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SOLITARY WAVE, BREATHER WAVE AND ROGUE WAVE SOLUTIONS OF AN INHOMOGENEOUS FIFTH-ORDER NONLINEAR SCHRÖDINGER EQUATION FROM HEISENBERG FERROMAGNETISM

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ABSTRACT. In this paper, we consider an inhomogeneous fifth-order nonlinear Schrödinger equation from Heisenberg ferromagnetism, which describes the dynamics of a site-dependent Hisenberg ferromagnetic spin chain. Based on its Lax pair, we study the determinant representation of the n-fold Darboux transformation (DT). Furthermore, by using the n-fold DT, we obtain the higher-order solitary wave, breather wave and rogue wave solutions of the equation, respectively. Finally, the dynamic characteristics of these exact solutions are discussed.

1. Introduction. In **[9]**, Fokas proposed an integrable generalization of the nonlinear Schrödinger (NLS) equation

$$iu_t - \nu u_{tx} + \gamma u_{xx} + \sigma |u|^2 (u + i\nu u_x) = 0, \quad \sigma \pm 1, \tag{1.1}$$

by using bi-Hamiltonian methods, where γ and ν are nonzero real parameters and $u(x,t)$ is a complex-valued function. When $\nu = 0$, (1.1) reduces to the NLS equation. Equation (1.1) arises as a model for nonlinear pulse propagation in monomode optical fibers and is the first negative member of the integrable hierarchy associated with the derivative NLS equation **[16]**. In **[18]**, Lenells and Fokas applied the bi-Hamiltonian structure to write the first few conservation laws of (1.1)

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and derive their Lax pair, by which they solve the initial value problem and analyze solitons. In [**39**], Tian and his collaborators obtained the quasi-periodic waves and rogue waves to a (4+1)-dimensional nonlinear Fokas equation.

Recently, rogue waves (RW), a special type of solitary wave, also known as monster waves, killer waves, extreme waves, and giant waves, have attracted much attention in the physical branch of mathematics. Rogue waves have been observed in many fields, such as oceanics [**2, 3, 14, 24, 26**], finance [**52**] and nonlinear optics [**15, 28, 54**], and there are several techniques, which can be used to investigate rogue waves, such as the dressing method, the Bäcklund transformation method, Darboux transformation (DT) method, bilinear method, etc., [**1, 8, 10, 11, 13, 17, 19–23, 25, 29, 30, 36–38, 47, 51, 53, 55–57**]. Recently, we have studied the breather wave, rogue wave and solitary wave solutions of some nonlinear differential equations by using the Hirota bilinear method [**5–7, 27, 31–35, 40–46, 48–50**].

In this paper, we mainly study the following inhomogeneous fifth-order nonlinear Schrödinger (NLS) equation [**4**]

$$\begin{aligned} &iq_t - i\varepsilon q_{xxxxx} - 10i\varepsilon|q|^2 q_{xxx} - 20i\varepsilon q_x q^* q_{xx} - 30i\varepsilon|q|^4 q_x \\ &- 10i\varepsilon(|q_x|^2 q)_x + q_{xx} + 2q|q|^2 - iq_x = 0, \end{aligned} \tag{1.2}$$

where $q = q(x,t)$ is a complex function, x and t denote the spatial coordinate and scaled time respectively, ε is a perturbation parameter, and the asterisk represents the complex conjugate.

As far as we know, the breather wave and rogue wave of eq. (1.2) have not previously been discovered. The primary purpose of the present paper is to employ the DT method to construct higher-order solitary wave, breather wave and rogue wave solutions of eq. (1.2), respectively.

The outline of this paper is as follows. In Section 2, we present a simple method to obtain the determinant representation of the n-fold DT. In Section 3, Based on the DT, we obtain the one- and two-soliton solutions, first-breather solution, first-rogue and second-rogue waves, respectively. Finally, some conclusions are discussed in the last section.

