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HOMOLOGICAL PROPERTIES OF THE ALGEBRA OF COMPACT OPERATORS ON A BANACH SPACE

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ABSTRACT. The conditions on a Banach space E under which the algebra $\mathcal{K}(E)$ of compact operators on E is right flat or homologically unital are investigated. These homological properties are related to factorization in the algebra, and, it is shown that, for $\mathcal{K}(E)$, they are closely associated with the approximation property for E. The class of spaces E such that $\mathcal{K}(E)$ is known to be right flat and homologically unital is extended to include spaces which do not have the bounded compact approximation property.

1. Introduction. This note is concerned with the flatness and H-unitality of certain algebras of operators on Banach spaces. The algebra of all operators on the Banach space E will be denoted $\mathcal{B}(E)$. Most of the paper will deal with the norm closure of the ideal of finite rank operators on E, which will be denoted $\mathcal{F}(E)$. The ideal of compact operators on E, which might not equal $\mathcal{F}(E)$ when E does not have the approximation property, will be denoted $\mathcal{K}(E)$.

Flatness and H-unitality have been studied by Wodzicki [**17, 18, 19**] for a number of algebras arising in functional analysis. These are important in the proof of Karoubi's conjecture of the equality of the algebraic and topological K-theory groups of stable C^*-algebras [**12, 13**]. Theorems 4.1 and 5.2 below give a partial solution to a question, posed in [**18**], concerning the H-unitality of $\mathcal{K}(E)$.

2. Some homological algebra. Let A be an algebra and X a right A-module. Then, X is said to be a *flat* module if, for every exact

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sequence of left A-modules,

$$0 \longrightarrow M' \xrightarrow{v} M \xrightarrow{w} M'' \longrightarrow 0,$$

the sequence

$$0 \longrightarrow X \otimes_A M' \xrightarrow{I\otimes v} X \otimes_A M \xrightarrow{I\otimes w} X \otimes_A M'' \longrightarrow 0$$

is exact [**4,** I.2.3, Definition 2]. The characterization of flat modules which will be most convenient for use here is that given in [**4,** I.2.11, Corollary 1]. The right A-module X is flat if and only if, whenever $\{e_i\}_{i\in\mathbb{I}}$ and $\{b_i\}_{i\in\mathbb{I}}$ are elements of X and A, respectively, such that $\Sigma_{i\in\mathbb{I}} e_i b_i = 0$, there are elements of X, $\{x_j\}_{j\in\mathbb{J}}$, and elements of A, $\{a_{j,i}\}_{(j\in\mathbb{J}, i\in\mathbb{I})}$, such that

$$\sum_{i\in\mathbb{I}} a_{j,i} b_i = 0 \quad \text{for each } j$$

and

$$e_i = \sum_{j\in\mathbb{J}} x_j a_{j,i} \quad \text{for each } i.$$

The algebra A is said to be *right universally flat* if, whenever A is embedded as a right ideal in an algebra B, A is a flat right B-module. It is easy to verify that, if A has a left unit, then it is right universally flat. There is a corresponding notion of flatness for left A-modules and left universally flat algebras.

For an algebra A, let $A^{\otimes n}$ denote the n-fold tensor product $A \otimes \cdots \otimes A$, and define a map

$$\delta_n : A^{\otimes n} \longrightarrow A^{\otimes(n-1)}$$

by

$$\delta_n(a_1 \otimes \cdots \otimes a_n) = \sum_{j=1}^{n-1} (-1)^{j-1} a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n.$$

Then, A is said to be *homologically unital*, abbreviated H-*unital*, if the sequence

$$0 \longleftarrow A \xleftarrow{\delta_2} A \otimes A \xleftarrow{\delta_3} A \otimes A \otimes A \xleftarrow{\delta_4} \cdots \tag{2.1}$$

is exact.

The notion of H-unitality was introduced by Wodzicki in [**17, 18**]. It was shown in those papers that the H-unitality of A is equivalent to A having the excision property in various homology theories, [**17,** Proposition 2], [**18,** Theorem 3.1].

