M^2\right)\frac{\partial^2 S}{\partial x^2} - M^2\left(\frac{\lambda}{U}\right)^2 S - 2M^2\,\frac{\lambda}{U}\,\frac{\partial S}{\partial x}\\ &= -\int_{-\infty}^{\infty} e^{ix\omega}\left((1-M^2)(\omega+id)^2 + r^2\right)\frac{e^{-z\left((1-M^2)(\omega+id)^2+r^2\right)^{1/2}}}{2\sqrt{(1-M^2)(\omega+id)^2+r^2}}\,d\omega\\ &\quad + \int_{-\infty}^{\infty} e^{ix\omega}\left(1-M^2\right)\omega^2\,\frac{e^{-z\left((1-M^2)(\omega+id)^2+r^2\right)^{1/2}}}{2\sqrt{(1-M^2)(\omega+id)^2+r^2}}\,d\omega \end{aligned}$$

$$+\int_{-\infty}^{\infty} e^{ix\omega} M^2 \left(\frac{\lambda}{U}\right)^2 \frac{e^{-z\left((1-M^2)(\omega+id)^2+r^2\right)^{1/2}}}{2\sqrt{(1-M^2)(\omega+id)^2+r^2}}\, d\omega$$
$$+\int_{-\infty}^{\infty} e^{ix\omega} 2M^2 \,\frac{\lambda}{U}\, i\omega\, \frac{e^{-z\left((1-M^2)(\omega+id)^2+r^2\right)^{1/2}}}{2\sqrt{(1-M^2)(\omega+id)^2+r^2}}\, d\omega = 0.$$

To prove that the function ϕ defined by formula (11) satisfies equation (1), we apply the inverse Laplace transform to equality

$$a^2\left(1-M^2\right)\frac{\partial^2\xi}{\partial x^2} + a^2\,\frac{\partial^2\xi}{\partial z^2} - \lambda^2\xi - 2M\lambda a\,\frac{\partial\xi}{\partial x} = 0$$

and obtain equation (1) for ϕ. □

3. Boundary conditions. In this section we reformulate the boundary conditions of Section 1 in terms of the function $v(y,\lambda)$ from formula (10).

To check the flow tangency condition (2), we use formulas (10) and (9) and obtain

$$\begin{aligned}\frac{\partial}{\partial z}\,\xi(x,z,\lambda)\Big|_{z=0} &= \frac{\partial}{\partial z}\int_{-\infty}^{\infty} S(x-y,z,\lambda)v(y,\lambda)\,dy\Big|_{z=0}\\ &= -\frac{\partial}{\partial z}\int_{-\infty}^{\infty} v(y,\lambda)\,dy\\ &\quad\times\int_{-\infty}^{\infty} e^{i(x-y)\omega}\,\frac{e^{-z\left((1-M^2)(\omega+id(\lambda))^2+r^2(\lambda)\right)^{1/2}}}{2\sqrt{(1-M^2)(\omega+id(\lambda))^2+r^2(\lambda)}}\,d\omega\Big|_{z=0}\\ &= \frac{1}{2}\int_{-\infty}^{\infty} e^{ix\omega}\,d\omega\int_{-\infty}^{\infty} e^{-iy\omega}v(y,\lambda)\,dy = \pi\cdot v(x,\lambda),\end{aligned}$$

which, after comparison with equality (2) leads to a unique choice

$$v(x,\lambda) = \frac{1}{\pi}\,\widehat{w}(x,\lambda) \quad \text{for} \quad |x|<1. \tag{13}$$

To satisfy the Kutta-Joukowski boundary condition (3) we should have

$$\left(\frac{\partial\phi}{\partial t} + U\,\frac{\partial\phi}{\partial x}\right)\Big|_{z=0} = 0 \quad \text{for} \quad 1<|x|<A,$$

or equality

$$\lambda\xi(x,0,\lambda) + U\,\frac{\partial\xi}{\partial x}\,(x,0,\lambda) = 0 \quad \text{for} \quad 1 < |x| < A$$

for the function ξ.

Substituting ξ from formula (10) into the equality above, we obtain the following condition for $1 < |x| < A$:

$$\begin{aligned}
0 &= \left(\lambda + U\frac{\partial}{\partial x}\right)\xi(x,0,\lambda)\\
&= -\left(\lambda + U\frac{\partial}{\partial x}\right)\frac{e^{d(\lambda)x}}{\sqrt{1-M^2}}\\
&\quad \times \int_{-\infty}^{\infty} e^{-d(\lambda)y} K_0\left(r(\lambda)\left(\frac{(x-y)^2}{1-M^2} + z^2\right)^{1/2}\right) v(y,\lambda)\,dy\Big|_{z=0}.
\end{aligned} \tag{14}$$

To reformulate the last condition as an integral equation, we use condition (13) and define

$$g(x,\lambda) = \frac{e^{d(\lambda)x}}{\pi}\int_{-1}^{1} e^{-d(\lambda)y} R(x-y,\lambda)\widehat{w}(y,\lambda)\,dy \quad \text{for} \quad 1 < |x| < A,$$

with kernel $R(x,\lambda)$ defined by the formula

$$R(x,\lambda) = \left[(\lambda + Ud(\lambda))\,K_0\left(\frac{r(\lambda)|x|}{\sqrt{1-M^2}}\right) + U\frac{\partial}{\partial x}\,K_0\left(\frac{r(\lambda)|x|}{\sqrt{1-M^2}}\right)\right]. \tag{15}$$

Then condition (14) will be satisfied if v satisfies the following integral equation

$$e^{d(\lambda)x}\int_{|y|>1} e^{-d(\lambda)y} R(x-y,\lambda)v(y,\lambda)\,dy = -g(x,\lambda) \quad \text{for} \quad 1 < |x| < A.$$

Further simplifying the equation above, we consider $h(y,\lambda) := e^{-d(\lambda)y}\cdot v(y,\lambda)$ as an unknown function and rewrite it as

$$\int_{|y|>1} R(x-y,\lambda)h(y,\lambda)\,dy = f(x,\lambda) \quad \text{for} \quad 1 < |x| < A, \tag{16}$$

where $f(x,\lambda) = -e^{-d(\lambda)x} \cdot \chi_A(x) g(x,\lambda)$ is defined for

$$\{(x,\lambda) \in \mathbf{R} \times \mathbf{C} : |x| > 1,\ \operatorname{Re}\lambda \in [\sigma_1, \sigma_2]\}$$

by the formula

$$f(x,\lambda) = -\frac{\chi_A(x)}{\pi} \int_{-1}^{1} e^{-d(\lambda)y} R(x-y,\lambda) \widehat{w}(y,\lambda)\, dy \tag{17}$$

with

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in [-A, A] \setminus [-1,1], \\ 0 & \text{otherwise.} \end{cases}$$

4. Cofinite Hilbert transform. As a first step in the analysis of solvability of (16), we prove solvability for the operator, closely related to operator $\mathcal{R}_\lambda$ from (16), and which in analogy with Tricomi's definition of the finite Hilbert transform [**23**] we call the *cofinite Hilbert transform.*

We define the cofinite Hilbert transform on the set of functions on

$$I^c(1) = \mathbf{R} \setminus I(1) = \mathbf{R} \setminus [-1,1]$$

by the formula

$$\mathcal{P}[h](x) = \frac{1}{\pi} \int_{|y|>1} \frac{h(y)}{y-x}\, dy \quad \text{for} \quad |x| > 1, \tag{18}$$

where the integral

$$\int_{|y|>1} = \int_{-\infty}^{-1} + \int_{1}^{\infty}$$

is understood in the sense of Cauchy's principal value.

In the proposition below we prove solvability for the nonhomogeneous integral equation with operator $\mathcal{P}$ in weighted spaces

$$\mathcal{L}^p\left(I^c(1)\right) = \left\{ f : \int_{|x|>1} |x|^{p-2} \left|f(x)\right|^p dx < \infty \right\}$$

with

$$\|f\|_{\mathcal{L}^p(I^c(1))} = \left(\int_{|x|>1} |x|^{p-2} \, |f(x)|^p \; dx \right)^{1/p}.$$

Proposition 4.1. *For any function $f \in \mathcal{L}^q\left(I^c(1)\right)$ with $q > 4/3$, there exists a solution h of equation*

$$\mathcal{P}[h] = f, \tag{19}$$

such that $h \in \mathcal{L}^p\left(I^c(1)\right)$ for any $p < 4/3$.