2. Darboux transformation. The Lax pairs corresponding to inhomogeneous fifth-order NLS equation (1.2) can be given by the two matrix spectral problems [4]

$$\begin{aligned} \psi_x &= U\psi, \\ \psi_t &= V\psi, \end{aligned} \tag{2.1}$$

where $\psi = (\phi_1, \phi_2)'$, and

$$U = \begin{pmatrix} -i\lambda & q \\ -q^* & i\lambda \end{pmatrix}, \qquad V = \begin{pmatrix} V_{11} & V_{12} \\ -V_{12}^* & -V_{11} \end{pmatrix}, \tag{2.2}$$

where

$$\begin{aligned} V_{11} = {} & -16i\lambda^5\varepsilon + 8i\lambda^3\varepsilon|q|^2 + 4\lambda^2\varepsilon(qq_x^* - q_xq^*) - 2i\lambda^2 \\ & - 2i\lambda\varepsilon(qq_{xx}^* + q_{xx}q^* - |q_x|^2 + 3|q|^4) \\ & - i\lambda + i|q|^2 + \varepsilon(q_{xxx}q^* - qq_{xxx}^* + q_xq_{xx}^* - q_{xx}q_x^* \\ & \qquad + 6|q|^2q^*q_x - 6|q|^2q_x^*q), \end{aligned} \tag{2.3}$$

$$\begin{aligned} V_{12} = {} & 16\lambda^4\varepsilon q + 8i\lambda^3\varepsilon q_x - 4\lambda^2\varepsilon(q_{xx} + 2|q|^2q) \\ & - 2i\lambda\varepsilon(q_{xxx} + 6|q|^2q_x) + 2\lambda q + iq_x + q \\ & + \varepsilon(q_{xxxx} + 8|q|^2q_{xx} + 2q^2q_{xx}^* + 4|q_x|^2q + 6q_x^2q^* + 6|q|^4q). \end{aligned} \tag{2.4}$$

Here, λ is a constant spectral parameter, ψ is called the eigenfunction associated with λ of eq. (1.2). In addition, eq. (1.2) is equivalent to the compatibility condition $U_t - V_x + [U, V] = 0$.

2.1. One-fold Darboux transformation. Now, we will introduce a simple gauge transformation

$$\psi^{[1]} = T^{[1]}\psi. \tag{2.5}$$

After this gauge transformation, we can transform linear problems (2.5) into a new one possessing the same matrix form, namely,

$$\psi_x^{[1]} = U^{[1]}\psi^{[1]}, \qquad U^{[1]}T^{[1]} = T_x^{[1]} + T^{[1]}U, \tag{2.6}$$

$$\psi_t^{[1]} = V^{[1]}\psi^{[1]}, \qquad V^{[1]}T^{[1]} = T_t^{[1]} + T^{[1]}V. \tag{2.7}$$

By cross differentiating (2.6) and (2.7), we obtain

$$U_t^{[1]} - V_x^{[1]} + [U^{[1]}, V^{[1]}] = T^{[1]}\left(U_t - V_x + [U, V]\right)(T^{[1]})^{-1}. \tag{2.8}$$

This means that, in order to make eq. (1.2) invariant under the transformation (2.5), it is necessary to search a matrix $T^{[1]}$ so that $U^{[1]}$ and $V^{[1]}$ have the same forms as U and V. At the same time, the old potential (or seed solution) (q, q^*) in spectral matrices U and V are mapped into new potentials (or new solution) $(q^{[1]}, q^{[1]*})$ in transformed spectral matrices $U^{[1]}$, $V^{[1]}$.

Next, we assume that the Darboux matrix $T^{[1]}$ in (2.5) is of the following form:

$$T^{[1]} = T^{[1]}(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \lambda + \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}, \tag{2.9}$$

where a_0, b_0, c_0 and d_0 are the functions of x and t, which will be expressed by the eigenfunctions associated with λ and seed solutions (q, q^*) in the Lax pair. Firstly, setting two eigenfunctions ψ_j as

$$\begin{aligned} &\psi_j = \begin{pmatrix} \phi_{j1} \\ \phi_{j2} \end{pmatrix}, \quad j = 1, 2, \ldots, 2n, \\ &\phi_{j1} = \phi_{j1}(x, t, \lambda_j), \qquad \phi_{j2} = \phi_{j2}(x, t, \lambda_j). \end{aligned} \tag{2.10}$$

Note that $\phi_1(x, t, \lambda)$ and $\phi_2(x, t, \lambda)$ are two components of eigenfunction ψ associated with λ in eq. (2.1). It should be pointed out that, since the eigenfunction

$$\psi_j = \begin{pmatrix} \phi_{j1} \\ \phi_{j2} \end{pmatrix} \tag{2.11}$$

is the solution of the eigenvalue *equations* (2.1) corresponding to λ_j, and the eigenfunction

$$\psi'_j = \begin{pmatrix} -\phi^*_{j2} \\ \phi^*_{j1} \end{pmatrix} \tag{2.12}$$

is also the solution of eq. (2.1) corresponding to λ^*_j, where $*$ denotes the complex conjugate.

