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# ON GENERALIZED WEAVING FRAMES IN HILBERT SPACES

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ABSTRACT. Generalized frames (in short, $g$-frames) are a natural generalization of standard frames in separable Hilbert spaces. Motivated by the concept of weaving frames in separable Hilbert spaces by [**1**] in the context of distributed signal processing, we study weaving properties of $g$-frames. Firstly, we present necessary and sufficient conditions for weaving $g$-frames in Hilbert spaces. We extend some results of [**1, 6**] regarding conversion of standard weaving frames to $g$-weaving frames. Some Paley-Wiener type perturbation results for weaving $g$-frames are obtained. Finally, we give necessary and sufficient conditions for weaving $g$-Riesz bases.

**1. Introduction.** Frames in Hilbert spaces were originally introduced by Duffin and Schaeffer [**13**] in 1952 in the context of nonharmonic Fourier series and popularized in 1986 by Daubechies, Grossmann and Meyer [**9**]. Frames are basis-like building blocks that span a vector space but allow for linear dependency, which is useful for reducing noise and finding sparse representations, spherical codes, compressed sensing, signal processing, wavelet analysis, etc., see [**5**]. Motivated by a problem regarding distributed signal processing where redundant building blocks, e.g., frames, play an important role, Bemrose, et al., [**1**] introduced weaving frames in separable Hilbert spaces. Weaving frames have potential applications in wireless sensor networks that require distributed processing under different frames, as well as

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preprocessing of signals using Gabor frames. Sun introduced the notion of *generalized frames* or *g-frames* in **[17]**. It is well known that $g$-frames include standard frames and bounded invertible linear operators, as well as many recent generalizations of frames, e.g., bounded quasi-projectors and frames of subspaces. It is of interest to find the weaving properties of $g$-frames in separable Hilbert spaces.

**1.1. Outline of the paper.** The paper is organized as follows. Section 2 contains basic definitions and results regarding frames, weaving frames and $g$-frames in Hilbert spaces. In Section 3, we study weaving $g$-frames. Necessary and sufficient conditions for weaving $g$-frames in Hilbert spaces are given. We present sufficient conditions in terms of lower $g$-frame bounds for a sequence of operators not to be weaving $g$-frames. Some Paley-Wiener type perturbation results for weaving $g$-frames are obtained. In Section 4, we discuss weaving properties of $g$-Riesz bases.

**2. Preliminaries.** In this section, we review the concepts of frames, $g$-frames and weaving frames. We begin with some notation. The set of all positive integers is denoted by $\mathbb{N}$, and $\mathbb{J}$ denotes a subset of $\mathbb{N}$. As is standard, $\ell^2(\mathbb{N})$ is the space of all square summable complex-valued sequences indexed by $\mathbb{N}$.

**2.1. Frames in Hilbert spaces.** A sequence $\{x_k\}_{k\in\mathbb{N}}$ in a separable Hilbert space $H$ is called a *frame* (or *Hilbert frame*) for $H$ if there exist positive numbers $A_0 \leq B_0 < \infty$ such that

$$A_0\|x\|^2 \leq \sum_{k\in\mathbb{N}} |\langle x, x_k\rangle|^2 \leq B_0\|x\|^2 \quad \text{for all } x \in H. \tag{2.1}$$

The numbers $A_0$ and $B_0$ are called *lower* and *upper frame bounds*, respectively. If the upper inequality in (2.1) is satisfied, then we say that $\{x_k\}_{k\in\mathbb{N}}$ is a *Bessel sequence* (or *Hilbert Bessel sequence*) with *Bessel bound* $B_0$. The frame $\{x_k\}_{k\in\mathbb{N}}$ is *tight* if it is possible to choose $A_0 = B_0$. The frame operator $S : H \to H$ for the frame $\{x_k\}_{k\in\mathbb{N}}$ is a bounded, linear, invertible and positive operator, given by

$$Sx = \sum_{k\in\mathbb{N}} \langle x, x_k\rangle x_k.$$

This gives the *reconstruction formula* for all $x \in H$,

$$x = SS^{-1}x = \sum_{k\in\mathbb{N}} \langle S^{-1}x, x_k\rangle x_k = \sum_{k\in\mathbb{N}} \langle x, S^{-1}x_k\rangle x_k.$$

The basic theory of frames may be found in Han, et al., [**14**], Christensen [**7, 8**], Casazza and Kutyniok [**5**], Casazza [**2, 3**] and Han and Larson [**15**].

**2.2. Weaving frames.** We recall some elementary facts about weaving frames. Let $m \in \mathbb{N}$ be fixed, and let

$$[m] = \{1, 2, \ldots, m\} \quad \text{and} \quad [m]^c = \mathbb{N} \setminus [m] = \{m+1, m+2, \ldots\}.$$

**Definition 2.1** ([**1**])**.** A family of frames $\{\phi_{ij}\}_{i\in\mathbb{N}, j\in[m]}$ for a Hilbert space $H$ is said to be *woven* if there are universal constants $A$ and $B$ so that, for every partition $\{\sigma_j\}_{j\in[m]}$ of $\mathbb{N}$, the family $\{\phi_{ij}\}_{i\in\sigma_j, j\in[m]}$ is a frame for $H$ with lower and upper frame bounds $A$ and $B$, respectively.

**Definition 2.2** ([**1**])**.** A family of frames $\{\phi_{ij}\}_{i\in\mathbb{N}, j\in[m]}$ for a Hilbert space $H$ is *weakly woven* if, for every partition $\{\sigma_j\}_{j\in[m]}$ of $\mathbb{N}$, the family $\{\phi_{ij}\}_{i\in\sigma_j, j\in[m]}$ is a frame for $H$.

It may be observed that weakly woven frames do not require universal frame bounds for each weaving.

It is proven in [**1**] that this weaker form of weaving, given in Definition 2.2, is equivalent to weaving. Bemrose, et al., in [**1**] proved necessary and sufficient conditions for weaving frames (which depend on frame bounds). They classified when Riesz bases and Riesz basic sequences can be woven and provided a characterization in terms of distances between subspaces. Furthermore, they proved that, if two Riesz bases are woven, then every weaving is, in fact, a Riesz basis and not just a frame. A geometric characterization of woven Riesz bases in terms of distance between subspaces of a Hilbert space $H$ is given in [**1**]. Casazza and Lynch [**6**] reviewed fundamental properties of weaving frames. They considered a relation of frames to projections and gave a better understanding of what it really means for two frames to be woven. Finally, they discussed a weaving equivalent of an unconditional basis.

Casazza, Freeman and Lynch [**4**] extended the concept of weaving Hilbert space frames to the Banach space setting. They introduced and studied *weaving Schauder frames* in Banach spaces. Deepshikha and Vashisht [**10**] studied weaving properties of an infinite family of frames in separable Hilbert spaces. They also studied vector-valued weaving frames [**11**] and weaving frames with respect to measure spaces in [**19**]. Deepshikha and Vashisht [**12**] studied weaving properties of $K$-frames in separable Hilbert spaces.