Proof. We consider the following diagram of transformations

$$\begin{array}{ccc} L^p\left(I(1)\right) & \xrightarrow{\ -\mathcal{T}\ } & L^p\left(I(1)\right) \\ \Big\downarrow \Theta & & \Big\downarrow \Theta \\ \mathcal{L}^p\left(I^c(1)\right) & \xrightarrow{\ \mathcal{P}\ } & \mathcal{L}^p\left(I^c(1)\right), \end{array} \tag{20}$$

where $\mathcal{T}$ is the finite Hilbert transform, $\mathcal{P}$ is the cofinite Hilbert transform, and

$$\Theta : L^p\left(I(1)\right) \longrightarrow \mathcal{L}^p\left(I^c(1)\right)$$

is defined by the formula

$$\Theta[f](x) = \frac{1}{x}\, f\left(\frac{1}{x}\right). \tag{21}$$

To prove that the maps in diagram (20) are well defined, we use equality

$$\begin{aligned} \|\Theta[f]\|^p_{\mathcal{L}^p} &= \int_{|x|>1} |x|^{p-2} \; |\Theta\,[f](x)|^p \; dx \\ &= \int_{|x|>1} |x|^{p-2} \, \frac{|f\left(1/x\right)|^p}{|x|^p} \, dx \\ &= -\int_1^{-1} |f\left(t\right)|^p \; dt = \|f\|^p_p, \end{aligned}$$

and notice that, for

$$\Theta^* : \mathcal{L}^p\left(I^c(1)\right) \to L^p\left(I(1)\right)$$

defined by the same formula

$$\Theta^*[f](x) = \frac{1}{x}\, f\left(\frac{1}{x}\right),$$

we have

$$\Theta \circ \Theta^*\left[f\right](x) = \Theta\left[\frac{1}{y}\, f\left(\frac{1}{y}\right)\right](x) = \frac{1}{x} \cdot x f(x) = f(x). \tag{22}$$

Diagram (20) is commutative, as can be seen from equality

$$\begin{aligned}
\mathcal{P}\left[\Theta\left[f\right]\right](x) &= \frac{1}{\pi}\int_{|y|>1} \frac{f\left(1/y\right)}{y(y-x)}\, dy \\
&= \frac{1}{\pi}\int_{-1}^{1} t\, \frac{f\left(t\right)}{((1/t)-x)t^2}\, dt \\
&= \frac{1}{\pi x}\int_{-1}^{1} \frac{f\left(t\right)}{(1/x)-t}\, dt \\
&= \Theta\left[-\mathcal{T}[f]\right].
\end{aligned}$$

To "invert" operator $\mathcal{P}$, we use commutativity of diagram (20), relation (22), and operator [**19, 23**]

$$\mathcal{T}^{-1} : L^p\left(I(1)\right) \to L^q\left(I(1)\right) \quad \text{with} \quad p > 4/3 \quad \text{and} \quad q < 4/3,$$

defined by the formula

$$\mathcal{T}^{-1}[g](x) = -\frac{1}{\pi}\int_{-1}^{1} \sqrt{\frac{1-y^2}{1-x^2}}\, \frac{g(y)}{y-x}\, dy,$$

and satisfying

$$\mathcal{T} \circ \mathcal{T}^{-1}[f] = f.$$

Namely, we define the operator

$$\mathcal{P}^{-1} : \mathcal{L}^p\left(I^c(1)\right) \to \mathcal{L}^q\left(I^c(1)\right) \quad \text{with} \quad p > 4/3 \quad \text{and} \quad q < 4/3,$$

by the formula

$$\mathcal{P}^{-1}[f] = -\,\Theta \circ \mathcal{T}^{-1} \circ \Theta^*[f].$$

Then

$$\begin{aligned}\mathcal{P} \circ \mathcal{P}^{-1}[f] &= -\,\mathcal{P} \circ \Theta \circ \mathcal{T}^{-1} \circ \Theta^*[f] = \Theta \circ \mathcal{T} \circ \mathcal{T}^{-1} \circ \Theta^*[f] \\ &= \Theta \circ \Theta^*[f] = f,\end{aligned}$$

and we obtain the statement of the proposition for

$$h = \mathcal{P}^{-1}[f].$$

To find an explicit formula for $\mathcal{P}^{-1}$, we use explicit formulas for Θ and $\mathcal{T}^{-1}$ and obtain

$$\begin{aligned}\mathcal{P}^{-1}[f](x) &= \frac{1}{\pi x}\int_{-1}^{1}\sqrt{\frac{1-y^2}{1-1/x^2}}\cdot\frac{f(1/y)}{y(y-1/x)}\,dy \\ &\frac{|x|}{\pi}\int_{-1}^{1}\sqrt{\frac{1-y^2}{x^2-1}}\left[\frac{1}{y}\,f\left(\frac{1}{y}\right)\right]\frac{dy}{xy-1}. \qquad \square\end{aligned} \tag{23}$$

Remark. Following [**23**] we notice that the solution of equation (19) is unique in $\mathcal{L}^2\left(I^c(1)\right)$ but is not unique in larger spaces. Namely, the function

$$h(x) = \frac{1}{\sqrt{x^2-1}}$$

is a solution, and the only one up to the linear dependence in $\mathcal{L}^p\left(I^c(1)\right)$, $1 < p < 2$, of the homogeneous equation

$$\mathcal{P}[h] = 0. \qquad \square$$

5. Solvability of equation (16). From the asymptotic expansions of $K_0(\zeta)$, see [**9, 11**], we obtain the following representations of the function $R(x,\lambda)$ for λ such that $\mathrm{Re}\,\lambda \in [\sigma_1, \sigma_2]$ with $\sigma_1 > \sigma_0$:

$$\begin{aligned}R(x,\lambda) &= -\frac{U}{x} + \lambda\log\left(\lambda|x|\right)\alpha(\lambda|x|) + \lambda\beta\left(\lambda|x|\right) + \gamma\left(\lambda|x|\right) \\ &\qquad\qquad\qquad\qquad \text{for} \quad |\lambda x| \le B, \\ R(x,\lambda) &= \lambda\delta\left(\lambda|x|\right)\frac{e^{-(\sigma+i\eta)|x|}}{\sqrt{|\lambda|\cdot|x|}} \quad \text{for} \quad |\lambda x| > B,\end{aligned} \tag{24}$$

where $\alpha(\zeta)$, $\beta(\zeta)$, $\gamma(\zeta)$, and $\delta(\zeta)$ are bounded analytic functions on $\operatorname{Re}\zeta > \varepsilon > 0$ and $B > 0$ is some constant.

Using representations (24) we introduce a function $M(x,\lambda)$, analytic with respect to $\lambda \in \{\operatorname{Re}\lambda > \sigma_0\}$, uniquely defined by (24), and such that

$$R(x,\lambda) = -\frac{U}{x} + M(x,\lambda).$$

We then consider operators

$$\mathcal{M}_\lambda[f](x) = \int_{|y|>1} \chi_A(x)\, M(x-y,\lambda) f(y)\, dy,$$

and

$$\mathcal{R}_\lambda = \pi U \cdot \mathcal{P} + \mathcal{M}_\lambda.$$

In the next proposition we prove compactness of the operator

$$\mathcal{N}_\lambda = \frac{1}{\pi U}\, \mathcal{M}_\lambda \circ \mathcal{P}^{-1} \tag{25}$$

on $\mathcal{L}^2\left(I^c(1)\right)$.

Proposition 5.1. *For any fixed $\lambda \in \mathbf{C}$, the operator $\mathcal{N}_\lambda = (1/\pi U)\mathcal{M}_\lambda \circ \mathcal{P}^{-1}$ is compact on $\mathcal{L}^2\left(I^c(1)\right)$, and therefore the operator*

$$\mathcal{G}_\lambda = \mathcal{R}_\lambda \circ \mathcal{P}^{-1} = \left(\pi U \cdot \mathcal{P} + \mathcal{M}_\lambda\right) \circ \mathcal{P}^{-1} = \pi U \left(\mathcal{I} + \mathcal{N}_\lambda\right), \tag{26}$$

where $\mathcal{I}$ is the identity operator, is a Fredholm operator on $\mathcal{L}^2\left(I^c(1)\right) = L^2\left(I^c(1)\right)$. In addition, the kernel $N(x,y,\lambda)$ of the operator $\mathcal{N}_\lambda$ admits estimate

$$\int_{\mathbf{R}^2} |N(x,y,\lambda)|^2\, dx\, dy < C|\lambda \log\lambda|^2 \tag{27}$$

with constant C independent of λ.