From now on, we assume that even number eigenfunctions and eigenvalues are given by odd ones, as in the following rule:

$$\begin{aligned} \lambda_{2j} &= \lambda^*_{2j-1}, \\ \phi_{2j,1} &= -\phi^*_{2j-1,2}(\lambda_{2j-1}), \\ \phi_{2j,2} &= \phi^*_{2j-1,1}(\lambda_{2j-1}), \\ j &= 1,2,\dots,n. \end{aligned} \tag{2.13}$$

For convenience and simple mathematical operation, we derive the following theorem.

Theorem 2.1. *The elements of a one-fold Darboux matrix are presented with the eigenfunction ψ_1 corresponding to the eigenvalue λ_1, as follows:*

(2.14)

$$a_0 = -\frac{1}{\Delta_2}\begin{vmatrix} \lambda_1\phi_{11} & \phi_{12} \\ \lambda_2\phi_{21} & \phi_{22}\end{vmatrix}, \qquad b_0 = \frac{1}{\Delta_2}\begin{vmatrix} \lambda_1\phi_{11} & \phi_{11} \\ \lambda_2\phi_{21} & \phi_{21}\end{vmatrix},$$

$$c_0 = \frac{1}{\Delta_2}\begin{vmatrix} \phi_{12} & \lambda_1\phi_{12} \\ \phi_{22} & \lambda_2\phi_{22}\end{vmatrix}, \qquad d_0 = -\frac{1}{\Delta_2}\begin{vmatrix} \phi_{11} & \lambda_1\phi_{12} \\ \phi_{21} & \lambda_2\phi_{22}\end{vmatrix},$$

$$\Longleftrightarrow T^{[1]}(\lambda;\lambda_1) = \begin{pmatrix} \lambda - \frac{1}{\Delta_2}\begin{vmatrix} \lambda_1\phi_{11} & \phi_{12} \\ \lambda_2\phi_{21} & \phi_{22}\end{vmatrix} & \frac{1}{\Delta_2}\begin{vmatrix} \lambda_1\phi_{11} & \phi_{11} \\ \lambda_2\phi_{21} & \phi_{21}\end{vmatrix} \\ \frac{1}{\Delta_2}\begin{vmatrix} \phi_{12} & \lambda_1\phi_{12} \\ \phi_{22} & \lambda_2\phi_{22}\end{vmatrix} & \lambda - \frac{1}{\Delta_2}\begin{vmatrix} \phi_{11} & \lambda_1\phi_{12} \\ \phi_{21} & \lambda_2\phi_{22}\end{vmatrix} \end{pmatrix},$$

with

$$\Delta_2 = \begin{vmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22}\end{vmatrix}, \tag{2.15}$$

and then the new solution $q^{[1]}$ is given by

$$q^{[1]} = q + 2i\frac{1}{\Delta_2}\begin{vmatrix} \lambda_1\phi_{11} & \phi_{11} \\ \lambda_2\phi_{21} & \phi_{21}\end{vmatrix}, \tag{2.16}$$

and the new eigenfunction $\psi_j^{[1]}$ corresponding to λ_j is

$$\psi_j^{[1]} = T^{[1]}(\lambda;\lambda_1)|_{\lambda=\lambda_j}\psi_j. \tag{2.17}$$

2.2. n-fold Darboux transformation. By n-times iteration of the one-fold DT $T^{[1]}$, we obtain the n-fold DT $T^{[n]}$ of eq. (1.2) with the special choice on λ_{2j} and ψ_{2j} in (2.13). In order to save space, we omit the tedious calculations of $T^{[n]}$ and its determinant representation. Then, we give $q^{[n]}$ in the following theorem.

Theorem 2.2. *Under the choice of eq.* (2.13)*, the n-fold* DT $T^{[n]}$ *generates a new solution of eq.* (1.2) *from a seed solution q, as*:

$$q^{[n]} = q - 2i\frac{N_{2n}}{D_{2n}}, \tag{2.18}$$

where

(2.19)