If A is right (or left) universally flat, then it is H-unital. In order to see this, suppose that A is right universally flat, and let $A^{\#}$ denote the algebra obtained by adjoining a unit 1 to A. Then, A is a right ideal in $A^{\#}$. Given

$$v = \sum_{j\in\mathbb{J}} a_{j,1} \otimes \cdots \otimes a_{j,n} \quad \text{in } A^{\otimes n},$$

choose, for each $k = 3, \ldots, n$, a basis $\{b_{1,k}, \ldots, b_{m_k,k}\}$ for $\operatorname{span}\{a_{j,k-1}, a_{j,k-1}a_{j,k}, a_{j,k} : j \in \mathbb{J}\}$. Let $\mathbb{B} = \{b_{j_3,3} \otimes \cdots \otimes b_{j_n,n} : j_k = 1, 2, \ldots, m_k;\ k = 3, 4, \ldots, n\}$. Then, $\mathbb{B}$ is a linearly independent set in $A^{\otimes(n-2)}$, and

$$\begin{aligned}\delta_n(v) &= \sum_{j\in\mathbb{J}} \sum_{k=1}^{n-1} (-1)^{k-1} a_{j,1} \otimes \cdots \otimes a_{j,k}a_{j,k+1} \otimes \cdots \otimes a_{j,n} \\ &= \sum_{b\in\mathbb{B}} \left(\sum_{j\in\mathbb{J}} a_{j,1}c_{j,b} \right) \otimes b,\end{aligned}$$

where each $c_{j,b}$ belongs to $A^{\#}$ for each $j \in \mathbb{J}$ and each b in $\mathbb{B}$. Hence, if we now suppose that $\delta_n(v) = 0$, then $\Sigma_{j\in\mathbb{J}} a_{j,1}c_{j,b} = 0$ for each b in $\mathbb{B}$. This is equivalent to $\Sigma_{j\in\mathbb{J}} a_{j,1} \otimes \widetilde{c}_j = 0$ in $A \otimes_{A^{\#}} F$, where F denotes the direct sum of $|\mathbb{B}|$ copies of $A^{\#}$ and $\widetilde{c}_j = (c_{j,b})_{b\in\mathbb{B}}$. Since A is right universally flat, it is a flat right $A^{\#}$-module, and thus, by [**4,** I.2.3, Proposition 1], A is F-flat. Hence, from [**4,** I.2.11, Proposition 13], there are a family $\{x_p\}_{p\in\mathbb{P}}$ of elements of A and a family $\{f_{p,j}\}_{(p\in\mathbb{P}, j\in\mathbb{J})}$ of elements of $A^{\#}$ such that

$$\sum_{j\in\mathbb{J}} f_{p,j}\widetilde{c}_j = 0 \quad \text{for all } p \in \mathbb{P}, \tag{2.2}$$

and

$$a_{j,1} = \sum_{p\in\mathbb{P}} x_p f_{p,j} \quad \text{for all } j \in \mathbb{J}. \tag{2.3}$$

Now, it follows from (2.2) that

$$\delta_n\bigg(\sum_{j\in\mathbb{J}} f_{p,j}\otimes a_{j,2}\otimes\cdots\otimes a_{j,n}\bigg) = \sum_{b\in\mathbb{B}}\bigg(\sum_{j\in\mathbb{J}} f_{p,j}c_{j,b}\bigg)\otimes b = 0,$$

for each p in $\mathbb{P}$; hence, if we define w in $A\otimes A^{\#}\otimes A^{\otimes(n-1)}$ by

$$w = \sum_{p\in\mathbb{P}}\sum_{j\in\mathbb{J}} x_p\otimes f_{p,j}\otimes a_{j,2}\otimes\cdots\otimes a_{j,n},$$

then, from (2.3),

$$\begin{aligned}\delta_{n+1}(w) &= \sum_{j\in\mathbb{J}}\bigg(\sum_{p\in\mathbb{P}} x_p f_{p,j}\bigg)\otimes a_{j,2}\otimes\cdots\otimes a_{j,n}\\ &\qquad - \sum_{p\in\mathbb{P}} x_p\otimes\delta_n\bigg(\sum_{j\in\mathbb{J}} f_{p,j}\otimes a_{j,2}\otimes\cdots\otimes a_{j,n}\bigg)\\ &= \sum_{j\in\mathbb{J}} a_{j,1}\otimes a_{j,2}\cdots\otimes a_{j,n} = v.\end{aligned}$$

Now, write $f_{p,j} = c1 + f'_{p,j}$, where c is a scalar and each $f'_{p,j}$ belongs to A. Next, set

$$w' = \sum_{p\in\mathbb{P}}\sum_{j\in\mathbb{J}} x_p\otimes f'_{p,j}\otimes a_{j,2}\otimes\cdots\otimes a_{j,n}.$$

Then, w' belongs to $A^{\otimes(n+1)}$ and, as is easily checked, $\delta_{n+1}(w') = v$. Therefore, the sequence (2.1) is exact at $A^{\otimes n}$, and so, A is H-unital.