**2.3. $g$-frames in Hilbert spaces.** Sun [**17**] introduced $g$-frames which are generalized frames and include ordinary frames and many recent generalizations of frames, e.g., bounded quasi-projectors and frames of subspaces. For stability of the $g$-frame, see [**18**]. Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces, and let $\{\mathcal{H}_n\}_{n\in\mathbb{N}}$ be a sequence of closed subspaces of $\mathcal{K}$. By $B(\mathcal{H},\mathcal{H}_n)$ we denote the space of bounded linear operators from $\mathcal{H}$ into $\mathcal{H}_n$.

**Definition 2.3.** A sequence $\Lambda \equiv \{\Lambda_n\}_{n\in\mathbb{N}}$, where $\Lambda_n \in B(\mathcal{H},\mathcal{H}_n)$ for each $n\in\mathbb{N}$, is a *generalized frame* (in short, *$g$-frame*) for $\mathcal{H}$ with respect to $\{\mathcal{H}_n\}_{n\in\mathbb{N}}$ if there exist positive constants $A \le B$ such that

$$A\|x\|^2 \le \sum_{n\in\mathbb{N}} \|\Lambda_n x\|^2 \le B\|x\|^2 \quad \text{for all } x\in\mathcal{H}. \tag{2.2}$$

As in the case of standard frames, the constants $A$ and $B$ are called *lower* and *upper $g$-frame bounds*, respectively. If the right-hand inequality of (2.2) holds, then $\Lambda$ is said to be a $g$-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{H}_n\}_{n\in\mathbb{N}}$. Associated with a $g$-Bessel sequence $\Lambda$, we shall denote the representation space as follows:

$$\Big(\sum_{n\in\mathbb{N}} \bigoplus \mathcal{H}_n\Big)_{\ell^2} = \Big\{\{z_n\}_{n\in\mathbb{N}} : z_n \in \mathcal{H}_n \ (n\in\mathbb{N}), \sum_{n\in\mathbb{N}} \|z_n\|^2 < +\infty\Big\}.$$

The operator

$$T_\Lambda : \Big(\sum_{n\in\mathbb{N}} \bigoplus \mathcal{H}_n\Big)_{\ell^2} \longrightarrow \mathcal{H}$$

defined by

$$T_\Lambda(\{z_n\}_{n\in\mathbb{N}}) = \sum_{n\in\mathbb{N}} \Lambda_n^* z_n,$$

is called the *pre-frame operator* or *synthesis operator*, and the adjoint of $T_\Lambda$, given by

$$T_\Lambda^*: \mathcal{H} \longrightarrow \left(\sum_{i\in\mathbb{N}} \bigoplus \mathcal{H}_i\right)_{\ell^2}$$
$$T_\Lambda^*: x \longrightarrow \{\Lambda_n x\}_{n\in\mathbb{N}}, \quad x \in \mathcal{H},$$

is called the *analysis operator* of $\Lambda$. The *frame operator* $S_\Lambda$ associated with $\Lambda$ is defined as

$$S_\Lambda = T_\Lambda T_\Lambda^* : \mathcal{H} \longrightarrow \mathcal{H}$$
$$S_\Lambda : x \longrightarrow \sum_{n\in\mathbb{N}} \Lambda_n^* \Lambda_n x, \quad x \in \mathcal{H}.$$

If $\Lambda$ is a $g$-frame for $\mathcal{H}$, then $S_\Lambda$ is a linear, bounded, positive and invertible operator.

**Definition 2.4** ([**17**])**.** A sequence $\Lambda \equiv \{\Lambda_n\}_{n\in\mathbb{N}}$, where $\Lambda_n \in B(\mathcal{H}, \mathcal{H}_n)$ for each $n \in \mathbb{N}$, is called a *generalized Riesz basis* (abbreviated $g$-Riesz basis) for $\mathcal{H}$ with respect to $\{\mathcal{H}_n\}_{n\in\mathbb{N}}$, if

(i) $\Lambda$ is complete in $\mathcal{H}$, i.e.,

$$\{x : \Lambda_n x = 0,\ n \in \mathbb{N}\} = \{0\},$$

and

(ii) there are positive constants $A_\Lambda$ and $B_\Lambda$ such that, for any finite subset $J \subset \mathbb{N}$,

$$A_\Lambda \sum_{j\in J} \|x_j\|^2 \leq \left\| \sum_{j\in J} \Lambda_j^* x_j \right\|^2 \leq B_\Lambda \sum_{j\in J} \|x_j\|^2, \quad x_j \in H_j,\ j \in J.$$

The reader is referred to [**16, 17, 18**] for basic properties about $g$-frames and $g$-Riesz bases.

## 3. Weaving $g$-frames.

We begin with the definition of weaving $g$-frames for separable Hilbert spaces.

**Definition 3.1.** A family of $g$-frames

$$\Big\{\{\Lambda_{ni}\}_{n\in\mathbb{N}} : i \in [m]\Big\}$$

for a separable Hilbert space $\mathcal{H}$ with respect to $\{\mathcal{H}_n : n \in \mathbb{N}\}$ is said to be *g-woven* if there are universal constants $A$ and $B$ so that, for every partition $\{\sigma_i\}_{i\in[m]}$ of $\mathbb{N}$, the family $\{\Lambda_{ni}\}_{n\in\sigma_i, i\in[m]}$ is a $g$-frame for $\mathcal{H}$ with lower and upper $g$-frame bounds $A$ and $B$, respectively.

Sun [**17**] obtained a characterization of $g$-frames in terms of ordinary frames in separable Hilbert spaces.

**Theorem 3.2** ([**17**])**.** *Let $\Lambda_n \in B(\mathcal{H}, \mathcal{H}_n)$ and $\{e_{n,m}\}_{m\in\mathbb{J}_n}$ be an orthonormal basis for $\mathcal{H}_n$, where $\mathbb{J}_n \subset \mathbb{N}$, $n \in \mathbb{N}$. Then, $\{\Lambda_n\}_{n\in\mathbb{N}}$ is a $g$-frame for $\mathcal{H}$ if and only if $\{\Lambda_n^* e_{n,m}\}_{m\in\mathbb{J}_n, n\in\mathbb{N}}$ is a frame for $\mathcal{H}$.*

As an immediate consequence, we have the following result for weaving $g$-frames.