Proof. Using formula (23) for $\mathcal{P}^{-1}$, we obtain

$$\begin{aligned}
\mathcal{N}_\lambda[g](x) &= \mathcal{M}_\lambda\left[\frac{|x|}{\pi^2 U}\int_{-1}^{1}\sqrt{\frac{1-u^2}{x^2-1}}\left[\frac{1}{u}\,g\left(\frac{1}{u}\right)\right]\frac{du}{xu-1}\right]\\
&= \mathcal{M}_\lambda\left[\frac{|x|}{\pi^2 U}\int_{|y|>1}\sqrt{\frac{1-(1/y^2)}{x^2-1}}\;y^2 g(y)\,\frac{dy}{y^2(x-y)}\right]\\
&= \frac{\chi_A(x)}{\pi^2 U}\int_{|u|>1} M(x-u,\lambda)\,du\int_{|y|>1}\frac{|u|\sqrt{y^2-1}}{|y|\sqrt{u^2-1}}\,g(y)\,\frac{dy}{(u-y)}\\
&= \frac{\chi_A(x)}{\pi^2 U}\int_{|y|>1} g(y)\,dy\int_{|u|>1} M(x-u,\lambda)\,\frac{|u|\sqrt{y^2-1}}{|y|\sqrt{u^2-1}}\,\frac{du}{(u-y)}\\
&= \int_{|y|>1} N(x,y,\lambda)g(y)\,dy
\end{aligned}$$

with kernel

$$N(x,y,\lambda) = \frac{\chi_A(x)}{\pi^2 U}\int_{|u|>1} M(x-u,\lambda)\,\frac{|u|\sqrt{y^2-1}}{|y|\sqrt{u^2-1}}\,\frac{du}{(u-y)}.$$

To prove compactness of $\mathcal{N}_\lambda$, we use the representation

$$N(x,y,\lambda) = \frac{1}{\pi^2 U}\,\left[N_1(x,y,\lambda)+N_2(x,y,\lambda)\right],$$

with

$$N_1(x,y,\lambda) = \frac{\chi_A(x)}{|y|}\int_{|u|>1} M(x-u,\lambda)|u|\,\frac{du}{(u-y)},$$

and

$$\begin{aligned}
&N_2(x,y,\lambda)\\
&\quad = \frac{\chi_A(x)}{|y|}\int_{|u|>1} M(x-u,\lambda)\,\frac{|u|\left(\sqrt{y^2-1}-\sqrt{u^2-1}\right)}{\sqrt{u^2-1}}\,\frac{du}{(u-y)}\\
&\quad = -\frac{\chi_A(x)}{|y|}\int_{|u|>1} M(x-u,\lambda)\,\frac{|u|\,(y+u)\,du}{\left(\sqrt{y^2-1}+\sqrt{u^2-1}\right)\sqrt{u^2-1}},
\end{aligned}$$

and prove the Hilbert-Schmidt property, cf. [**12**], of the kernels $N_1(x,y,\lambda)$ and $N_2(x,y,\lambda)$.

For $N_1(x,y,\lambda)$, we notice that, for fixed x satisfying $1<|x|<A$,

$$\int_{|u|>1} M(x-u,\lambda)|u|\,\frac{du}{(u-y)}$$

is a multiple of the Hilbert transform of an $L^2\,(I^c(1))$-function $M(x-u,\lambda)|u|$ with

$$\|M(x-u,\lambda)|u|\,\|_{L^2_u}<\infty.$$

Therefore we have

$$\begin{aligned}
&\int_{1<|x|<A} dx \int_{|y|>1} dy\,|N_1(x,y,\lambda)|^2 \\
&\quad= \int_{1<|x|<A} dx \int_{|y|>1} dy \left|\frac{1}{|y|}\int_{|u|>1} M(x-u,\lambda)|u|\,\frac{du}{(u-y)}\right|^2 \qquad (28)\\
&\quad< C\int_{1<|x|<A} dx\,\|M(x-u,\lambda)|u|\|^2_{L^2_u}<\infty.
\end{aligned}$$

For $N_2(x,y,\lambda)$, we have

$$\begin{aligned}
&\int_{1<|x|<A} dx \int_{|y|>1} dy\,|N_2(x,y,\lambda)|^2 \\
&\quad= \int_{1<|x|<A} dx \int_{|y|>1} \frac{dy}{|y|^2} \\
&\qquad\times\left|\int_{|u|>1} M(x-u,\lambda)\,\frac{|u|\,(y+u)\,du}{\big(\sqrt{y^2-1}+\sqrt{u^2-1}\big)\sqrt{u^2-1}}\right|^2 \\
&\quad\le 2\int_{1<|x|<A} dx \int_{|y|>1} \frac{dy}{|y|^2}\left|\int_0^\infty M(x-\sqrt{t^2+1},\lambda)\,\frac{\big(y+\sqrt{t^2+1}\big)\,dt}{\big(\sqrt{y^2-1}+t\big)}\right|^2 \\
&\qquad+2\int_{1<|x|<A} dx \int_{|y|>1} \frac{dy}{|y|^2} \\
&\qquad\times\left|\int_0^\infty M(x+\sqrt{t^2+1},\lambda)\,\frac{\big(y-\sqrt{t^2+1}\big)\,dt}{\big(\sqrt{y^2-1}+t\big)}\right|^2,
\end{aligned}$$

where we changed the variable of integration to $t=\sqrt{u^2-1}$.

Both integrals of the right-hand side of (29) are estimated analogously; therefore, we will present an estimate of the first of them only.

For $|y| > 2$, we have inequality

$$\left|\frac{y+\sqrt{t^2+1}}{\sqrt{y^2-1}+t}\right| < C \tag{30}$$

for some C independent of y and t, and therefore, using representations (24), we obtain

(31)

$$\int_{1<|x|<A} dx \int_{|y|>2} \frac{dy}{|y|^2}\left|\int_0^{\infty} M(x-\sqrt{t^2+1},\lambda)\,\frac{\left(y+\sqrt{t^2+1}\right)dt}{\left(\sqrt{y^2-1}+t\right)}\right|^2$$
$$\leq C^2\int_{1<|x|<A} dx \int_{|y|>2} \frac{dy}{|y|^2}\left|\int_0^{\infty} M(x-\sqrt{t^2+1},\lambda)\,dt\right|^2 < \infty.$$

For $1<|x|<A$, $1<|y|<2$, and $t > A+B$, we again use inequality (30) and obtain

(32)

$$\int_{1<|x|<A} dx \int_{1<|y|<2} \frac{dy}{|y|^2}\left|\int_{A+B}^{\infty} M(x-\sqrt{t^2+1},\lambda)\,\frac{\left(y+\sqrt{t^2+1}\right)dt}{\left(\sqrt{y^2-1}+t\right)}\right|^2$$
$$\leq C^2\int_{1<|x|<A} dx \int_{1<|y|<2} \frac{dy}{|y|^2}\left|\int_0^{\infty} M(x-\sqrt{t^2+1},\lambda)\,dt\right|^2 < \infty.$$

For $1<|x|<A$, $1<|y|<2$ and $t<A+B$, we have

$$\int_{1<|x|<A} dx \int_{1<|y|<2} \frac{dy}{|y|^2}\left|\int_0^{A+B} M(x-\sqrt{t^2+1},\lambda)\,\frac{\left(y+\sqrt{t^2+1}\right)dt}{\left(\sqrt{y^2-1}+t\right)}\right|^2$$
$$< C\int_{1<|x|<A} dx \int_{1<|y|<2} dy\left|\int_0^{A+B} \frac{M(x-\sqrt{t^2+1},\lambda)dt}{\left(\sqrt{y^2-1}+t\right)}\right|^2$$
$$< C|\lambda\log\lambda|^2\int_{1<|x|<A} dx \int_{1<|y|<2} dy\left|\int_0^{A+B} \frac{dt}{\left(\sqrt{y^2-1}+t\right)}\right|^2$$

$$+ C|\lambda|^2 \int_{1<|x|<A} dx \int_{1<|y|<2} dy \left| \int_0^{A+B} \frac{\log|x - \sqrt{t^2+1}|}{\left(\sqrt{y^2-1}+t\right)} dt \right|^2$$

$$< C|\lambda|^2 \Bigg(|\log\lambda|^2 + \int_{1<|x|<A} dx \times \int_{1<|y|<2} dy \left| \int_0^{A+B} \frac{\log|x - \sqrt{t^2+1}|}{\left(\sqrt{y^2-1}+t\right)} dt \right|^2 \Bigg),$$

where we used the representation

$$M\left(x-\sqrt{t^2+1}, \lambda\right) = \lambda\left(\log\lambda + \log|x-\sqrt{t^2+1}|\right)\,\alpha\left(\lambda|x-\sqrt{t^2+1}|\right) + \lambda\beta\left(\lambda|x-\sqrt{t^2+1}|\right) + \gamma\left(\lambda|x-\sqrt{t^2+1}|\right)$$

for $1 < |x| < A$ and $0 < t < A+B$, which is a corollary of (24).

To estimate the last integral we represent it as

$$\int_{1<|x|<A} dx \int_{1<|y|<2} dy \left| \int_0^{A+B} \frac{\log|x - \sqrt{t^2+1}|}{\left(\sqrt{y^2-1}+t\right)} dt \right|^2$$
$$= \int_{1<|x|<A} dx \int_{1<|y|<2} dy \left| \int_{S(x,y)} \frac{\log|x - \sqrt{t^2+1}|}{\left(\sqrt{y^2-1}+t\right)} dt \right|^2$$
$$+ \int_{1<|x|<A} dx \int_{1<|y|<2} dy \left| \int_{\{[0,A+B]\setminus S(x,y)\}} \frac{\log|x - \sqrt{t^2+1}|}{\left(\sqrt{y^2-1}+t\right)} dt \right|^2$$

where $S(x,y) = \left\{ t : \left|x - \sqrt{t^2+1}\right| \geq 1/2\,(x-1)\,\sqrt{y^2-1} \right\}$.