In [**18,** Section 8], it is asked for which Banach spaces E is it the case that $\mathcal{K}(E)$ is H-unital. This question is partially answered in [**18, 19**]. The stronger result is in [**19**], where a certain factorization property (Φ) is given which, if satisfied by an algebra, implies that the algebra is right universally flat, and hence, H-unital. It follows from the proof of Cohen's factorization theorem [**3,** Theorem 11.10] that a Banach algebra with a bounded left approximate identity satisfies (Φ). Now, $\mathcal{K}(E)$ has a bounded left approximate identity if and only if E has the bounded compact approximation property [**5,** Theorem 2.6], and thus, $\mathcal{K}(E)$ is right universally flat if E has this property [**19,** Theorem 8(f)].

It is shown below in Theorem 4.1 that $\mathcal{K}(E)$ can be right universally flat when E does not have the bounded compact approximation property. However, in the circumstances considered, E must have the approximation property, in which case $\mathcal{F}(E) = \mathcal{K}(E)$. Consequently, all of the following discussion will be concerned with the conditions under which $\mathcal{F}(E)$ is right universally flat. Note that the bounded compact approximation property does not imply the approximation property [**14**], and thus, there are spaces to which the result of Wodzicki applies that are not covered by Theorem 4.1.

For the new cases covered by Theorem 4.1, E does not have the bounded approximation property; therefore, $\mathcal{F}(E)$ does not have a bounded left approximate identity [**5,** Theorem 2.6]. It is unclear whether $\mathcal{F}(E)$ satisfies the factorization property (Φ) in these cases. Right universal flatness is proved by another method, which at one point requires an automatic continuity lemma.

3. An automatic continuity lemma. Lemma 3.1 concerns right multipliers, where a right multiplier on an algebra A is a linear map

$$T : A \longrightarrow A$$

such that $T(ab) = aT(b)$ for all a and b in A. Note that, if an algebra A is a right ideal in a larger algebra B, then, for each b in B, the map

$$a \longmapsto ab : A \longrightarrow A$$

is a right multiplier on A, and it is in this context that we will be applying Lemma 3.1.

Lemma 3.1. *Let E be a Banach space, and let T be a right multiplier on $\mathcal{F}(E)$. Then, T is continuous.*

Proof. Following [**10,** Section 1], let $\mathfrak{G}(T) = \{a \in \mathcal{F}(E) :$ there are $a_n \in \mathcal{F}(E)$, $n = 1, 2, \ldots$, such that $\lim_{n\to\infty} ||a_n|| = 0$ and $\lim_{n\to\infty} ||a - T(a_n)|| = 0\}$ be the separating space of T and $\mathfrak{J}_T = \{a \in \mathcal{F}(E) : a\mathfrak{G}(T) = (0)\}$ the continuity ideal of T. Then, since T is a right multiplier, $\mathfrak{G}(T)$ is a closed left ideal and $\mathfrak{J}_T$ is a closed two-sided ideal in $\mathcal{F}(E)$.

Lemma 3.1 is straightforward if E is finite-dimensional, and thus, we may suppose that E is infinite-dimensional and choose sequences

$\{e_n\}_{n=1}^\infty$ in E and $\{e_n^*\}_{n=1}^\infty$ in E^* such that $\langle e_i^*, e_j\rangle = \delta_{i,j}$. Then, $\{e_n \otimes e_n^*\}_{n=1}^\infty$ is a sequence of mutually orthogonal rank one projections on E. It follows from an application of [**10,** Lemma 1.6] that there is an n such that $e_n \otimes e_n^*$ belongs to $\mathfrak{J}_T$. Since $\mathcal{F}(E)$ is topologically simple, it follows that $\mathfrak{J}_T = \mathcal{F}(E)$, whence the separating space of T is zero. Therefore, by [**10,** Lemma 1.2], T is continuous. □

For another proof of Lemma 3.1 in the cases in which we will apply it, see Corollary 6.7 below. Although it will not be required in what follows, note that the set of all multipliers on A is an algebra under composition, and we have the following.