**Corollary 3.3.** *Let $\Lambda \equiv \{\Lambda_n\}_{n\in\mathbb{N}}$ and $\Omega \equiv \{\Omega_n\}_{n\in\mathbb{N}}$ be $g$-frames for $\mathcal{H}$ with respect to $\{\mathcal{H}_n : n \in \mathbb{N}\}$ and, for every $n \in \mathbb{N}$, let $\{e_{n,m}\}_{m\in\mathbb{J}_n}$ be an orthonormal basis for $\mathcal{H}_n$. Then, $\Lambda$ and $\Omega$ are $g$-woven if and only if $\{\Lambda_n^* e_{n,m}\}_{m\in\mathbb{J}_n, n\in\mathbb{N}}$ and $\{\Omega_n^* e_{n,m}\}_{m\in\mathbb{J}_n, n\in\mathbb{N}}$ are woven frames for $\mathcal{H}$.*

*Proof.* Since $\Lambda_n, \Omega_n \in B(\mathcal{H}, \mathcal{H}_n)$ for all $n \in \mathbb{N}$, the mappings

$$x \longmapsto \langle \Lambda_n x, e_{n,m}\rangle \quad \text{and} \quad x \longmapsto \langle \Omega_n x, e_{n,m}\rangle$$

define bounded linear functionals on $\mathcal{H}$ for every $m \in \mathbb{J}_n$, $n \in \mathbb{N}$. Consequently, we can find some $v_{n,m} \in \mathcal{H}$ and $w_{n,m} \in \mathcal{H}$ such that, for all $x \in \mathcal{H}$,

$$\langle x, v_{n,m}\rangle = \langle \Lambda_n x, e_{n,m}\rangle \quad \text{and} \quad \langle x, w_{n,m}\rangle = \langle \Omega_n x, e_{n,m}\rangle.$$

Hence, for all $x \in \mathcal{H}$, we have

$$\Lambda_n x = \sum_{m\in\mathbb{J}_n} \langle x, v_{n,m}\rangle e_{n,m} \quad \text{and} \quad \Omega_n x = \sum_{m\in\mathbb{J}_n} \langle x, w_{n,m}\rangle e_{n,m}.$$

Let $\{\sigma, \sigma^c\}$ be any partition of $\mathbb{N}$, and write $\{\Gamma_n\}_{n\in\mathbb{N}} = \{\Lambda_n\}_{n\in\sigma} \cup \{\Omega_n\}_{n\in\sigma^c}$. Then,

$$\Gamma_n x = \begin{cases} \Lambda_n x & n \in \sigma, \\ \Omega_n x & n \in \sigma^c \end{cases} = \begin{cases} \sum_{m\in\mathbb{J}_n} \langle x, v_{n,m}\rangle e_{n,m} & n \in \sigma, \\ \sum_{m\in\mathbb{J}_n} \langle x, w_{n,m}\rangle e_{n,m} & n \in \sigma^c. \end{cases}$$

This gives

$$\sum_{n\in\mathbb{N}}\|\Gamma_n x\|^2=\sum_{n\in\sigma}\sum_{m\in\mathbb{J}_n}|\langle x,v_{n,m}\rangle|^2 + \sum_{n\in\sigma^c}\sum_{m\in\mathbb{J}_n}|\langle x,w_{n,m}\rangle|^2 \quad \text{for all } x\in\mathcal{H}.$$

Hence, $\{\Lambda_n\}_{n\in\sigma}\cup\{\Omega_n\}_{n\in\sigma^c}$ is a $g$-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_n : n\in\mathbb{N}\}$ if and only if

$$\{u_{n,m} : m\in\mathbb{J}_n, n\in\mathbb{N}\}=\{v_{n,m} : m\in\mathbb{J}_n, n\in\sigma\} \cup\{w_{n,m} : m\in\mathbb{J}_n, n\in\sigma^c\}$$

is a frame for $\mathcal{H}$. Furthermore, for any $x\in\mathcal{H}$ and for any $y_n\in\mathcal{H}_n$, we have

$$\begin{aligned}\langle x,\Lambda_n^* y_n\rangle=\langle\Lambda_n x,y_n\rangle&=\sum_{m\in\mathbb{J}_n}\langle x,v_{n,m}\rangle\langle e_{n,m},y_n\rangle\\&=\Big\langle x,\sum_{m\in\mathbb{J}_n}\langle y_n,e_{n,m}\rangle v_{n,m}\Big\rangle,\end{aligned}$$

and

$$\begin{aligned}\langle x,\Omega_n^* y_n\rangle=\langle\Omega_n x,y_n\rangle&=\sum_{m\in\mathbb{J}_n}\langle x,w_{n,m}\rangle\langle e_{n,m},y_n\rangle\\&=\Big\langle x,\sum_{m\in\mathbb{J}_n}\langle y_n,e_{n,m}\rangle w_{n,m}\Big\rangle.\end{aligned}$$

This gives

$$\Lambda_n^* y_n=\sum_{m\in\mathbb{J}_n}\langle y_n,e_{n,m}\rangle v_{n,m}$$

and

$$\Omega_n^* y_n=\sum_{m\in\mathbb{J}_n}\langle y_n,e_{n,m}\rangle w_{n,m} \text{ for all } y_n\in\mathcal{H}_n,\ n\in\mathbb{N}.$$

In particular,

$$v_{n,m}=\Lambda_n^* e_{n,m}$$

and

$$w_{n,m}=\Omega_n^* e_{n,m} \text{ for any } m\in\mathbb{J}_n,\ n\in\mathbb{N}.$$

This completes the proof. □

**3.1. Application of Corollary 3.3.** Let $\mathcal{H} = \ell^2(\mathbb{N})$ and $\{e_k\}_{k\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$. Choose $\mathcal{H}_n = \overline{\text{span}}\{e_k\}_{k=n}^{\infty}$ for $n \in \mathbb{N}$. Then, $\{e_{n,m}\}_{m=1}^{\infty} = \{e_{n+m-1}\}_{m=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}_n$, $n \in \mathbb{N}$.

(i) Let $\Lambda \equiv \{\Lambda_n\}_{n=1}^{\infty}$ and $\Omega \equiv \{\Omega_n\}_{n=1}^{\infty}$, where $\Lambda_n \in B(\mathcal{H}, \mathcal{H}_n)$ is the orthogonal projection of $\mathcal{H}$ onto $\overline{\text{span}}\{e_n\}$ and $\Omega_n \in B(\mathcal{H}, \mathcal{H}_n)$ is the orthogonal projection of $\mathcal{H}$ onto $\overline{\text{span}}\{e_n, e_{n+1}\}$. Clearly,

$$\Lambda_n^* e_{n,m} = \begin{cases} e_n & m = 1, \\ 0 & m > 1, \end{cases}$$

and

$$\Omega_n^* e_{n,m} = \begin{cases} e_n & m = 1, \\ e_{n+1} & m = 2, \\ 0 & m > 2. \end{cases}$$

Note that $\{\Lambda_n^* e_{n,m}\}_{n,m=1}^{\infty}$ and $\{\Omega_n^* e_{n,m}\}_{n,m=1}^{\infty}$ are frames for $\mathcal{H}$.