Then, for $S(x,y)$, we have

$$\int_{1<|x|<A} dx \int_{1<|y|<2} dy \left| \int_{S(x,y)} \frac{\log|x-\sqrt{t^2+1}|}{\left(\sqrt{y^2-1}+t\right)}\, dt \right|^2$$
$$\leq C \int_{1<|x|<A} dx \int_{1<|y|<2} dy \left| \int_0^{A+B} \frac{\log|x-1| + \log\left(\sqrt{y^2-1}\right)}{\left(\sqrt{y^2-1}+t\right)}\, dt \right|^2$$
$$\leq C \int_{1<|x|<A} dx \int_{1<|y|<2} dy \log^2\sqrt{y^2-1}\left(\log|x-1| + \log\sqrt{y^2-1}\right)^2$$
$$< C < \infty.$$

For $t \in [0, A+B] \setminus S(x,y)$, we have

$$1 + \frac{t^2}{2} \geq \sqrt{t^2+1} > x - \frac{1}{2}\,(x-1)\sqrt{y^2-1},$$

and therefore

$$\frac{t^2}{2} \geq (x-1)\left[1 - \frac{1}{2}\sqrt{y^2-1}\right],$$

or

$$t \geq C\sqrt{x-1}.$$

Using the last inequality we obtain

$$|dt| \leq \left|\frac{\sqrt{t^2+1}}{t}\, du\right| \leq C\left|\frac{1}{\sqrt{x-1}}\, du\right|,$$

and, switching to variable $u = \sqrt{t^2+1}$ for $[0, A+B]\backslash S(x,y)$, we obtain

$$\int_{1<|x|<A} dx \int_{1<|y|<2} dy \left| \int_{\{[0,A+B]\setminus S(x,y)\}} \frac{\log|x-\sqrt{t^2+1}|}{\left(\sqrt{y^2-1}+t\right)}\, dt \right|^2$$
$$\leq C \int_{1<|x|<A} dx \int_{1<|y|<2} dy \left| \int_{x-(1/2)(x-1)\sqrt{y^2-1}}^{x+(1/2)(x-1)\sqrt{y^2-1}} \frac{\log|u-x|}{\sqrt{y^2-1}\cdot\sqrt{x-1}}\, du \right|^2$$

$$\leq C \int_{1<|x|<A} dx \int_{1<|y|<2} dy$$
$$\times \left| \frac{(x-1)\sqrt{y^2-1} \cdot \left(\log(x-1) + \log\sqrt{y^2-1}\right)}{\sqrt{y^2-1} \cdot \sqrt{x-1}} \right|^2 < C < \infty.$$

Combining the last two estimates above, we obtain

$$\int_{1<|x|<A} dx \int_{1<|y|<2} dy \left| \int_0^{A+B} \frac{\log|x - \sqrt{t^2+1}|}{\left(\sqrt{y^2-1}+t\right)} dt \right|^2 < C < \infty,$$

and therefore,

(33)
$$\int_{1<|x|<A} dx \int_{1<|y|<2} \frac{dy}{|y|^2} \left| \int_0^{A+B} M(x- \sqrt{t^2+1}, \lambda) \frac{\left(y+ \sqrt{t^2+1}\right)\, dt}{\left(\sqrt{y^2-1}+t\right)} \right|^2$$
$$< C|\lambda \log \lambda|^2.$$

To prove estimate (27), we use the following lemma.

Lemma 5.2. *The following estimates hold for* $1 < |x| < A$ *and* $\operatorname{Re} \lambda \in [\sigma_1, \sigma_2]$:

(34)
$$\int_{\mathbf{R}} |M(x-u, \lambda)|^2 u^2 \, du < C(\varepsilon)|\lambda|^{1+\varepsilon} \quad \textit{for arbitrary} \quad \varepsilon > 0,$$
$$\left| \int_0^\infty M(x - \sqrt{t^2+1}, \lambda) \, dt \right| < C|\lambda|| \log \lambda|.$$

Proof. Using representation (24) for $1 < |x| < A$ and $|\lambda(x-u)| \leq B$, we obtain

$$M(x-u, \lambda) \cdot u$$
$$= [\lambda \log(\lambda|x-u|)\, \alpha(\lambda|x-u|) + \lambda\beta(\lambda|x-u|) + \gamma(\lambda|x-u|)]\, u,$$

and therefore

$$\begin{aligned}&\int_{|x-u|\leq B/|\lambda|}|M(x-u,\lambda)|^2u^2\,du\\&\quad<C\int_{|x-u|\leq B/|\lambda|}\left[|\lambda|^2\left(|\log\lambda|^2+\log^2|x-u|\right)|\alpha(\lambda|x-u|)|^2\right.\\&\qquad\qquad\left.+|\lambda|^2|\beta\left(\lambda|x-u|\right)|^2+|\gamma\left(\lambda|x-u|\right)|^2\right]u^2\,du\\&\quad<C|\lambda||\log\lambda|^2.\end{aligned}$$

For $1<|x|<A$ and $|\lambda(x-u)|\geq B$ from (24), we have

$$M(x-u,\lambda)=\lambda\,\delta\left(\lambda|x-u|\right)\,\frac{e^{-(\sigma+i\eta)|x-u|}}{\sqrt{|\lambda|\cdot|x-u|}},$$

and therefore

$$\begin{aligned}&\int_{|x-u|\geq B/|\lambda|}|M(x-u,\lambda)|^2u^2\,dx\\&\quad<C\int_{\mathbf{R}}|\lambda|^{1+\varepsilon}|\delta\left(\lambda|x-u|\right)|^2\frac{e^{-2\sigma|x-u|}u^2\,du}{\left(|\lambda(x-u)|\right)^{\varepsilon}|x-u|^{1-\varepsilon}}\\&\quad<C(\varepsilon)|\lambda|^{1+\varepsilon}\end{aligned}$$

for any $\varepsilon>0$.

Combining the estimates above we obtain the first estimate of (34).

For the second integral in (34), we use representation (24) and obtain, for $1<|x|<A$ and $|\lambda(x-\sqrt{t^2+1})|\leq B$, after the change of variable $u=\sqrt{t^2+1}$,

$$\begin{aligned}&\left|\int_{|x-\sqrt{t^2+1}|\leq B/|\lambda|}M(x-\sqrt{t^2+1},\lambda)\,dt\right|\\&\quad\leq\int_{\left\{\substack{|x-u|\leq B/|\lambda|\\2>|u|>1}\right\}}\Big|\lambda\log\left(\lambda|x-u|\right)\alpha(\lambda|x-u|)\\&\qquad\qquad+\lambda\beta\left(\lambda|x-u|\right)+\gamma\left(\lambda|x-u|\right)\Big|\frac{|u|du}{\sqrt{u^2-1}}\\&\qquad+\int_{\left\{\substack{|x-u|\leq B/|\lambda|\\|u|>2}\right\}}\Big|\lambda\log\left(\lambda|x-u|\right)\alpha(\lambda|x-u|)\\&\qquad\qquad+\lambda\beta\left(\lambda|x-u|\right)+\gamma\left(\lambda|x-u|\right)\Big|\frac{|u|du}{\sqrt{u^2-1}}.\end{aligned}$$

For the first integral of the right-hand side above, we obtain, using Hölder inequality:

$$\int_{\left\{\substack{|x-u|\le B/|\lambda|\\ 2>|u|>1}\right\}} \Big|\lambda \log(\lambda|x-u|)\,\alpha(\lambda|x-u|) + \lambda\beta(\lambda|x-u|) + \gamma(\lambda|x-u|)\Big| \frac{|u|\,du}{\sqrt{u^2-1}}$$
$$\le C|\lambda|\cdot|\log\lambda|\cdot\|\log|x-u|\|_{L^4[1,2]}\left\|\frac{1}{\sqrt{u^2-1}}\right\|_{L^{4/3}[1,2]}$$
$$\le C|\lambda|\cdot|\log\lambda|.$$

For the second integral of the right-hand side above, we have

$$\int_{\left\{\substack{|x-u|\le B/|\lambda|\\ |u|>2}\right\}} \Big|\lambda \log(\lambda|x-u|)\,\alpha(\lambda|x-u|) + \lambda\beta(\lambda|x-u|) + \gamma(\lambda|x-u|)\Big| \frac{|u|du}{\sqrt{u^2-1}} \le C|\log\lambda|,$$

where we used the estimate $|u|/\sqrt{u^2-1} < C$ and the fact that the length of the interval of integration is bounded by $B/|\lambda|$.