Corollary 3.2. *The algebra of right multipliers of $\mathcal{F}(E)$ is isomorphic to $\mathcal{B}(E^*)$.*

Proof. For each operator T in $\mathcal{B}(E^*)$, define a right multiplier $\phi(T)$ on $\mathcal{F}(E)$ by $\phi(T)(e \otimes e^*) = e \otimes Te^*$, for rank one operators $e \otimes e^*$, and then extend it to all of $\mathcal{F}(E)$ by linearity and continuity. Then, ϕ is clearly an injective algebra homomorphism from $\mathcal{B}(E^*)$ into the algebra of right multipliers on $\mathcal{F}(E)$.

In order to show that ϕ is a surjection, let $p = e \otimes e^*$ be a rank one projection on E and M a right multiplier on $\mathcal{F}(E)$. Then, since $M(pa) = pM(a)$ for all a in $\mathcal{F}(E)$, it follows that M leaves the space of $e \otimes E^*$ invariant. This subspace is isomorphic to E^*, and thus, the restriction of M to the subspace determines an operator T on E^* which is continuous by Lemma 3.1. It is easily verified that $\phi(T) = M$. □

Similar arguments to the above show that all left multipliers on $\mathcal{F}(E)$ are continuous and that the algebra of all left multipliers on $\mathcal{F}(E)$ is isomorphic to $\mathcal{B}(E)$.

4. Right universal flatness of $\mathcal{F}(E)$. As has already been pointed out, the factorization property Φ implies right universal flatness. On the other hand, if an algebra A is H-unital, then exactness of the sequence (2.1) at A implies that A is the span of its products. It is thus clear that factorization is closely connected with the homological properties in which we are interested. For this reason, the spaces C_p, $1 \le p \le \infty$, introduced by Johnson [**8**] will be important.[1] These spaces have the property that, for any Banach space E, every operator in $\mathcal{F}(E)$ factors through C_p, [**8,** Theorem 1].

In order to describe these spaces, first define a distance $d(X,Y)$ between isomorphic Banach spaces X and Y by

$$d(X,Y) = \inf\{||T||\,||T^{-1}|| : T \text{ is an isomorphism from } X \text{ to } Y\}.$$

Now, let $\{G_i\}_{i=1}^{\infty}$ be a sequence of finite-dimensional Banach spaces such that:

(i) for each finite dimensional space E and each $\epsilon > 0$ there is an i such that $d(E, G_i) < 1 + \epsilon$; and

(ii) for each i, there are infinitely many $j \neq i$ such that G_j is isometric to G_i.

Next, define, for $1 \leq p \leq \infty$,

$$C_p = \left(\bigoplus_{i=1}^{\infty} G_i \right)_p ,$$

where the subscript p indicates the ℓ_p direct sum. Up to isomorphism, the space C_p is independent of the choice of sequence of finite-dimensional spaces $\{G_i\}_{i=1}^{\infty}$ [**11,** Definition 12.3].

Many theorems regarding the spaces C_p describe what are called in [**11**] their 'universal complement properties,' for examples, see [**8,** Theorem 4] and [**11,** Theorem 12.4, Proposition 13.13]. The content of all of these results is that, if a Banach space E has some variant of the approximation property, then $E \oplus C_p$ has a better approximation property. The theorem we now prove is of a similar nature.

Theorem 4.1. *For each Banach space E the following are equivalent*:

(1) *E has the approximation property*;
(2) *$\mathcal{F}(E \oplus C_1)$ is right flat*; *and*
(3) *$\mathcal{F}(E \oplus C_1)$ is right universally flat.*

Proof. We begin by showing that (1) implies (3). Suppose that E has the approximation property. Then, $E \oplus C_1$ has the approximation property.