Next, we show that $\{\Lambda_n^* e_{n,m}\}_{n,m=1}^{\infty}$ and $\{\Omega_n^* e_{n,m}\}_{n,m=1}^{\infty}$ are woven. Let $\sigma \subset \mathbb{N}$ be any arbitrary subset. We compute

$$\begin{aligned} \|x\|^2 &\leq \sum_{n\in\sigma}\sum_{m\in\mathbb{N}} |\langle x, \Lambda_n^* e_{n,m}\rangle|^2 + \sum_{n\in\sigma^c}\sum_{m\in\mathbb{N}} |\langle x, \Omega_n^* e_{n,m}\rangle|^2 \\ &= \sum_{n\in\sigma} |\langle x, \Lambda_n^* e_{n,1}\rangle|^2 + \sum_{n\in\sigma^c} |\langle x, \Omega_n^* e_{n,1}\rangle|^2 \\ &\quad + \sum_{n\in\sigma^c} |\langle x, \Omega_n^* e_{n,2}\rangle|^2 \\ &= \sum_{n\in\sigma} |\langle x, e_n\rangle|^2 + \sum_{n\in\sigma^c} |\langle x, e_n\rangle|^2 \\ &\quad + \sum_{n\in\sigma^c} |\langle x, e_{n+1}\rangle|^2 \leq 2\sum_{n\in\mathbb{N}} |\langle x, e_n\rangle|^2 \\ &= 2\|x\|^2 \quad \text{for all } x \in \mathcal{H}. \end{aligned}$$

Thus,

$$\{\Lambda_n^* e_{n,m}\}_{\substack{n\in\sigma \\ m\in\mathbb{N}}} \cup \{\Omega_n^* e_{n,m}\}_{\substack{n\in\sigma^c \\ m\in\mathbb{N}}}$$

is a frame for $\mathcal{H}$ for any $\sigma \subset \mathbb{N}$. Hence, by Corollary 3.3, $\Lambda$ and $\Omega$ are $g$-woven.

(ii) Let $\Lambda \equiv \{\Lambda_n\}_{n=1}^{\infty}$ and $\Omega \equiv \{\Omega_n\}_{n=2}^{\infty}$ be the same as in part (i) except for $\Omega_1$ which is the zero mapping. Then, $\{\Lambda_n^* e_{n,m}\}_{n,m=1}^{\infty}$ and $\{\Omega_n^* e_{n,m}\}_{n,m=1}^{\infty}$ are not woven. Indeed, let $\{\Lambda_n^* e_{n,m}\}_{m,n=1}^{\infty}$ and $\{\Omega_n^* e_{n,m}\}_{n,m=1}^{\infty}$ be woven with universal frame bounds $A$ and $B$. Choose $\sigma = \mathbb{N} \setminus \{1\}$. Then, compute

$$\begin{aligned}
&\sum_{n\in\sigma}\sum_{m\in\mathbb{N}} |\langle e_1, \Lambda_n^* e_{n,m}\rangle|^2 + \sum_{n\in\sigma^c}\sum_{m\in\mathbb{N}} |\langle e_1, \Omega_n^* e_{n,m}\rangle|^2 \\
&\quad = \sum_{n\in\mathbb{N}\setminus\{1\}} |\langle e_1, \Lambda_n^* e_{n,1}\rangle|^2 + |\langle e_1, 0\rangle|^2 \\
&\quad = \sum_{n\in\mathbb{N}\setminus\{1\}} |\langle e_1, e_n\rangle|^2 + |\langle e_1, 0\rangle|^2 \\
&\quad = 0 < A\|e_1\|^2.
\end{aligned}$$

This is a contradiction. Hence, by Corollary 3.3, $\Lambda$ and $\Omega$ are not $g$-woven.

Inspired by **[1,** Lemma 4.3], the next theorem provides sufficient conditions for a sequence of operators not to be woven $g$-frames for $\mathcal{H}$.

**Theorem 3.4.** *Suppose that $\Lambda \equiv \{\Lambda_n\}_{n\in\mathbb{N}}$ and $\Omega \equiv \{\Omega_n\}_{n\in\mathbb{N}}$ are $g$-frames for $\mathcal{H}$ with respect to $\{\mathcal{H}_n : n \in \mathbb{N}\}$. Assume that, for every two disjoint finite sets $I, J \subset \mathbb{N}$ and every $\epsilon > 0$, there are subsets $\sigma, \delta \subset \mathbb{N} \setminus (I \cup J)$ with $\delta = \mathbb{N} \setminus (I \cup J \cup \sigma)$ such that the lower $g$-frame bound of*

$$\{\Lambda_n\}_{n\in I\cup\sigma} \cup \{\Omega_n\}_{n\in J\cup\delta}$$

*is less than $\epsilon$. Then, there exists a subset $\mathcal{M} \subset \mathbb{N}$ so that*

$$\{\Lambda_n\}_{n\in\mathcal{M}} \cup \{\Omega_n\}_{n\in\mathcal{M}^c}$$

*is not a $g$-frame. Hence, $\Lambda$ and $\Omega$ are not $g$-woven.*

*Proof.* Let $\epsilon > 0$ be arbitrary. By hypothesis, for $I_0 = J_0 = \varnothing$, we can choose $\sigma_1 \subset \mathbb{N}$ such that, if $\delta_1 = \sigma_1^c$, then a lower $g$-frame bound of $\{\Lambda_n\}_{n\in\sigma_1} \cup \{\Omega_n\}_{n\in\delta_1}$ is less than $\epsilon$. Thus, there exists an $x_1 \in \mathcal{H}$ with $\|x_1\| = 1$ such that

$$\sum_{n\in\sigma_1} \|\Lambda_n x_1\|^2 + \sum_{n\in\delta_1} \|\Omega_n x_1\|^2 < \epsilon.$$

Since
$$\sum_{n=1}^{\infty}\|\Lambda_n x_1\|^2+\sum_{n=1}^{\infty}\|\Omega_n x_1\|^2<\infty,$$
there is a positive integer $k_1$ such that
$$\sum_{n=k_1+1}^{\infty}\|\Lambda_n x_1\|^2+\sum_{n=k_1+1}^{\infty}\|\Omega_n x_1\|^2<\epsilon.$$
Let $I_1=\sigma_1\cap[k_1]$ and $J_1=\delta_1\cap[k_1]$. Then, $I_1\cap J_1=\varnothing$ and $I_1\cup J_1=[k_1]$.