For $1<|x|<A$ and $|\lambda(x-\sqrt{t^2+1})|>B$, using representation (24) and changing to variable $u=\sqrt{t^2+1}$, we obtain

$$\left|\int_{|x-\sqrt{t^2+1}|>B/|\lambda|} M(x-\sqrt{t^2+1},\lambda)\,dt\right|$$
$$= \left|\int_{|x-u|>B/|\lambda|} M(x-u,\lambda)\,\frac{|u|\,du}{\sqrt{u^2-1}}\right|$$
$$< C\left|\int_{|x-u|>B/|\lambda|} \lambda\delta(\lambda|x-u|)\,\frac{e^{-(\sigma+i\eta)|x-u|}}{\sqrt{|\lambda|\cdot|x-u|}}\,\frac{|u|\,du}{\sqrt{u^2-1}}\right|$$
$$< C|\lambda|\int_{|u|>1} e^{-(\sigma+i\eta)|x-u|}\,\frac{|u|\,du}{\sqrt{u^2-1}} < C|\lambda|.$$

Combining these estimates, we obtain the second estimate of (34).

Using now estimates (34) from the lemma above in estimates (28), (31) and (32), and combining them with estimate (33), we obtain estimate (27) of Proposition 5.1. □

Proposition 5.1 allows us to reduce the question of solvability of (16) to the solvability of the corresponding equation for $\mathcal{G}_\lambda$. Namely, calling those λ for which the operator $\mathcal{G}_\lambda$ is not invertible by *characteristic values of* $\mathcal{G}_\lambda$, we have

Proposition 5.3. *If* λ_0 *is not a characteristic value of* $\mathcal{G}_\lambda$*, then for an arbitrary function* $f \in \mathcal{L}^2\left(I^c(1)\right)$ *and* $\lambda = \lambda_0$*, there exists a solution* h *of equation* (16) *such that* $h \in \mathcal{L}^p\left(I^c(1)\right)$ *for any* $p < 4/3$*.*

Proof. Considering a solution of

$$\mathcal{G}_\lambda[g] = \mathcal{R}_\lambda \circ \mathcal{P}^{-1}[g] = f,$$

we define $h = \mathcal{P}^{-1}[g]$, which satisfies equation (16) and belongs to $\mathcal{L}^p\left(I^c(1)\right)$ for any $p < 4/3$ according to Proposition 4.1. □

6. The resolvent of operator $\mathcal{G}_\lambda$. In this section we construct the resolvent of the operator $\mathcal{G}_\lambda$ and show that it is a Fredholm operator depending analytically on $\lambda \in \{\operatorname{Re}\lambda > \sigma_1\}$.

Let $\mathcal{T} : L^2(\mathbf{R}) \to L^2(\mathbf{R})$ be an integral operator with kernel $T(x,y)$ satisfying the Hilbert-Schmidt condition. Following [**7**], we consider for operator $\mathcal{T}$ Hilbert's modification of the original Fredholm's determinants:

(35)

$$\mathcal{D}_{T,m}\left(t_1,\ldots,t_m\right) = \begin{vmatrix} 0 & T(t_1,t_2) & \cdots & T(t_1,t_m) \\ T(t_2,t_1) & 0 & \cdots & T(t_2,t_m) \\ \vdots & & & \vdots \\ T(t_m,t_1) & \cdots & T(t_m,t_{m-1}) & 0 \end{vmatrix},$$

$$\begin{aligned} \mathcal{D}_T &= 1 + \sum_{m=1}^{\infty} \frac{1}{m!} \int_{\mathbf{R}} \cdots \int_{\mathbf{R}} \mathcal{D}_{T,m}\left(t_1,\ldots,t_m\right) dt_1 \cdots dt_m \\ &= 1 + \sum_{m=1}^{\infty} \delta_m, \end{aligned}$$

$$\mathcal{D}_{T,m}\begin{pmatrix} x \\ y \end{pmatrix} t_1,\ldots,t_m\Big) = \begin{vmatrix} T(x,y) & T(x,t_1) & \cdots & T(x,t_m) \\ T(t_1,y) & 0 & \cdots & T(t_1,t_m) \\ \vdots & & & \vdots \\ T(t_m,y) & \cdots & T(t_m,t_{m-1}) & 0 \end{vmatrix},$$

and

$$\mathcal{D}_T\begin{pmatrix} x \\ y \end{pmatrix} = T(x,y) + \sum_{m=1}^{\infty}\frac{1}{m!}\int_{\mathbf{R}}\cdots\int_{\mathbf{R}}\mathcal{D}_{T,m}\begin{pmatrix} x \\ y \end{pmatrix} t_1,\ldots,t_m\Big)\,dt_1\cdots dt_m = T(x,y) + \sum_{m=1}^{\infty}\delta_m\begin{pmatrix} x \\ y \end{pmatrix}. \tag{36}$$

We start with the following proposition which summarizes the results from [**7**], cf. also [**13**], that will be used in the construction of the resolvent of $\mathcal{G}_\lambda$.

Proposition 6.1 [**7**]. *Let the function $T(x,y):\mathbf{R}^2\to\mathbf{C}$ satisfy Hilbert-Schmidt condition*

$$\|T\|^2 = \int_{\mathbf{R}^2}|T(x,y)|^2\,dx\,dy < \infty.$$

Then the function $\mathcal{D}_T\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\in L^2(\mathbf{R}^2)$ is well defined, and the following estimates hold:

$$|\delta_m| \le \left(\frac{e}{m}\right)^{m/2}\|T\|^m,\ |\mathcal{D}_T| \le e^{\|T\|^2/2}, \tag{37}$$

$$\left|\mathcal{D}_T\begin{pmatrix} x \\ y \end{pmatrix}\right| \le e^{\|T\|^2/2}\left(|T(x,y)| + \sqrt{e}\,\alpha(x)\beta(y)\right), \tag{38}$$

where

$$\alpha^2(x) = \int_{\mathbf{R}}|T(x,t)|^2\,dt, \qquad \beta^2(y) = \int_{\mathbf{R}}|T(t,y)|^2\,dt.$$

If $\mathcal{D}_T \neq 0$, then the kernel

$$H(x,y) = [\mathcal{D}_T]^{-1} \cdot \mathcal{D}_T \begin{pmatrix} x \\ y \end{pmatrix} \tag{39}$$

defines the resolvent of operator $\mathcal{I} - \mathcal{T}$, i.e., it satisfies the following equations

$$\begin{aligned} H(x,y) + \int_{\mathbf{R}} T(x,t) \cdot H(t,y)\, dt = T(x,y), \\ H(x,y) + \int_{\mathbf{R}} T(t,y) \cdot H(x,t)\, dt = T(x,y), \end{aligned} \tag{40}$$

and therefore the operator $\mathcal{I} - \mathcal{H}$ is the inverse of operator $\mathcal{I} + \mathcal{T}$.

Using Proposition 6.1, we construct the resolvent of operator $\mathcal{G}_\lambda = \pi U\,(\mathcal{I} + \mathcal{N}_\lambda)$, defined in (26), and prove the estimate that will be necessary in the proof of Theorem 1.

Proposition 6.2. *The set of characteristic values of operator $\mathcal{G}_\lambda$ coincides with the set*

$$E(\mathcal{G}) = \{\lambda \in \mathbf{C} : \ \mathrm{Re}\,\lambda > \sigma_1,\ \mathcal{D}_{N_\lambda} = 0\}$$

and consists of at most countably many isolated points.

For $\lambda \notin E(\mathcal{G})$, there exists an operator $\mathcal{H}_\lambda$ with kernel $H(x,y,\lambda)$ satisfying the Hilbert-Schmidt condition and such that $\mathcal{I} - \mathcal{H}_\lambda$ is the inverse of $\mathcal{I} + \mathcal{N}_\lambda$, and therefore $1/(\pi U)\,(\mathcal{I} - \mathcal{H}_\lambda)$ is the inverse of $\mathcal{G}_\lambda$.

If the function $\mathcal{D}_N(\lambda) = \mathcal{D}_{N_\lambda}$ has no zeros in a strip $\{\lambda : \ \sigma_1 < \mathrm{Re}\,\lambda < \sigma_2\}$, then the operator $\mathcal{H}_\lambda$ admits estimate

$$||\mathcal{H}_\lambda|| < C(\varepsilon) \cdot \exp\left\{ e^{|\eta|} \cdot (1 + |\eta|)^{2+\varepsilon} \right\} \tag{41}$$

for $\lambda \in \{\sigma_1 + \gamma < \mathrm{Re}\,\lambda < \sigma_2 - \gamma\}$, arbitrary $\varepsilon > 0$, and some $C(\varepsilon) > 0$.