$$N_{2n} = \begin{vmatrix} \phi_{11} & \phi_{12} & \lambda_1\phi_{11} & \lambda_1\phi_{12} & \cdots & \lambda_1^{n-1}\phi_{11} & \lambda_1^{n}\phi_{11} \\ \phi_{21} & \phi_{22} & \lambda_2\phi_{21} & \lambda_2\phi_{22} & \cdots & \lambda_2^{n-1}\phi_{21} & \lambda_2^{n}\phi_{21} \\ \phi_{31} & \phi_{32} & \lambda_3\phi_{31} & \lambda_3\phi_{32} & \cdots & \lambda_3^{n-1}\phi_{31} & \lambda_3^{n}\phi_{31} \\ \phi_{41} & \phi_{42} & \lambda_4\phi_{41} & \lambda_4\phi_{42} & \cdots & \lambda_4^{n-1}\phi_{41} & \lambda_4^{n}\phi_{41} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi_{2n1} & \phi_{2n2} & \lambda_{2n}\phi_{2n1} & \lambda_{2n}\phi_{2n2} & \cdots & \lambda_{2n}^{n-1}\phi_{2n1} & \lambda_{2n}^{n}\phi_{2n1} \end{vmatrix},$$

(2.20)

$$D_{2n} = \begin{vmatrix} \phi_{11} & \phi_{12} & \lambda_1\phi_{11} & \lambda_1\phi_{12} & \cdots & \lambda_1^{n-1}\phi_{11} & \lambda_1^{n-1}\phi_{12} \\ \phi_{21} & \phi_{22} & \lambda_2\phi_{21} & \lambda_2\phi_{22} & \cdots & \lambda_2^{n-1}\phi_{21} & \lambda_2^{n-1}\phi_{22} \\ \phi_{31} & \phi_{32} & \lambda_3\phi_{31} & \lambda_3\phi_{32} & \cdots & \lambda_3^{n-1}\phi_{31} & \lambda_3^{n-1}\phi_{32} \\ \phi_{41} & \phi_{42} & \lambda_4\phi_{41} & \lambda_4\phi_{42} & \cdots & \lambda_4^{n-1}\phi_{41} & \lambda_4^{n-1}\phi_{42} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi_{2n1} & \phi_{2n2} & \lambda_{2n}\phi_{2n1} & \lambda_{2n}\phi_{2n2} & \cdots & \lambda_{2n}^{n-1}\phi_{2n1} & \lambda_{2n-1}^{n}\phi_{2n2} \end{vmatrix}.$$

3. The explicit solutions. In this section, we will use Theorem 2.2 to construct the explicit solutions of eq. (1.2), including the solitary wave, breather wave and rogue wave solutions.

3.1. Solitary wave solutions.

(i) Let the seed $q = 0$ and $\lambda_1 = \alpha + i\beta$. Then,

$$\begin{aligned} \phi_{11} &= e^{-i(\lambda_1 x + (16\varepsilon\lambda_1^5 + 2\lambda_1^2 + \lambda_1)t)}, \\ \phi_{12} &= e^{i(\lambda_1 x + (16\varepsilon\lambda_1^5 + 2\lambda_2^2 + \lambda_1)t)}, \end{aligned} \tag{3.1}$$

where $\phi_{11} = \phi_{22}^*, \phi_{12} = -\phi_{21}^*$.

Taking ϕ_{11} and ϕ_{12} given by eq. (3.1) into (2.16), we obtain one soliton solution

$$q^{[1]} = \frac{2\beta e^{-2if_2}}{\cosh(2f_1)}, \tag{3.2}$$

with

$$\begin{aligned} f_1 &= -16\beta^5\varepsilon t + 160\alpha^2\beta^3\varepsilon t - 80\alpha^4\beta\varepsilon t - 4\alpha\beta t - \beta t - \beta x, \\ f_2 &= 80\alpha\beta^4\varepsilon t - 160\alpha^3\beta^2\varepsilon t + 16\alpha^5\varepsilon t - 2\beta^2 t + 2\alpha^2 t + \alpha t + \alpha x. \end{aligned} \tag{3.3}$$

(ii) Let the seed $q = 0$ and $\lambda_1 = \alpha + i\beta$, $\lambda_3 = \xi + i\eta$. By solving linear problems (2.1), the eigenfunctions can be obtained as following:

$$\begin{aligned} \phi_{11} &= e^{-i(\lambda_1 x + (16\varepsilon\lambda_1^5 + 2\lambda_1^2 + \lambda_1)t)}, \\ \phi_{12} &= e^{i(\lambda_1 x + (16\varepsilon\lambda_1^5 + 2\lambda_1^2 + \lambda_1)t)}, \\ \phi_{31} &= e^{-i(\lambda_3 x + (16\varepsilon\lambda_3^5 + 2\lambda_3^2 + \lambda_3)t)}, \\ \phi_{32} &= e^{i(\lambda_3 x + (16\varepsilon\lambda_3^5 + 2\lambda_3^2 + \lambda_3)t)}, \end{aligned} \tag{3.4}$$

where $\phi_{11} = \phi_{22}^*$, $\phi_{12} = -\phi_{21}^*$, $\phi_{31} = \phi_{42}^*$ and $\phi_{32} = -\phi_{41}^*$.