By assumption, there are subsets $\sigma_2,\delta_2\subset[k_1]^c$ with $\delta_2=[k_1]^c\setminus\sigma_2$ such that a lower $g$-frame bound of
$$\{\Lambda_n\}_{n\in I_1\cup\sigma_2}\cup\{\Omega_n\}_{n\in J_1\cup\delta_2}$$
is less than $\epsilon/2$, that is, there exists a vector $x_2\in\mathcal{H}$ with $\|x_2\|=1$ such that
$$\sum_{n\in I_1\cup\sigma_2}\|\Lambda_n x_2\|^2+\sum_{n\in J_1\cup\delta_2}\|\Omega_n x_2\|^2<\frac{\epsilon}{2}.$$
Similar to the above, there is a $k_2>k_1$ such that
$$\sum_{n=k_2+1}^{\infty}\|\Lambda_n x_2\|^2+\sum_{n=k_2+1}^{\infty}\|\Omega_n x_2\|^2<\frac{\epsilon}{2}.$$
Set $I_2=I_1\cup(\sigma_2\cap[k_2])$ and $J_2=J_1\cup(\delta_2\cap[k_2])$. Note that $I_2\cap J_2=\varnothing$ and $I_2\cup J_2=[k_2]$. Thus, by the induction method, we obtain:

(i) a sequence of positive integers $\{k_n\}_{n\in\mathbb{N}}\subset\mathbb{N}$ with $k_n<k_{n+1}$ for all $n\in\mathbb{N}$;
(ii) a sequence of vectors $\{x_n\}_{n\in\mathbb{N}}\subset\mathcal{H}$ with $\|x_n\|=1$ for all $n\in\mathbb{N}$;
(iii) subsets $\sigma_n\subset[k_{n-1}]^c$, $\delta_n=[k_{n-1}]^c\setminus\sigma_n$, $n\in\mathbb{N}$; and
(iv) $I_n=I_{n-1}\cup(\sigma_n\cap[k_n])$, $J_n=J_{n-1}\cup(\delta_n\cap[k_n])$, $n\in\mathbb{N}$,

which satisfy both
$$\sum_{i\in I_{n-1}\cup\sigma_n}\|\Lambda_i x_n\|^2+\sum_{i\in J_{n-1}\cup\delta_n}\|\Omega_i x_n\|^2<\frac{\epsilon}{n},\tag{3.1}$$
and
$$\sum_{i=k_n+1}^{\infty}\|\Lambda_i x_n\|^2+\sum_{i=k_n+1}^{\infty}\|\Omega_i x_n\|^2<\frac{\epsilon}{n}.\tag{3.2}$$

By construction, $I_n \cap J_n = \varnothing$ and $I_n \cup J_n = [k_n]$ for all $n \in \mathbb{N}$ such that

$$\left(\bigcup_{i=1}^{\infty} I_i\right) \bigsqcup \left(\bigcup_{j=1}^{\infty} J_j\right) = \mathbb{N},$$

where $\sqcup$ represents disjoint union. Choose $\mathcal{M} = \cup_{i=1}^{\infty} I_i$. Note that

$$\mathcal{M}^c = \bigcup_{j=1}^{\infty} J_j.$$

We compute

$$\begin{aligned}
&\sum_{i\in\mathcal{M}} \|\Lambda_i x_n\|^2 + \sum_{i\in\mathcal{M}^c} \|\Omega_i x_n\|^2 \\
&\quad = \left(\sum_{i\in I_n} \|\Lambda_i x_n\|^2 + \sum_{i\in J_n} \|\Omega_i x_n\|^2\right) \\
&\quad\quad + \left(\sum_{i\in A\cap[k_n]^c} \|\Lambda_i x_n\|^2 + \sum_{i\in A^c\cap[k_n]^c} \|\Omega_i x_n\|^2\right) \\
&\quad \leqslant \left(\sum_{i\in I_{n-1}\cup\sigma_n} \|\Lambda_i x_n\|^2 + \sum_{i\in J_{n-1}\cup\delta_n} \|\Omega_i x_n\|^2\right) \\
&\quad\quad + \left(\sum_{i=k_n+1}^{\infty} \|\Lambda_i x_n\|^2 + \sum_{i=k_n+1}^{\infty} \|\Omega_i x_n\|^2\right) \\
&\quad < \frac{\epsilon}{n} + \frac{\epsilon}{n} = \frac{2\epsilon}{n}.
\end{aligned}$$

This shows that a lower $g$-frame bound of $\{\Lambda_n\}_{n\in\mathcal{M}} \cup \{\Omega_n\}_{n\in\mathcal{M}^c}$ is zero, a contradiction. Hence, the $g$-frames $\Lambda$ and $\Omega$ are not $g$-woven. $\square$

Theorem 3.4 gives a necessary condition for weaving $g$-frames in terms of lower frame bounds.

**Proposition 3.5.** *Suppose that the family of $g$-frames*

$$\{\{\Lambda_{ni}\}_{n\in\mathbb{N}} : i \in [m]\}$$

*for $\mathcal{H}$ with respect to $\{\mathcal{H}_n : n \in \mathbb{N}\}$ is $g$-woven. Then, there exists a partition $\{\tau_i\}_{i\in[m]}$ of some finite subset of $\mathbb{N}$ and $A > 0$ such that, for* *any partition* $\{\sigma_i\}_{i\in[m]}$ *of* $\mathbb{N}\setminus\{\tau_i\}_{i\in[m]}$*, the family*

$$\bigcup_{i\in[m]}\{\Lambda_{in}\}_{n\in\sigma_i\cup\tau_i}$$

*has a lower g-frame bound A.*

The next proposition gives a universal $g$-Bessel bound for a family of $g$-Bessel sequences. This is an adaptation of [**1,** Proposition 3.1].

**Proposition 3.6.** *For each* $i\in[m]$*, let* $\{\Lambda_{ni}\}_{n\in\mathbb{N}}$ *be a g-Bessel sequence for* $\mathcal{H}$ *with respect to* $\{\mathcal{H}_n : n\in\mathbb{N}\}$ *and with g-Bessel bounds* $B_i$*. Then, every weaving is a g-Bessel sequence with*

$$\sum_{i=1}^{m}B_i$$

*as a g-Bessel bound.*

*Proof.* Let $\{\Lambda_{ni}\}_{n\in\sigma_i,i\in[m]}$ be a weaving for any partition $\{\sigma_i\}_{i\in[m]}$ of $\mathbb{N}$. Then,

$$\begin{aligned}\sum_{i=1}^{m}\sum_{n\in\sigma_i}\|\Lambda_{ni}x\|^2 &\leqslant \sum_{i=1}^{m}\sum_{n\in\mathbb{N}}\|\Lambda_{ni}x\|^2\\ &\leqslant \left(\sum_{i=1}^{m}B_i\right)\|x\|^2 \quad \text{for all } x\in\mathcal{H}.\end{aligned}$$

The proof is complete. □

As in the case of standard weaving frames [**6,** Proposition 15], it is enough to check $g$-weaving on smaller sets than the original.