Proof. Applying Proposition 6.1 to the operator $\mathcal{N}_\lambda$ defined in (25), we obtain the existence of functions

$$\mathcal{D}_N(\lambda) = \mathcal{D}_{N_\lambda} \tag{42}$$

and

$$\mathcal{D}_N\left(\begin{matrix} x \\ y \end{matrix}\middle| \lambda\right) = \mathcal{D}_{N_\lambda}\begin{pmatrix} x \\ y \end{pmatrix}$$

such that, for any fixed λ, satisfying $\mathcal{D}_N(\lambda) \neq 0$, kernel

$$H(x, y, \lambda) = [\mathcal{D}_N(\lambda)]^{-1} \cdot \mathcal{D}_N\left(\begin{matrix} x \\ y \end{matrix}\middle| \lambda\right) \in L^2(\mathbf{R}^2),$$

and the operator $\mathcal{I} - \mathcal{H}_\lambda$ is the inverse of operator $\mathcal{I} + \mathcal{N}_\lambda$.

Terms of the series (35) for $\mathcal{N}_\lambda$ depend analytically on λ, and, according to estimates (37), this series converges uniformly with respect to λ on compact subsets of $\{\lambda \in \mathbf{C} : \operatorname{Re}\lambda > \sigma_1\}$. Therefore, $\mathcal{D}_N(\lambda)$ is an analytic function on $\{\lambda \in \mathbf{C} : \operatorname{Re}\lambda > \sigma_1\}$, and the set $E(\mathcal{G})$ consists of at most countably many isolated points.

Analyticity of $\mathcal{I} - \mathcal{H}_\lambda$ with respect to λ on

$$\{\lambda \in \mathbf{C} : \operatorname{Re}\lambda > \sigma_1\} \setminus E(\mathcal{G})$$

follows from [**18**, Theorem VI.14]. It is proved by approximating the kernel by degenerate kernels and employing an argument that can be traced back to at least [**13**].

To prove estimate (41) we use the well-known estimate [**12**]

$$\|T\|^2 \leq \int_{\mathbf{R}^2} |T(x, y)|^2 \, dx \, dy$$

for integral operators. Using this estimate, and estimates (38) and (27), we obtain

$$\left\|\mathcal{D}_N\left(\begin{matrix} x \\ y \end{matrix}\middle| \lambda\right)\right\| < \exp\left\{C(1 + |\eta|)^2 \cdot \log^2 |\eta|\right\}(1 + |\eta|)^4 \cdot \log^4 |\eta|.$$

To estimate the function $[\mathcal{D}_N(\lambda)]^{-1}$ for $\lambda \in \{\sigma_1 + \gamma < \operatorname{Re}\lambda < \sigma_2 - \gamma\}$ we use the following lemma.

Lemma 6.3. *If the function $\mathcal{D}_N(\lambda) = \mathcal{D}_{N_\lambda}$ has no zeros in the strip $\{\lambda : \ \sigma_1 < \operatorname{Re}\lambda < \sigma_2\}$, then the estimate*

$$|1/\mathcal{D}_N(\lambda)| < C(\varepsilon) \exp\left\{e^{|\eta|} \cdot (1 + |\eta|)^{2+\varepsilon}\right\} \tag{43}$$

holds for $\lambda \in \{\sigma_1 + \gamma < \mathrm{Re}\,\lambda < \sigma_2 - \gamma\}$ *with fixed* $\gamma > 0$ *and arbitrary* $\varepsilon > 0$.

Proof. We consider a biholomorphic map

$$\Psi : \ \{\lambda : \ \sigma_1 < \mathrm{Re}\,\lambda < \sigma_2\} \longrightarrow \mathbf{D}(1) = \{z \in \mathbf{C} : |z| < 1\},$$

defined by the formula

$$\Psi(\lambda) = \frac{e^{i\pi(\lambda-\sigma_1)/(\sigma_2-\sigma_1)} - i}{e^{i\pi(\lambda-\sigma_1)/(\sigma_2-\sigma_1)} + i}.$$

Denoting

$$w = u + iv = e^{i\pi(\lambda-\sigma_1)/(\sigma_2-\sigma_1)},$$

we obtain for the circle $C(r) = \{z : \ |z| = r\}$,

$$\begin{aligned}\Psi^{-1}\left(C(r)\right) &= \Big\{\sigma + i\eta : \ \left|e^{i\pi(\lambda-\sigma_1)/(\sigma_2-\sigma_1)} - i\right| \\ &= r\left|e^{i\pi(\lambda-\sigma_1)/(\sigma_2-\sigma_1)} + i\right|\Big\} \\ &= \big\{u + iv : \ \big(u^2 + v^2 - 2v + 1\big) \\ &= \ \ r^2\big(u^2 + v^2 + 2v + 1\big)\big\} \\ &= \left\{u + iv : \ u^2 + \left(v - \frac{1+r^2}{1-r^2}\right)^2 = \frac{4r^2}{(1-r^2)^2}\right\}.\end{aligned}$$

Introducing coordinates

$$t = \mathrm{Re}\,\frac{\pi(\lambda-\sigma_1)}{\sigma_2-\sigma_1}, \qquad s = \mathrm{Im}\,\frac{\pi(\lambda-\sigma_1)}{\sigma_2-\sigma_1},$$

such that

$$w = u + iv = e^{i(\lambda-\sigma_1)(\pi/\sigma_2-\sigma_1)} = e^{it-s} = e^{-s}\left(\cos t + i\sin t\right),$$

we can rewrite the last condition as a quadratic equation with respect to e^{-s} for fixed t

$$\left(e^{-s} - \sin t\,\frac{1+r^2}{1-r^2}\right)^2 + \cos^2 t\left(\frac{1+r^2}{1-r^2}\right)^2 - \frac{4r^2}{(1-r^2)^2} = 0.$$

Solving the equation above we obtain

$$e^{-s} = \sin t \, \frac{1+r^2}{1-r^2} \pm \sqrt{\frac{4r^2}{(1-r^2)^2} - \cos^2 t \left(\frac{1+r^2}{1-r^2}\right)^2}$$

with solutions existing for t such that

$$|\cos t| \leq \frac{2r}{1-r^2}\,\frac{1-r^2}{1+r^2} = \frac{2r}{1+r^2}.$$

The maximal value for e^{-s} is achieved at $t = \frac{\pi}{2}$ and it is

$$e^{-s} = \frac{1+r^2}{1-r^2} + \frac{2r}{1-r^2} = \frac{1+r^2+2r}{1-r^2} = \frac{(1+r)^2}{1-r^2} = \frac{1+r}{1-r}.$$

Therefore the maximal value for $|s|$ is achieved at $t = \pi/2$, is equal to $|s| = \log(1+r/1-r)$, and for $r = 1-\delta$, we have the maximal value

$$\max|s| = \log\left(\frac{1+r}{1-r}\right) = -\log\delta + \log(2-\delta). \tag{44}$$

Since the function $\mathcal{D}_N(\lambda)$ has no zeros in $\{\lambda : \ \sigma_1 < \operatorname{Re}\lambda < \sigma_2\}$, we can consider the analytic function $\log(\mathcal{D}_N(\lambda))$ in this strip and, using estimates (37)and (27), and equality (44), we obtain the following estimate for $z = (1-\delta)e^{i\theta}$,

$$\begin{aligned}\log\left|\mathcal{D}_N\left(\Psi^{-1}(z)\right)\right| &\leq \frac{\left\|N_{\Psi^{-1}(z)}\right\|^2}{2} \leq C\left|\Psi^{-1}(z)\cdot\log\left(\Psi^{-1}(z)\right)\right|^2 \\ &\leq C|\log\delta\cdot\log(\log\delta)|^2.\end{aligned}$$

Using the Borel-Caratheodory inequality [**4, 21**] on disks with radii

$$1-2\delta = r < R = 1-\delta,$$

we then obtain

$$\begin{aligned}\left|\log\left(\mathcal{D}_N\left(\Psi^{-1}(z)\right)\right)\right|_{\{|z|=1-2\delta\}} &\leq \frac{2-4\delta}{\delta}\max_{|z|=R}\operatorname{Re}\left\{\log\left(\mathcal{D}_N\left(\Psi^{-1}(z)\right)\right)\right\} \\ &\quad + \frac{1-\delta+1-2\delta}{\delta}\left|\log\left(\mathcal{D}_N\left(\Psi^{-1}(0)\right)\right)\right| \\ &< \frac{C}{\delta}\log^2\delta\cdot\log^2(\log\delta),\end{aligned}$$

or

$$-\frac{C}{\delta}\log^2\delta\cdot\log^2(\log\delta) < \mathrm{Re}\left\{\log\left(\mathcal{D}_N\left(\Psi^{-1}(z)\right)\right)\right\}\Big|_{\{|z|=1-2\delta\}} < \frac{C}{\delta}\log^2\delta\cdot\log^2(\log\delta).$$

From the last estimate we obtain an estimate for the function $\left|1/\mathcal{D}_N\left(\Psi^{-1}(z)\right)\right|$ in the disk $\mathbf{D}(1-2\delta)$:

$$\left|1/\mathcal{D}_N\left(\Psi^{-1}(z)\right)\right|\Big|_{\{|z|\leq 1-2\delta\}} < C(\varepsilon)\exp\left\{\frac{|\log\delta|^{2+\varepsilon}}{\delta}\right\} \tag{45}$$

for arbitrary $\varepsilon > 0$.