Let $\mathcal{F}(E \oplus C_1)$ be a right ideal in the algebra B, and let $T_1, T_2, \ldots, T_n$ in $\mathcal{F}(E \oplus C_1)$ and $b_1, b_2, \ldots, b_n$ in B be such that

$$\sum_{k=1}^{n} T_k b_k = 0.$$

Define the subspace Y of $\mathcal{F}(E \oplus C_1)^{\oplus n}$ by

$$Y = \Big\{(S_1, S_2, \ldots, S_n) : S_k \in \mathcal{F}(E \oplus C_1) \text{ and } \sum_{k=1}^{n} S_k b_k = 0\Big\}.$$

Then, $(T_1, T_2, \ldots, T_n)$ belongs to Y and, since, by Lemma 3.1, each map $T \mapsto Tb_k$ is continuous on $\mathcal{F}(E \oplus C_1)$, Y is a closed subspace of $\mathcal{F}(E \oplus C_1)^{\oplus n}$. It is clear, furthermore, that Y is a left $\mathcal{F}(E \oplus C_1)$-submodule of $\mathcal{F}(E \oplus C_1)^{\oplus n}$.

Since $E \oplus C_1$ has the approximation property, for each $\epsilon > 0$, there is a finite rank operator U in $\mathcal{F}(E \oplus C_1)$ such that

$$\sum_{k=1}^{n} ||UT_k - T_k|| < \epsilon,$$

where $(UT_1, UT_2, \ldots, UT_n)$ belongs to Y since Y is a left $\mathcal{F}(E \oplus C_1)$-module. Therefore, for each k, there is a sequence $\{T_k^{(p)}\}_{p=1}^{\infty}$ of finite rank operators on $E \oplus C_1$ such that:

(i) for each k,

$$\sum_{p=1}^{\infty} ||T_k^{(p)}|| < \infty \quad \text{and} \quad \sum_{p=1}^{\infty} T_k^{(p)} = T_k;$$

and,

(ii) for each p, $(T_1^{(p)}, T_2^{(p)}, \ldots, T_n^{(p)})$ belongs to Y.

For each p, choose a finite-dimensional subspace E_p of $E \oplus C_1$ such that the range of $T_k^{(p)}$ is contained in E_p for each k. Then, choose a finite-dimensional space G_{i_p} such that $d(E_p, G_{i_p}) < 2$ and an isomorphism

$$U_p : E_p \longrightarrow G_{i_p}$$

such that $||U_p|| = 1$ and $||U_p^{-1}|| < 2$. Finally, choose a sequence of positive numbers $\{\lambda_p\}_{p=1}^{\infty}$ which converges to zero and is such that

$\sum_{p=1}^{\infty} ||\lambda_p^{-1} T_k^{(p)}|| < \infty$ for each k. The spaces G_{i_p} are subspaces of C_1, and hence of $E \oplus C_1$, and there is a projection Q_p from $E \oplus C_1$ onto G_{i_p} with $||Q_p|| = 1$. Define the operators $R_k, k = 1, 2, \ldots, n$, and V in $\mathcal{F}(E \oplus C_1)$ by

$$R_k = \sum_{p=1}^{\infty} \lambda_p^{-1} U_p T_k^{(p)} \quad \text{and} \quad V = \sum_{p=1}^{\infty} \lambda_p U_p^{-1} Q_p.$$

Let, for each P,

$$R_k^{(P)} = \sum_{p=1}^{P} \lambda_p^{-1} U_p T_k^{(p)}.$$

Then, since Y is a left $\mathcal{F}(E \oplus C_1)$-module, $(R_1^{(P)}, R_2^{(P)}, \ldots, R_n^{(P)})$ belongs to Y for each P. Since Y is closed, it follows that $(R_1, R_2, \ldots, R_n)$ belongs to Y, that is, that

$$\sum_{k=1}^{n} R_k b_k = 0.$$

Also, since $Q_p U_q = 0$ if $p \neq q$, we have

$$V R_k = \sum_{p=1}^{\infty} U_p^{-1} Q_p U_p T_k^{(p)} = \sum_{p=1}^{\infty} T_k^{(p)} = T_k.$$

Therefore, $\mathcal{F}(E \oplus C_1)$ satisfies the characterization of right universal flatness quoted above.