**Proposition 3.7.** *Let* $\Lambda\equiv\{\Lambda_n\}_{n\in\mathbb{N}}$ *and* $\Omega\equiv\{\Omega_n\}_{n\in\mathbb{N}}$ *be g-Bessel sequences in* $\mathcal{H}$ *with respect to* $\{\mathcal{H}_n : n\in\mathbb{N}\}$ *with g-Bessel bounds* $B_1$ *and* $B_2$*, respectively. If* $J\subset\mathbb{N}$*, and* $\Lambda_J\equiv\{\Lambda_i\}_{i\in J}$ *and* $\Omega_J\equiv\{\Omega_i\}_{i\in J}$ *are g-woven frames, then* $\Lambda$ *and* $\Omega$ *are g-woven frames for* $\mathcal{H}$*.*

*Proof.* Let $A$ be a lower universal $g$-frame bound for $\Lambda_J$ and $\Omega_J$, and let $\sigma \subset \mathbb{N}$ be an arbitrary subset. Then,

$$\begin{aligned} A\|x\|^2 &\leqslant \sum_{i\in\sigma\cap J} \|\Lambda_i x\|^2 + \sum_{i\in\sigma^c\cap J} \|\Omega_i x\|^2 \\ &\leqslant \sum_{i\in\sigma} \|\Lambda_i x\|^2 + \sum_{i\in\sigma^c} \|\Omega_i x\|^2 \\ &\leqslant (B_1 + B_2)\|x\|^2 \quad \text{for all } x \in \mathcal{H} \end{aligned}$$

(by Proposition 3.6). Hence, $\Lambda$ and $\Omega$ are $g$-woven frames for $\mathcal{H}$. □

Recall that, after removal of a vector from a discrete frame, the resultant family is either a frame or an incomplete set, see [**8,** Theorem 5.4.7]. Casazza and Lynch [**6**] proved that removal of vectors from woven frames leaves them woven. In the direction of $g$-frames we have following result.

**Proposition 3.8.** *Let $\Lambda \equiv \{\Lambda_n\}_{n\in\mathbb{N}}$ and $\Omega \equiv \{\Omega_n\}_{n\in\mathbb{N}}$ be $g$-woven frames for $\mathcal{H}$ with respect to $\{\mathcal{H}_n : n \in \mathbb{N}\}$ with universal $g$-frame bounds $A$ and $B$. If $J \subset \mathbb{N}$ and*

$$\sum_{i\in J} \|\Lambda_i x\|^2 \leq D_0\|x\|^2$$

*for all $x \in \mathcal{H}$ and for some $0 < D_0 < A$, then $\Lambda_0 \equiv \{\Lambda_i\}_{i\in\mathbb{N}\setminus J}$ and $\Omega_0 \equiv \{\Omega_i\}_{i\in\mathbb{N}\setminus J}$ are $g$-woven frames for $\mathcal{H}$ with universal $g$-frame bounds $A - D_0$ and $B$.*

*Proof.* Let $\sigma \subset \mathbb{N} \setminus J$ be arbitrary. We compute

$$\begin{aligned} &\sum_{i\in\sigma} \|\Lambda_i x\|^2 + \sum_{i\in(\mathbb{N}\setminus J)\setminus\sigma} \|\Omega_i x\|^2 \\ &\quad = \left( \sum_{i\in\sigma\cup J} \|\Lambda_i x\|^2 - \sum_{i\in J} \|\Lambda_i x\|^2 \right) + \sum_{i\in(\mathbb{N}\setminus J)\setminus\sigma} \|\Omega_i x\|^2 \\ &\quad = \left( \sum_{i\in\sigma\cup J} \|\Lambda_i x\|^2 + \sum_{i\in(\mathbb{N}\setminus J)\setminus\sigma} \|\Omega_i x\|^2 \right) - \sum_{i\in J} \|\Lambda_i x\|^2 \\ &\quad \geqslant (A - D_0)\|x\|^2 \quad \text{for all } x \in \mathcal{H}. \end{aligned}$$

On the other hand, for all $x \in \mathcal{H}$, we have

$$\sum_{i\in\sigma} \|\Lambda_i x\|^2 + \sum_{i\in(\mathbb{N}\setminus J)\setminus\sigma} \|\Omega_i x\|^2 \leqslant \sum_{i\in\sigma\cup J} \|\Lambda_i x\|^2 + \sum_{i\in(\mathbb{N}\setminus J)\setminus\sigma} \|\Omega_i x\|^2 \leqslant B\|x\|^2.$$

Hence, $\Lambda_0$ and $\Omega_0$ are $g$-woven frames for $\mathcal{H}$ with the required universal $g$-frame bounds. □

**4. Perturbation of weaving $g$-frames.** It is well known that perturbation theory is an important area in applied mathematics. For applications of perturbation theory for frames in various directions, the reader is referred to **[2, 5, 7, 8]** and the references therein. Bemrose, et al., **[1]** proved sufficient conditions for weaving frames by means of perturbation theory and diagonal dominance. We begin this section with the following Paley-Wiener type perturbation of weaving $g$-frames.

**Theorem 4.1.** *Let $\Lambda \equiv \{\Lambda_i\}_{i\in\mathbb{N}}$ and $\Omega \equiv \{\Omega_i\}_{i\in\mathbb{N}}$ be g-frames for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in \mathbb{N}\}$ with g-frame bounds $A_1, B_1$ and $A_2, B_2$, respectively. Assume that there are constants $0 < \lambda_1,\ \lambda_2,\ \mu < 1$ such that*

$$\lambda_1\sqrt{B_1} + \lambda_2\sqrt{B_2} + \mu \leqslant \frac{A_1}{2(\sqrt{B_1} + \sqrt{B_2})}$$

*and*

(4.1)
$$\Big\|\sum_{i\in\mathbb{N}}(\Lambda_i^* x_i - \Omega_i^* x_i)\Big\| \leqslant \lambda_1\Big\|\sum_{i\in\mathbb{N}}\Lambda_i^* x_i\Big\| + \lambda_2\Big\|\sum_{i\in\mathbb{N}}\Omega_i^* x_i\Big\| + \mu\|\{x_i\}_{i\in\mathbb{N}}\|,$$

*for all*

$$\{x_i\}_{i\in\mathbb{N}} \in \Big(\sum_{i\in\mathbb{N}}\bigoplus \mathcal{H}_i\Big)_{\ell^2}.$$

*Then, $\Lambda$ and $\Omega$ are g-woven with universal g-frame bounds $A_1/2$, $B_1 + B_2$.*

*Proof.* Let $T$ and $R$ be the synthesis operators for the frames $\{\Lambda_i\}_{i\in\mathbb{N}}$ and $\{\Omega_i\}_{i\in\mathbb{N}}$, respectively. For each $\sigma \subset \mathbb{N}$, define bounded operators

$$T_\sigma, R_\sigma : \Big(\sum_{i\in\mathbb{N}}\bigoplus \mathcal{H}_i\Big)_{\ell^2} \longrightarrow \mathcal{H},$$

$$T_\sigma(\{x_i\}_{i\in\mathbb{N}}) = \sum_{i\in\sigma} \Lambda_i^*(x_i),$$

and

$$R_\sigma(\{x_i\}_{i\in\mathbb{N}}) = \sum_{i\in\sigma} \Omega_i^*(x_i).$$

Note that $\|T_\sigma\| \leqslant \|T\|$, $\|R_\sigma\| \leqslant \|R\|$ and $\|T_\sigma - R_\sigma\| \leqslant \|T - R\|$.