For a fixed $t \in (0,\pi)$ and arbitrary s, we have that $t+is \in \Psi^{-1}(\mathbf{D}(r))$ with $r = 1-2\delta$ if

$$\begin{aligned} e^{|s|} &\leq \sin t\cdot\frac{1+r^2}{1-r^2} + \sqrt{\frac{4r^2}{(1-r^2)^2} - \cos^2 t\cdot\left(\frac{1+r^2}{1-r^2}\right)^2} \\ &= \sin t\cdot\frac{2-4\delta+4\delta^2}{2\delta(2-2\delta)} + \frac{\sqrt{4(1-2\delta)^2 - \cos^2 t\cdot(2-4\delta+4\delta^2)^2}}{2\delta(2-2\delta)}, \end{aligned}$$

and therefore, for any interval $[\gamma', \pi-\gamma']$, there exist constants $C_1,\ C_2$ such that conditions

$$t \in [\gamma', \pi-\gamma'], \qquad \frac{C_1}{\delta} < e^{|s|} < \frac{C_2}{\delta}$$

imply that $t+is \in \Psi^{-1}(\mathbf{D}(1-2\delta))$.

Using estimate (45) we then obtain for λ with

$$\mathrm{Re}\,\lambda \in \left[\sigma_1 + \frac{\gamma'(\sigma_2-\sigma_1)}{\pi}, \sigma_2 - \frac{\gamma'(\sigma_2-\sigma_1)}{\pi}\right]$$

the estimate

$$|1/\mathcal{D}_N(\lambda)| < C(\varepsilon)\exp\left\{e^{|s|}\cdot(1+|s|)^{2+\varepsilon}\right\}$$

for arbitrary $\varepsilon > 0$, which leads to estimate (43). □

Combining the estimate for $\left\| \mathcal{D}_N \left(\begin{smallmatrix} x \\ y \end{smallmatrix} \middle| \lambda \right) \right\|$ with (43), we now obtain the estimate (41). $\square$

7. Proof of Theorem 1. Before proving Theorem 1 we will prove two lemmas that will be used in the proof of this theorem.

In order to assure applicability of Proposition 5.3 to f, defined in (17), we have to prove that

$$f \in \mathcal{L}^2 \left(I^c(1) \right)$$

for $\widehat{w}$ satisfying (4). In the lemma below we prove the necessary property of f.

Lemma 7.1. *If $\widehat{w}$ satisfies condition (4), then $f(x, \lambda)$ defined by formula (17) is a function in $\mathcal{L}^2 \left(I^c(1) \right)$ for any fixed λ, which satisfies the estimate*

$$\|f(\cdot, \sigma + i\eta)\|_{\mathcal{L}^2(I^c(1))} < C \exp \left\{ -e^{|\eta|} \cdot (1 + |\eta|)^{2+\varepsilon} \right\} \tag{46}$$

with some $\varepsilon > 0$ for $\sigma \in [\sigma_1, \sigma_2]$.

Proof. For a fixed $\lambda = \sigma + i\eta$ with $\sigma \in [\sigma_1, \sigma_2]$, we choose $B > 1$ and, using the second representation from (24) of $R(x, \lambda)$ for $|\lambda x| > B$, obtain an estimate

$$|R(x - y, \lambda)| < C \, \frac{|\lambda|^{1/2} e^{-\lambda |x-y|}}{\sqrt{|x - y|}}.$$

Using condition (4), we then have

$$\begin{aligned} &\left(\int_{|x|>B/|\lambda|} |f(x, \lambda)|^2 \, dx \right)^{1/2} \\ &\quad = \frac{1}{\pi^2} \left| \int_{|x|>B/|\lambda|} \left| \int_{-1}^{1} e^{-d(\lambda) y} R(x - y, \lambda) \widehat{w}(y, \lambda) \, dy \right|^2 dx \right|^{1/2} \\ &\quad < C |\lambda|^{1/2} \left(\int_{|x|>B/|\lambda|} \left(\int_{-1}^{1} e^{-\sigma |x|} \left| \widehat{w}(y, \lambda) \right| \, dy \right)^2 dx \right)^{1/2} \end{aligned}$$

$$< C|\lambda|^{1/2}\int_{-1}^{1}|\widehat{w}(y,\lambda)|\ dy < C|\lambda|^{1/2}\int_{-1}^{1}|\widehat{w}(y,\lambda)|^2\ dy$$
$$< C\exp\Big\{-e^{|\eta|}\cdot(1+|\eta|)^{2+\varepsilon}\Big\}.$$

For $|\lambda x| < B$, we use the first representation from (24) for $R(x-y,\lambda)$. Since the Hilbert transform is a bounded linear operator from L^q into L^q, see [**22, 23**], and kernels $\alpha\,(\lambda(x-y))$, $\beta\,(\lambda(x-y))$, and $\gamma\,(\lambda(x-y))$ from (24) are bounded, we obtain

$$\left(\int_{|x|<|B/\lambda|}|f(x,\lambda)|^2\ dx\right)^{1/2} < C\,|\lambda\log\lambda|\cdot\|\widehat{w}(y,\lambda)\|_{L^2(I(1))} < C\exp\Big\{-e^{|\eta|}\cdot(1+|\eta|)^{2+\varepsilon}\Big\}, \tag{48}$$

where in the last inequality we used condition (4).

Combining estimates (47) and (48), we obtain (46). □

Lemma 7.2. *If a function* $h(y,\lambda)$ *satisfies*

$$\int_{-\infty}^{\infty}e^{-\sigma_1\cdot|y|}|h(y,\sigma+i\eta)|\,dy < \frac{C}{(1+|\eta|)^{(5/2)+\varepsilon}} \tag{49}$$

for some $\varepsilon > 0$ *and* $\sigma_1 < \operatorname{Re}\lambda < \sigma_2$*, then the function*

$$\xi(x,z,\lambda) = e^{d(\sigma+i\eta)x} \times\left[\int_{-\infty}^{\infty}K_0\left(r(\sigma+i\eta)\left(\frac{(x-y)^2}{1-M^2}+z^2\right)^{1/2}\right)h(y,\sigma+i\eta)\,dy\right] \tag{50}$$

lies in $L^1(\mathbf{R}_\eta)$ *for* $\sigma\in[\sigma_1,\sigma_2]$ *and satisfies*

$$\frac{\partial^2\xi(x,z,\sigma+i\eta)}{\partial x^2},\qquad \frac{\partial^2\xi(x,z,\sigma+i\eta)}{\partial z^2},\qquad |\eta|^2\xi(x,z,\sigma+i\eta),\qquad |\eta|\frac{\partial\xi(x,z,\sigma+i\eta)}{\partial x}\in L^1(\mathbf{R}_\eta). \tag{51}$$

The function

$$\phi(x,z,t) = -\frac{1}{2\pi\sqrt{1-M^2}}\int_{-\infty}^{\infty} e^{d(\sigma+i\eta)x} \times \left[\int_{-\infty}^{\infty} K_0\left(r(\sigma+i\eta)\left(\frac{(x-y)^2}{1-M^2}+z^2\right)^{1/2}\right)h(y,\sigma+i\eta)\,dy\right] \times e^{(\sigma+i\eta)t}\,d\eta$$

is then well defined for $z > 0$ *and doesn't depend on* $\sigma \in [\sigma_1, \sigma_2]$.

Proof. To prove that the function $\xi(x,z,\lambda)$ defined in (50) lies in $L^1(\mathbf{R}_\eta)$, it suffices to prove that under the conditions above

(52)
$$\left\| e^{d(\sigma+i\eta)x}\left[\int_{-\infty}^{\infty} K_0\left(r(\sigma+i\eta)\left(\frac{(x-y)^2}{1-M^2}+z^2\right)^{1/2}\right)h(y,\sigma+i\eta)\,dy\right]\right\|_{L^1(\mathbf{R}_\eta)} < C(M,z)$$

holds uniformly with respect to $\sigma \in [\sigma_1, \sigma_2]$ for fixed x, fixed $z > 0$ and for some $\sigma_1 > \sigma_0$. Applying then Theorem 47 from [**5**] we will obtain the second part of the lemma.

Using asymptotics of $K_0(\zeta)$ for large and for small $|\zeta|$ ([**9**]), we obtain the existence for fixed $z > 0$ of a constant $A(z) > 0$, large enough, such that the estimates

(53)
$$\left|K_0\left(r(\sigma+i\eta)\left(\frac{(x-y)^2}{1-M^2}+z^2\right)^{1/2}\right)\right| < \frac{C(M,z)e^{-\sigma|x-y|}}{\sqrt{|\sigma+i\eta|\cdot|x-y|}} \quad \text{for} \quad |x-y| > A(z),$$

and

(54)
$$\left|K_0\left(r(\sigma+i\eta)\left(\frac{(x-y)^2}{1-M^2}+z^2\right)^{1/2}\right)\right| < \frac{C(M,z)}{\sqrt{|\sigma+i\eta|}} \quad \text{for} \quad |x-y| < A(z),$$

hold uniformly for $\sigma \in [\sigma_1, \sigma_2]$, with a constant C depending on M and z.