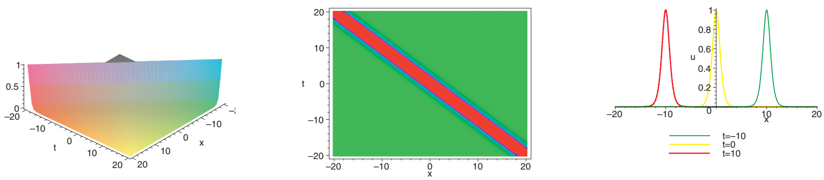
Choose $n = 2$ in eq. (2.18). Then, we have

$$q^{[2]} = -2i\frac{N_4}{D_4}, \tag{3.5}$$

where

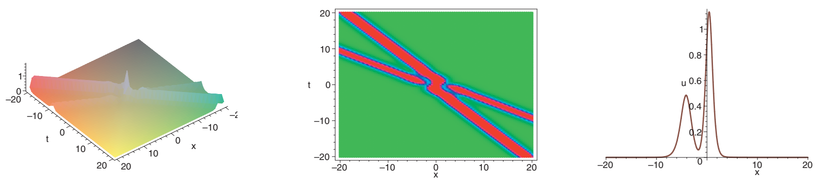
$$\begin{aligned} N_4 &= \begin{vmatrix} \phi_{11} & \phi_{12} & \lambda_1\phi_{11} & \lambda_1^2\phi_{11} \\ \phi_{21} & \phi_{22} & \lambda_2\phi_{21} & \lambda_2^2\phi_{21} \\ \phi_{31} & \phi_{32} & \lambda_3\phi_{31} & \lambda_3^2\phi_{31} \\ \phi_{41} & \phi_{42} & \lambda_4\phi_{41} & \lambda_4^2\phi_{41} \end{vmatrix}, \\ D_4 &= \begin{vmatrix} \phi_{11} & \phi_{12} & \lambda_1\phi_{11} & \lambda_1\phi_{12} \\ \phi_{21} & \phi_{22} & \lambda_2\phi_{21} & \lambda_2\phi_{22} \\ \phi_{31} & \phi_{32} & \lambda_3\phi_{31} & \lambda_3\phi_{32} \\ \phi_{41} & \phi_{42} & \lambda_4\phi_{41} & \lambda_4\phi_{42} \end{vmatrix}. \end{aligned} \tag{3.6}$$

Based on this, we can obtain the two-soliton solution of eq. (1.2). In addition, through iterations of the DT, we can directly obtain the n-soliton solution of eq. (1.2) from a trivial solution.



(a) (b) (c)

Figure 1. (Color online). One-soliton wave (3.2) for eq. (1.2) by choosing suitable parameters: $\alpha = 0.5$, $\beta = 0.5$, $\varepsilon = 0.5$. (a) Perspective view of the real part of the wave. (b) Overhead view of the wave. (c) Wave propagation pattern of the x axis.



(a) (b) (c)

Figure 2. (Color online). Two-soliton wave (3.5) for eq. (1.2) by choosing suitable parameters: $\alpha = 0.5$, $\beta = 0.5$, $\xi = 1/3$, $\eta = 1/3$, $\varepsilon = 1/2$. (a) Perspective view of the real part of the wave. (b) Overhead view of the wave. (c) Wave propagation pattern of the x axis.

Figures 1 and 2 describe the one- and two- soliton solutions, respectively. Figure 1 describes the one-soliton solution. By choosing suitable parameters, we can observe the amplitude of the one-soliton $|q^{[1]}|^2$ of eq. (1.2). Figure 2 describes the two-soliton solution. Applying all of these effects for $|q^{[2]}|^2$, we obtain something similar to Figure 1.

3.2. The first-order breather wave solution. In this section, we first solve the eigenfunctions associated with a periodic seed q and use it to obtain a first-order breather by using the determinant representation of one-fold DT in (2.16).