By using (4.1), we have

$$\begin{aligned}
&\lambda_1\|T(\{x_i\}_{i\in\mathbb{N}})\| + \lambda_2\|R(\{x_i\}_{i\in\mathbb{N}})\| + \mu\|\{x_i\}_{i\in\mathbb{N}}\| \\
&\quad\geqslant \left\| \sum_{i\in\mathbb{N}} (\Lambda_i^* - \Omega_i^*)(x_i) \right\| \\
&\quad= \|(T-R)(\{x_i\}_{i\in\mathbb{N}})\|, \quad \{x_i\}_{i\in\mathbb{N}} \in \left( \sum_{i\in\mathbb{N}} \bigoplus \mathcal{H}_i \right)_{\ell^2}.
\end{aligned}$$

This gives $\|T - R\| \leq \lambda_1\|T\| + \lambda_2\|R\| + \mu$. Using this, for any $\sigma \subset \mathbb{N}$, we compute

$$\begin{aligned}
\left\| \sum_{i\in\sigma} \Lambda_i^*\Lambda_i x - \sum_{i\in\sigma} \Omega_i^*\Omega_i x \right\| &= \|T_\sigma(\{\Lambda_i x\}_{i\in\sigma}) - R_\sigma(\{\Omega_i x\}_{i\in\sigma})\| \\
&= \|T_\sigma T_\sigma^* x - R_\sigma R_\sigma^* x\| \\
&\leqslant \|(T_\sigma T_\sigma^* - T_\sigma R_\sigma^*)(x)\| + \|(T_\sigma R_\sigma^* - R_\sigma R_\sigma^*)(x)\| \\
&\leqslant \|T_\sigma\|\|T_\sigma^* - R_\sigma^*\|\|x\| + \|T_\sigma - R_\sigma\|\|R_\sigma^*\|\|x\| \\
&\leqslant \|T\|\|T-R\|\|x\| + \|T-R\|\|R\|\|x\| \\
&\leqslant (\lambda_1\|T\| + \lambda_2\|R\| + \mu)(\|T\| + \|R\|)\|x\| \\
&\leqslant (\lambda_1\sqrt{B_1} + \lambda_2\sqrt{B_2} + \mu)(\sqrt{B_1} + \sqrt{B_2})\|x\| \\
&< \left( \frac{A_1}{2(\sqrt{B_1} + \sqrt{B_2})} \right)(\sqrt{B_1} + \sqrt{B_2})\|x\| \\
&= \frac{A_1}{2}\|x\| \quad \text{for all } x \in \mathcal{H}.
\end{aligned} \tag{4.2}$$

By using (4.2), it follows that

$$\begin{aligned}
&\Big\| \sum_{i\in\sigma^c} \Lambda_i^*\Lambda_i x + \sum_{i\in\sigma} \Omega_i^*\Omega_i x \Big\| \\
&\quad = \Big\| \sum_{i\in\sigma^c} \Lambda_i^*\Lambda_i x + \sum_{i\in\sigma} \Lambda_i^*\Lambda_i x - \sum_{i\in\sigma} \Lambda_i^*\Lambda_i x + \sum_{i\in\sigma} \Omega_i^*\Omega_i x \Big\| \\
&\quad = \Big\| \sum_{i\in\mathbb{N}} \Lambda_i^*\Lambda_i x + \sum_{i\in\sigma} \Omega_i^*\Omega_i x - \sum_{i\in\sigma} \Lambda_i^*\Lambda_i x \Big\| \\
&\quad \geqslant \Big\| \sum_{i\in\mathbb{N}} \Lambda_i^*\Lambda_i x \Big\| - \Big\| \sum_{i\in\sigma} \Omega_i^*\Omega_i x - \sum_{i\in\sigma} \Lambda_i^*\Lambda_i x \Big\| \\
&\quad \geqslant A_1\|x\| - \Big\| \sum_{i\in\sigma} \Lambda_i^*\Lambda_i x - \sum_{i\in\sigma} \Omega_i^*\Omega_i x \Big\| \\
&\quad \geqslant A_1\|x\| - \frac{A_1}{2}\|x\| \\
&\quad = \frac{A_1}{2}\|x\| \quad \text{for all } x \in \mathcal{H}.
\end{aligned}$$

This gives a universal lower $g$-frame bound. The upper universal $g$-frame bound can be obtained from Proposition 3.6. Hence, $\Lambda$ and $\Omega$ are $g$-woven. □

The next theorem gives another variant of Paley-Wiener type perturbation of weaving $g$-frames in terms of frame operators associated with $\Lambda$ and $\Omega$.

**Theorem 4.2.** *Let $\Lambda \equiv \{\Lambda_i\}_{i\in\mathbb{N}}$ and $\Omega \equiv \{\Omega_i\}_{i\in\mathbb{N}}$ be g-frames for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in \mathbb{N}\}$ with frame bounds $A_1, B_1$ and $A_2, B_2$, respectively. Assume that there are constants $0 < \lambda,\ \mu,\ \gamma < 1$ such that*

$$\lambda B_1 + \mu B_2 + \gamma\sqrt{B_1} < A_1$$

*and*

$$\begin{aligned}
\Big\| \sum_{i\in\sigma} (\Lambda_i^*\Lambda_i x - \Omega_i^*\Omega_i x) \Big\| &\leqslant \lambda \Big\| \sum_{i\in\sigma} \Lambda_i^*\Lambda_i x \Big\| \\
&\quad + \mu \Big\| \sum_{i\in\sigma} \Omega_i^*\Omega_i x \Big\| + \gamma \Big( \sum_{i\in\sigma} \|\Lambda_i x\|^2 \Big)^{1/2},
\end{aligned} \tag{4.3}$$

*for all $x \in \mathcal{H}$ and for every $\sigma \subset \mathbb{N}$. Then, $\Lambda$ and $\Omega$ are $g$-woven with universal $g$-frame bounds $(A_1 - \lambda\sqrt{B_1} - \mu B_2 - \gamma)$ and $(B_1 + \lambda\sqrt{B_1} + \mu B_2 + \gamma)$.*