Combining estimates (53) and (54) with the estimate for $h(y, \lambda)$, we obtain

$$\left| e^{d(\sigma+i\eta)x} \left[\int_{-\infty}^{\infty} K_0\left(r(\sigma+i\eta) \left(\frac{(x-y)^2}{1-M^2} + z^2 \right)^{1/2} \right) h(y, \sigma+i\eta)\, dy \right] \right| \\ < \frac{C(M,z)}{\sqrt{|\sigma+i\eta|}} \int_{-\infty}^{\infty} e^{-\sigma_1 \cdot |y|} |h(y, \sigma+i\eta)|\, dy < \frac{C(M,z)}{(1+|\eta|)^{3+\varepsilon}},$$

for $z > 0$, which leads to estimate (52).

Again, using estimates (53) and (54) and analogous estimates for

$$\frac{\partial}{\partial x} K_0\left(r(\sigma+i\eta) \left(\frac{(x-y)^2}{1-M^2} + z^2 \right)^{1/2} \right),$$
$$\frac{\partial^2}{\partial x^2} K_0\left(r(\sigma+i\eta) \left(\frac{(x-y)^2}{1-M^2} + z^2 \right)^{1/2} \right),$$

and

$$\frac{\partial^2}{\partial z^2} K_0\left(r(\sigma+i\eta) \left(\frac{(x-y)^2}{1-M^2} + z^2 \right)^{1/2} \right)$$

we obtain inclusions (51). □

To prove Theorem 1 we consider w satisfying condition (4) and define f by the formula (17). Using Lemma 7.1 we obtain that f satisfies estimate (46). Applying Proposition 5.3 to f and using estimate (41) from Proposition 6.2, we obtain the existence of h satisfying equation (16) and such that

$$\|h(\cdot, \sigma+i\eta)\|_{\mathcal{L}^p(I^c(1))} < C(\varepsilon) \exp\left\{ e^{|\eta|} \cdot (1+|\eta|)^{2+\varepsilon} \right\} \|f(\cdot, \sigma+i\eta)\|_{\mathcal{L}^2(I^c(1))} \\ < \frac{C(m)}{(1+|\eta|)^m}$$

for arbitrary m, arbitrary $p < 4/3$ and $\sigma \in [\sigma_1, \sigma_2]$, with $\sigma_0 < \sigma_1$.

Using the estimate above for $p = 1$, we obtain

$$\int_{|x|>1} |h(x, \sigma + i\eta)| \cdot |x|^{-1}\, dx < \frac{C(m)}{(1 + |\eta|)^m}. \tag{55}$$

From the definition of h on $[-1, 1]$ as

$$h(x, \lambda) = \frac{e^{-d(\lambda)x} \cdot \widehat{w}(x, \lambda)}{\pi}$$

and from condition (4), we obtain

$$\begin{aligned}
\|h(x, \sigma + i\eta)\|_{L^p(I(1))} &= \left\| e^{-d(\lambda)x} \cdot \widehat{w}(x, \sigma + i\eta) \right\|_{L^p(I(1))} \\
&< C\, \|\widehat{w}(\cdot, \sigma + i\eta)\|_{L^2(I(1))} \\
&< \frac{C(m)}{(1 + |\eta|)^m}
\end{aligned}$$

$$\text{for} \quad p < \frac{4}{3},\ \sigma \in [\sigma_1, \sigma_2] \quad \text{with } \sigma_0 < \sigma_1,$$

and therefore

$$\|h(\cdot, \sigma + i\eta)\|_{L^1(I(1))} < \frac{C(m)}{(1 + |\eta|)^m} \tag{56}$$

for arbitrary $m > 0$.

From the estimates (55) and (56) we conclude that the function h satisfies estimate (49), and therefore, applying Lemma 7.2 and Proposition 2.3, we obtain that the function $\phi(x, z, t)$ in formula (5) is well defined and satisfies equation (1).

To prove that $\phi(x, z, t)$ satisfies boundary condition (2), we fix $x \in [-1, 1]$ and denote $\delta = \min\{x + 1, 1 - x\}$. Then we have

$$\begin{aligned}
\lim_{z\to 0} \frac{\partial}{\partial z}\, \xi(x, z, \lambda) &= \lim_{z\to 0} \frac{\partial}{\partial z} \int_{-\infty}^{\infty} S(x-y, z, \lambda) e^{d(\lambda)y} h(y, \lambda)\, dy \\
&= \lim_{z\to 0} \frac{\partial}{\partial z} \int_{x-\frac{\delta}{2}}^{x+\frac{\delta}{2}} S(x-y, z, \lambda) e^{d(\lambda)y} h(y, \lambda)\, dy \\
&\quad + \lim_{z\to 0} \frac{\partial}{\partial z} \int_{\mathbf{R}\setminus[x-\frac{\delta}{2}, x+\frac{\delta}{2}]} S(x-y, z, \lambda) e^{d(\lambda)y} h(y, \lambda)\, dy.
\end{aligned}$$

For the first integral in the right-hand side of (57), we obtain using Lemma 2.2

$$\lim_{z\to 0}\frac{\partial}{\partial z}\int_{x-\frac{\delta}{2}}^{x+\frac{\delta}{2}} S(x-y,z,\lambda)e^{d(\lambda)y}h(y,\lambda)\,dy$$
$$= -\lim_{z\to 0}\frac{\partial}{\partial z}\int_{x-\frac{\delta}{2}}^{x+\frac{\delta}{2}} e^{d(\lambda)y}h(y,\lambda)\,dy$$
$$\times\int_{-\infty}^{\infty} e^{i(x-y)\omega}\,\frac{e^{-z\left((1-M^2)(\omega+id(\lambda))^2+r^2(\lambda)\right)^{1/2}}}{2\sqrt{(1-M^2)(\omega+id(\lambda))^2+r^2(\lambda)}}\,d\omega$$
$$= \widehat{w}(x,\lambda).$$

For the second integral in the right-hand side of (57), we have

$$\lim_{z\to 0}\frac{\partial}{\partial z}\int_{\mathbf{R}\setminus[x-(\delta/2),x+(\delta/2)]} S(x-y,z,\lambda)e^{d(\lambda)y}h(y,\lambda)\,dy$$
$$= -\frac{e^{d(\lambda)x}}{\sqrt{1-M^2}}$$
$$\times\lim_{z\to 0}\int_{\mathbf{R}\setminus[x-(\delta/2),x+(\delta/2)]}\left[\frac{\partial}{\partial z}K_0\left(r(\lambda)\left(\frac{(x-y)^2}{1-M^2}+z^2\right)^{1/2}\right)\right]h(y,\lambda)\,dy.$$

Using estimate (55) and equality

$$\lim_{z\to 0}\left[\frac{\partial}{\partial z}K_0\left(r(\lambda)\left(\frac{(x-y)^2}{1-M^2}+z^2\right)^{1/2}\right)\right]=0$$

for $y\in\mathbf{R}\setminus[x-(\delta/2),x+(\delta/2)]$, we then obtain that the second integral in the right-hand side of (57) is equal to zero.

From the equalities above we conclude that

$$\lim_{z\to 0}\frac{\partial}{\partial z}\,\xi(x,z,\lambda)=\widehat{w}(x,\lambda)$$

for $x\in[-1,1]$ and $\operatorname{Re}\lambda\in[\sigma_1,\sigma_2]$, and therefore

$$\lim_{z\to 0}\frac{\partial}{\partial z}\,\phi(x,z,t)=w(x,t).$$

Straightforward substitution of $v(x, \lambda) = e^{d(\lambda)x} h(x, \lambda)$ into the formula (10), with $h(x, \lambda)$ defined as

$$h(x, \lambda) = \begin{cases} (1/\pi) e^{-d(\lambda)x} \cdot \widehat{w}(x, \lambda) & \text{for } x \in [-1, 1], \\ \text{solution of equation (16)} & \text{for } x \in \mathbf{R} \setminus [-1, 1], \end{cases}$$

shows that $\xi(x, z, \lambda)$ defined by this formula satisfies equation (14) for $1 < |x| < A$. Then for $\phi(x, z, t)$ defined by formula (11) we will have

$$\begin{aligned} \frac{\partial \phi(x, 0, t)}{\partial t} + U \frac{\partial \phi(x, 0, t)}{\partial x} &= \frac{1}{2\pi} \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \int_{-\infty}^{\infty} e^{\lambda t} \xi(x, 0, \lambda) \, d\eta \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\lambda t} \left(\lambda + U \frac{\partial}{\partial x} \right) \xi(x, 0, \lambda) \, d\eta \\ &= 0 \end{aligned}$$

for $1 < |x| < A$. □

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