Starting with a non-zero seed

$$q = ce^{i\rho}, \tag{3.7}$$

with $\rho = ax+bt$, $b = 2c^2+a-a^2+\varepsilon(a^5-20a^3c^2+30ac^4)$, $a, b, c \in \mathcal{R}$. By using the principle of superposition of the linear differential equations, then the new eigenfunctions corresponding to λ_j can be provided by

$$\psi_j = \begin{pmatrix} d_1ce^{i(\rho/2+d)} + d_2i(a/2+c_1+\lambda_j)e^{i(\rho/2-d)} \\ d_2ce^{-i(\rho/2+d)} + d_1i(a/2+c_1+\lambda_j)e^{-i(\rho/2-d)} \end{pmatrix}, \tag{3.8}$$

with

$$\begin{aligned} c_1 &= \sqrt{c^2 + \left(\lambda_j + \frac{a}{2}\right)^2} \\ &= h_R + ih_I, \qquad d = (x + c_2t)c_1, \\ d_1 &= e^{ic_1(s_0+s_1\delta+\cdots+s_{n-1}\delta^{n-1})}, \\ d_2 &= e^{-ic_1(s_0+s_1\delta+\cdots+s_{n-1}\delta^{n-1})}, \\ c_2 &= a^4\varepsilon - 2a^3\varepsilon\lambda_j - 12a^2c^2\varepsilon + 4a^2\varepsilon\lambda_j^2 + 12ac^2\varepsilon\lambda_j \\ &\quad - 8a\varepsilon\lambda_j^3 + 6c^4\varepsilon - 8c^2\varepsilon\lambda_j^2 + 16\varepsilon\lambda_j^4 - a + 2\lambda_j + 1 \\ &= d_R + id_I. \end{aligned} \tag{3.9}$$

Here, $s_i \in \mathcal{C}$, $i = 0, 1, 2, \ldots, n-1$, δ is an infinitesimal parameter.

For convenience, let $a = -2\alpha$. Using the one-fold DT and $\lambda_1 = \alpha+i\beta$ $(j = 1)$, then we obtain the following first-order breather:

$$q_{br}^{[1]} = \left(c + \frac{2\beta\{[\omega_1\cos(2G)-\omega_2\cosh(2F)]-i[(\omega_1-2c^2)\sin(2G)-\omega_3\sinh(2F)]\}}{\omega_1\cosh(2F)-\omega_2\cos(2G)}\right)e^{i\rho}, \tag{3.10}$$

with

$$\begin{aligned} \omega_1 &= c^2 + (h_I+\beta)^2 + \left(\alpha + h_R + \frac{a}{2}\right)^2, \\ \omega_2 &= 2c(h_I+\eta), \qquad \omega_3 = 2c\left(\alpha + h_R + \frac{a}{2}\right), \\ F &= xh_I + (d_Rh_I + d_Ih_R)\,t, \\ G &= xh_R + (d_Rh_R - d_Ih_I)\,t. \end{aligned} \tag{3.11}$$

This is a periodic traveling wave. The coefficient ε can affect the period of the breather wave through G.

It is not difficult to find that

$$|q_{br}^{[1]}|^2 = (c+2\beta)^2, \tag{3.12}$$

which is the height of peaks of this breather wave. From eq. (3.12), we can easily find that height has nothing to do with a, α and ε, but this does not mean that ε cannot affect the properties of the breather. As a matter of fact, it is not hard to see from eq. (3.10) that ε actually controls the period of the breather wave. This observation can clearly be seen in Figure 3.

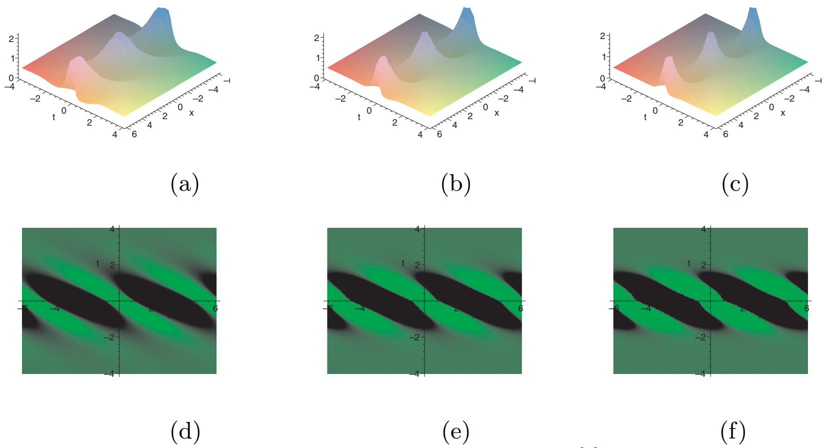


Figure 3. (Color online). Dynamical evolution of $|q_{br}^{[1]}|^2$ of eq. (3.10) with specific parameter $\alpha = 0.2$, $\beta = 0.4$, $c = 0.7$, $s_0 = 0$. Perspective view of the real part of the wave: (a) $\varepsilon = 1$. (b) $\varepsilon = 1.5$. (c) $\varepsilon = 2$. Overhead view of the wave: (d) $\varepsilon = 1$. (e) $\varepsilon = 1.5$. (f) $\varepsilon = 2$.