It is clear that (3) implies (2). In order to show that (2) implies (1), suppose that E does not have the approximation property. From a theorem of Grothendieck [**9,** Theorem 1.e.4], there are a Banach space F and a compact operator

$$T : F \longrightarrow E \oplus C_1$$

such that T cannot be approximated in norm by finite rank operators. It may be seen from [**9,** Proof of Theorem 1.e.4] that F may be chosen to be separable and thus be a quotient space of ℓ_1. Now, $E \oplus C_1 \simeq G \oplus \ell_1$ for some space G, and thus, the quotient map may be extended to yield a surjection

$$Q : E \oplus C_1 \longrightarrow F$$

by defining the extension to be zero on G. Put $R = TQ$. Then, R belongs to $\mathcal{F}(E \oplus C_1)$ since it is compact and effectively an operator from ℓ_1. Let S be any compact operator on ℓ_1 whose range is a dense subspace of the kernel of the quotient map

$$\ell_1 \longrightarrow F,$$

and extend S to be an operator on $E \oplus C_1$ by defining its extension to be zero on G. Then, S belongs to $\mathcal{F}(E \oplus C_1)$ and $RS = 0$.

If $\mathcal{F}(E \oplus C_1)$ were right flat there would be $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ in $\mathcal{F}(E \oplus C_1)$ such that

$$R = \sum_{k=1}^{n} A_k B_k$$

and $B_k S = 0$ for each k. The condition $B_k S = 0$ implies that the kernel of B_k would contain the kernel of the quotient map $\ell_1 \to F$, and thus, B_k would induce a map

$$B_k' : F \longrightarrow E \oplus C_1.$$

We would then have that

$$T = \sum_{k=1}^{n} A_k B_k',$$

and the requirement that each A_k belong to $\mathcal{F}(E \oplus C_1)$ would imply that T belongs to $\mathcal{F}(F, E \oplus C_1)$. This would contradict the choice of T, and thus, $\mathcal{F}(E \oplus C_1)$ is not right flat. $\square$

The reason that C_1 appears in the theorem rather than C_p for general p is to guarantee in the second part of the proof that ℓ_1 is a direct summand in $E \oplus C_1$. Thus, E having the approximation property is also equivalent to the right (universal) flatness of $\mathcal{F}(F, E \oplus C_p \oplus \ell_1)$. We also have the next corollary.

Corollary 4.2. *Let E be a Banach space with the approximation property. Then, $\mathcal{F}(E \oplus C_p)$ is right universally flat.*

5. Left universal flatness of $\mathcal{F}(E)$. An argument dual to the first part of the proof of Theorem 4.1 proves the next corollary.

Corollary 5.1. *Let E be a Banach space such that E^* has the approximation property. Then $\mathcal{F}(E \oplus C_p)$ is left universally flat.*

The approximation property for the dual space may also be characterized in terms of a homological property. Spaces whose duals do not have the approximation property are characterized in [**9,** Theorem 1.e.5]. This theorem states that E^* does not have the approximation property if and only if there are a space F and a compact operator $T : E \to F$ which cannot be approximated by finite rank operators. The theorem may be used in a dual way to that in which Theorem 1.e.4 was used above. However, to carry this through, it is necessary to embed F in some space which has the approximation property. None of the spaces C_p have the property that any separable space can be embedded in them. The space $C[0,1]$ of continuous functions on the unit interval does have this property [**21,** Theorem II.B.4] and thus, as a dual to Theorem 4.1, we have:

Theorem 5.2. *For each Banach space E the following are equivalent*:

(1) *E^* has the approximation property*;
(2) *$\mathcal{F}(E \oplus C_p \oplus C[0,1])$ is left flat*; *and*
(3) *$\mathcal{F}(E \oplus C_p \oplus C[0,1])$ is left universally flat.*

If E^* has the approximation property, then E also has the approximation property, see [**9,** Theorem 1.e.7]. Hence, it follows from Theorems 4.1 and 5.2 that, if $\mathcal{F}(E \oplus C_1 \oplus C[0,1])$ is left flat, then it is right universally flat. There are Banach spaces E such that E has the approximation property but E^* does not, see [**9,** Theorem 1.e.7]. It follows that the right flatness of $\mathcal{F}(E \oplus C_1 \oplus C[0,1])$ does not imply its left flatness.

If $\mathcal{K}(E)$ has a bounded right approximate identity, then it is left universally flat. It is shown in [**7,** Corollary 2.7] that, if $\mathcal{K}(E)$ has a bounded right approximate identity, then it has a bounded left approximate identity, and thus, it will also be right universally flat. There are spaces for which $\mathcal{K}(E)$ has a bounded left approximate identity but not a bounded right approximate identity.