3.3. Higher-order rogue wave solutions. In this section, we shall construct higher-order rogue waves of eq. (1.2). We mainly use (3.8) to study the higher-order rogue waves. Generally, it is difficult to derive higher-order rogue waves from multi-breather solutions, since the explicit expression of the nth order breather is very challenging to calculate when $n \geq 2$. We can overcome this problem by using the coefficient of the Taylor expansion in the determinant representation of a higher-order breather $q^{[n]}$ [**12**].

When $n = 1$, the first-order rogue wave of eq. (1.2) follows from

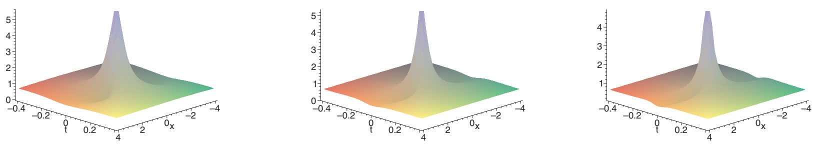
$$q_{rw}^{[1]} = -\left(\frac{T - 16ic^2t + 160ic^2a^3t\varepsilon - 480ic^4at\varepsilon - 3}{T + 1}\right)ce^{i\rho}, \tag{3.13}$$

with

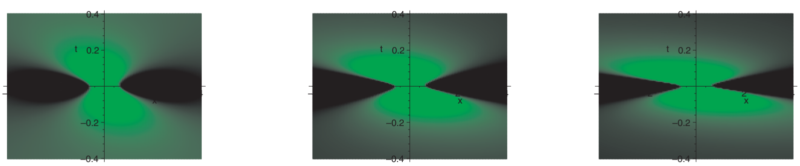
(3.14)

$$T = 100a^8c^2t^2\varepsilon^2 - 800a^6c^4t^2\varepsilon^2 + 6000a^4c^6t^2\varepsilon^2 + 3600c^{10}t^2\varepsilon^2 - 80a^5c^2t^2\varepsilon$$
$$+ 640a^3c^4t^2\varepsilon + 480ac^6t^2\varepsilon + 40a^4c^2s_0t\varepsilon + 40a^4c^2t^2\varepsilon + 40a^4c^2tx\varepsilon$$
$$- 480a^2c^4s_0t\varepsilon - 480a^2c^4t^2\varepsilon - 480a^2c^4tx\varepsilon + 240c^6s_0t\varepsilon + 240c^6t^2\varepsilon$$
$$+ 240c^6tx\varepsilon + 16a^2c^2t^2 + 16c^4t^2 - 16ac^2s_0t - 16ac^2t^2 - 16ac^2tx$$
$$+ 4c^2s_0^2 + 8c^2s_0t + 8c^2s_0x + 4c^2t^2 + 8c^2tx + 4c^2x^2.$$

It is trivial to find that $|q_{rw}^{[1]}|^2 = c^2$ when $x \to \infty$ and $t \to \infty$. This also means that the asymptotic plane of $|q_{rw}^{[1]}|^2$ has the height c^2. In Figure 4, for larger values of ε, it is clear that the compressions in the t direction are quite high.



(a) (b) (c)



(d) (e) (f)

Figure 4. (Color online). Dynamical evolution of the first-order rogue wave $|q_{rw}^{[1]}|^2$ of eq. (3.13) with specific parameters $c = 0.8$, $a = 0.5$, $s_0 = 0$. Perspective view of the real part of the wave: (a) $\varepsilon = 1$. (b) $\varepsilon = 2$. (c) $\varepsilon = 3$. Overhead view of the wave: (d) $\varepsilon = 1$. (e) $\varepsilon = 2$. (f) $\varepsilon = 3$.

When $n = 2$, we construct the analytical formulas for the second-order rogue wave. However, due to their long expressions in describing the solution, we do not present them here. The second-order rogue wave consists of two patterns. The first part is the fundamental pattern; it has a highest peak surrounded by four small equal peaks in two sides. Its evolution is presented in Figure 5 with the condition $s_0 = s_1 = 0$. From Figure 5, we can see that ε can affect high compression in the t direction. The second part is a triangular pattern, which consists of three equal peaks. As is shown in Figure 6, when taking $d_1 \neq 1$, and $d_2 \neq 1$, the large peak of the second-order rational solution is completely separated and forms a set of three first-order rational solution for sufficiently large s_1, while $s_0 = 0$, and actually forms an equilateral triangle. From Figure 6, we can see that, as the value of ε increases, the rogue wave compression increases.

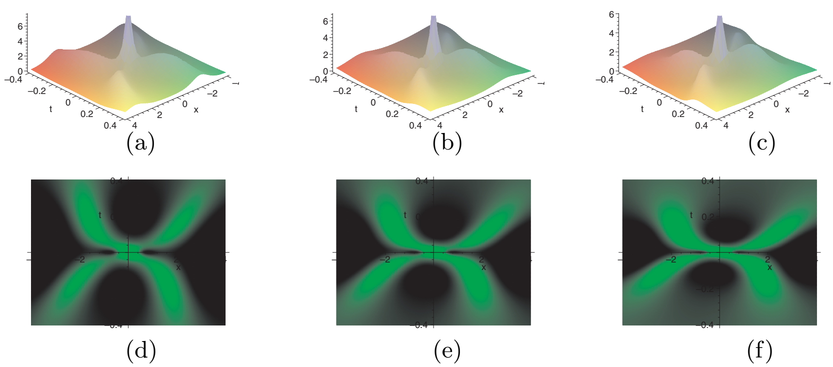


Figure 5. (Color online). Dynamical evolution of the second-order rogue wave $|q_{rw}^{[2]}|^2$ with specific parameters $c = 0.6$, $a = 0.5$, $s_1 = 0$. Perspective view of the real part of the wave: (a) $\varepsilon = 1$. (b) $\varepsilon = 1.5$. (c) $\varepsilon = 2$. Overhead view of the wave: (d) $\varepsilon = 1$. (e) $\varepsilon = 1.5$. (f) $\varepsilon = 2$.

4. Conclusion and discussions. In this work, we have investigated an inhomogeneous fifth-order nonlinear Schrödinger equation from Heisenberg ferromagnetism. From its Lax pair, we obtained the n-fold Darboux matrix of eq. (1.2). On the basis of the Darboux transformation and some periodic seed solutions, we obtained the first-order

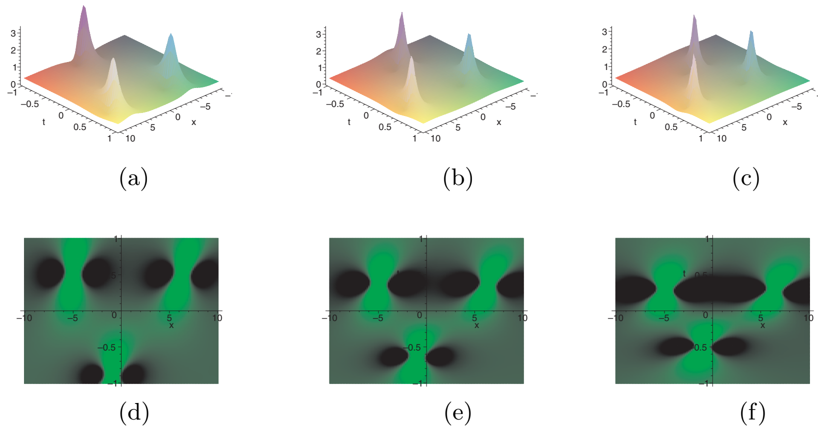


Figure 6. (Color online). Dynamical evolution of the second-order rogue wave $|q_{rw}^{[2]}|^2$ with specific parameters $c = 0.6$, $a = 0.5$, $s_1 = 100$. Perspective view of the real part of the wave: (a) $\varepsilon = 1$. (b) $\varepsilon = 1.5$. (c) $\varepsilon = 2$. Overhead view of the wave: (d) $\varepsilon = 1$. (e) $\varepsilon = 1.5$. (f) $\varepsilon = 2$.

breather wave solution. In addition, by the Taylor expansion, we constructed the first- and second-order rogue wave solutions. All of these solutions have parameter ε denoting the contribution of higher-order nonlinear terms. The compressed effects of these solutions were discussed through numerical plots by increasing the value of ε in Figures 3–6. We hope that the results obtained in this paper will help to better study breather and rogue waves in Heisenberg ferromagnetism.

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