6. Remarks and open questions. Flatness of algebras was proved in [**19**] by means of a factorization property (Φ), and H-unitality was proven in **[18]** by means of another factorization property (F). Note that (Φ) is a purely algebraic condition, whereas (F) applies to topological algebras. If E has the bounded approximation property, then $\mathcal{F}(E)$ has a bounded left approximate identity, and it follows that both (Φ) and (F) hold in $\mathcal{F}(E)$. The above arguments do not establish that either (Φ) or (F) holds in $\mathcal{F}(E \oplus C_1)$ when E has the approximation property.

Question 6.1. *If E has the approximation property, must $\mathcal{F}(E \oplus C_1)$ satisfy either (Φ) or (F)?*

In the proofs of the equivalences in Theorem 4.1, the direct summand C_1 was required in both directions. These results say nothing about $\mathcal{F}(E)$ when E does not have a direct summand isomorphic to C_1.

Question 6.2. *Is $\mathcal{F}(E)$ right flat or right universally flat whenever E has the approximation property?*

Question 6.3. *Does E have the approximation property whenever $\mathcal{F}(E)$ is right flat or right universally flat?*

Theorem 4.1 adds to the class of Banach spaces for which it is known that $\mathcal{K}(E)$ is H-unital. It also shows that there are Banach spaces that are not right flat. However, there are no spaces known to the author for which $\mathcal{K}(E)$ is not H-unital.

Question 6.4. *Are $\mathcal{K}(E)$ and $\mathcal{F}(E)$ always* H-*unital?*

A special case of this question is whether the sequence (2.1) is exact at A for these algebras.

Question 6.5. *Is there a Banach space such that the linear span of all products in $\mathcal{F}(E)$ or $\mathcal{K}(E)$ is not equal to $\mathcal{F}(E)$ or $\mathcal{K}(E)$, respectively?*

Theorem 1 in **[8]** shows that $\mathcal{F}(E \oplus C_p)$ is equal to the span of its products for any Banach space E. This makes the question as to whether $\mathcal{F}(E \oplus C_1)$ is H-unital particularly interesting.

The idea behind **[8,** Theorem 1] may be used to prove much more than factorization of single elements. We have the following:

Proposition 6.6. *Let E be a Banach space and $\{T_n\}_{n=1}^{\infty}$ a sequence of operators in $\mathcal{F}(E \oplus C_p)$ such that $||T_n|| \to 0$ as $n \to \infty$. Then, there are*:

(1) *an element U and a sequence $\{S_n\}_{n=1}^{\infty}$ in $\mathcal{F}(E \oplus C_p)$ such that $T_n = US_n$ for each n and $||S_n|| \to 0$ as $n \to \infty$; and*
(2) *an element U' and a sequence $\{S'_n\}_{n=1}^{\infty}$ in $\mathcal{F}(E \oplus C_p)$ such that $T_n = S'_n U'$ for each n and $||S'_n|| \to 0$ as $n \to \infty$.*

The proof of (1) is essentially the same argument as used in [**8**], while the proof of (2) is merely the dual of that argument. In general, $\mathcal{F}(E \oplus C_p)$ does not have a right or left bounded approximate identity, and thus, Proposition 6.6 provides more examples of Banach algebras which satisfy a strong factorization property without having a bounded approximate identity [**6, 15**].

The next result is an immediate consequence of this proposition.

Corollary 6.7. *Let E be a Banach space and T either a right or left Banach $\mathcal{F}(E \oplus C_p)$-module homomorphism from $\mathcal{F}(E \oplus C_p)$. Then, T is continuous.*

A right multiplier on an algebra is just a left module homomorphism from the algebra to itself. The corollary thus provides another proof of the lemma for spaces which have C_p as a direct summand. It was to such spaces that we applied the lemma, and thus, we may as well just have used this proof.

Note. This paper is essentially the preprint of [**16**] with the bibliography updated and some other minor changes. The original version was withdrawn from submission with the aim of revising it under the guidance of Wodzicki. That intention was not carried through, however, since the author was involved in other projects and Wodzicki came to see it as part of a much larger and deeper project on categories of Banach spaces [**20**]. Following further discussion with Wodzicki, it has been decided to publicize the answer to the original question in this version, pending what is hoped to be a more complete development in a subsequent publication.

ENDNOTES

1. These spaces have also played an important role in the work of A. Blanco, in which he shows that the algebras of approximable operators considered in Theorem 4.1 are weakly amenable whether or not E has the approximation property [**1, 2**].

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