*Proof.* By using the fact that

$$\Big\|\sum_{i\in\sigma} \Lambda_i^*\Lambda_i x\Big\| \leqslant B_1\|x\| \quad \text{and} \quad \Big\|\sum_{i\in\sigma} \Omega_i^*\Omega_i x\Big\| \leqslant B_2\|x\|$$

for any $\sigma \subset \mathbb{N}$ and $x \in \mathcal{H}$, we compute

$$\begin{aligned}
\Big\|\sum_{i\in\sigma^c} \Lambda_i^*\Lambda_i x + \sum_{i\in\sigma} \Omega_i^*\Omega_i x\Big\| &= \Big\|\sum_{i\in\mathbb{N}} \Lambda_i^*\Lambda_i x + \sum_{i\in\sigma} \Omega_i^*\Omega_i x - \sum_{i\in\sigma} \Lambda_i^*\Lambda_i x\Big\| \\
&\geqslant \Big\|\sum_{i\in\mathbb{N}} \Lambda_i^*\Lambda_i x\Big\| - \Big\|\sum_{i\in\sigma} \Omega_i^*\Omega_i x - \sum_{i\in\sigma} \Lambda_i^*\Lambda_i x\Big\| \\
&\geqslant A_1\|x\| - \lambda\Big\|\sum_{i\in\sigma} \Lambda_i^*\Lambda_i x\Big\| - \mu\Big\|\sum_{i\in\sigma} \Omega_i^*\Omega_i x\Big\| \\
&\quad - \gamma\Big(\sum_{i\in\sigma} \|\Lambda_i x\|^2\Big)^{1/2} \\
&\geq (A_1 - \lambda B_1 - \mu B_2 - \gamma\sqrt{B_1})\|x\|, \qquad (4.4)
\end{aligned}$$

and

$$\begin{aligned}
\Big\|\sum_{i\in\sigma^c} \Lambda_i^*\Lambda_i x + \sum_{i\in\sigma} \Omega_i^*\Omega_i x\Big\| &= \Big\|\sum_{i\in\mathbb{N}} \Lambda_i^*\Lambda_i x + \sum_{i\in\sigma} \Omega_i^*\Omega_i x - \sum_{i\in\sigma} \Lambda_i^*\Lambda_i x\Big\| \\
&\leqslant \Big\|\sum_{i\in\mathbb{N}} \Lambda_i^*\Lambda_i x\Big\| + \Big\|\sum_{i\in\sigma} \Omega_i^*\Omega_i x - \sum_{i\in\sigma} \Lambda_i^*\Lambda_i x\Big\| \\
&\leqslant B_1\|x\| + \lambda\Big\|\sum_{i\in\sigma} \Lambda_i^*\Lambda_i x\Big\| + \mu\Big\|\sum_{i\in\sigma} \Omega_i^*\Omega_i x\Big\| \\
&\quad + \gamma\Big(\sum_{i\in\sigma} \|\Lambda_i x\|^2\Big)^{1/2} \\
&\leq (B_1 + \lambda B_1 + \mu B_2 + \gamma\sqrt{B_1})\|x\|. \qquad (4.5)
\end{aligned}$$

Therefore, by (4.4) and (4.5), the $g$-frames $\Lambda$ and $\Omega$ are $g$-woven with the required universal $g$-frame bounds. $\square$

We end this section with perturbation of weaving $g$-frames in terms of certain closeness between the vectors in $\mathcal{H}_i$.

**Theorem 4.3.** *Let $\Lambda \equiv \{\Lambda_i\}_{i\in\mathbb{N}}$ and $\Omega \equiv \{\Omega_i\}_{i\in\mathbb{N}}$ be g-frames for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in \mathbb{N}\}$ and with g-frame bounds $A_1, B_1$ and $A_2, B_2$, respectively. Assume that there is a constant $M > 0$ such that, for every $J \subset \mathbb{N}$,*

$$\sum_{i\in J} \|\Lambda_i x - \Omega_i x\|^2 \leqslant M \min\left\{ \sum_{i\in J} \|\Lambda_i x\|^2, \sum_{i\in J} \|\Omega_i x\|^2 \right\}, \quad x \in \mathcal{H}. \tag{4.6}$$

*Then, $\Lambda$ and $\Omega$ are g-woven with universal g-frame bounds $(A_1 + A_2)/(2M + 3)$ and $B_1 + B_2$.*

*Proof.* Let $\sigma \subset \mathbb{N}$ be arbitrary. Then, by using (4.6), we compute

$$\begin{aligned}
(A_1 + A_2)\|x\|^2 &\leqslant \sum_{i\in\mathbb{N}} \|\Lambda_i x\|^2 + \sum_{i\in\mathbb{N}} \|\Omega_i x\|^2 \\
&= \sum_{i\in\sigma} \|\Lambda_i x\|^2 + \sum_{i\in\sigma^c} \|\Lambda_i x\|^2 + \sum_{i\in\sigma} \|\Omega_i x\|^2 + \sum_{i\in\sigma^c} \|\Omega_i x\|^2 \\
&\leqslant \sum_{i\in\sigma} \|\Lambda_i x\|^2 + 2\bigg( \sum_{i\in\sigma^c} \|(\Lambda_i - \Omega_i)(x)\|^2 + \sum_{i\in\sigma^c} \|\Omega_i x\|^2 \bigg) \\
&\quad + 2\bigg( \sum_{i\in\sigma} \|(\Lambda_i - \Omega_i)(x)\|^2 + \sum_{i\in\sigma} \|\Lambda_i x\|^2 \bigg) + \sum_{i\in\sigma^c} \|\Omega_i x\|^2 \\
&\leqslant \sum_{i\in\sigma} \|\Lambda_i x\|^2 + 2\bigg( M \sum_{i\in\sigma^c} \|\Omega_i x\|^2 + \sum_{i\in\sigma^c} \|\Omega_i x\|^2 \bigg) \\
&\quad + 2\bigg( M \sum_{i\in\sigma} \|\Lambda_i x\|^2 + \sum_{i\in\sigma} \|\Lambda_i x\|^2 \bigg) + \sum_{i\in\sigma^c} \|\Omega_i x\|^2 \\
&= (2M+3)\bigg( \sum_{i\in\sigma} \|\Lambda_i x\|^2 + \sum_{i\in\sigma^c} \|\Omega_i x\|^2 \bigg) \quad \text{for all } x \in \mathcal{H}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{A_1 + A_2}{2M + 3}\|x\|^2 &\leq \sum_{i\in\sigma} \|\Lambda_i x\|^2 + \sum_{i\in\sigma^c} \|\Omega_i x\|^2 \\
&\leq (B_1 + B_2)\|x\|^2, \quad x \in \mathcal{H}.
\end{aligned}$$

Hence, $\Lambda$ and $\Omega$ are $g$-woven with the desired universal $g$-frame bounds. $\square$

**5. Weaving $g$-Riesz bases.** Bemrose, et al., [**1**] classified when Riesz bases and Riesz basic sequences can be woven and proved a characterization in terms of distances between subspaces. We present a necessary and sufficient condition for weaving $g$-Riesz bases in terms of standard woven Riesz bases. The proof is based upon the technique developed by Sun [**17**], which may be found in the following theorem.

**Theorem 5.1** ([**17**]). *Let $\Lambda_n \in B(\mathcal{H}, \mathcal{H}_n)$ and $\{e_{n,m}\}_{m\in\mathbb{J}_n}$ be an orthonormal basis for $\mathcal{H}_n$, where $\mathbb{J}_n \subset \mathbb{N}$, $n \in \mathbb{N}$. Then, $\{\Lambda_n\}_{n\in\mathbb{N}}$ is a g-Riesz basis for $\mathcal{H}$ if and only if $\{\Lambda_n^* e_{n,m}\}_{m\in\mathbb{J}_n, n\in\mathbb{N}}$ is a Riesz basis for $\mathcal{H}$.*

As a corollary, we have the next result for weaving $g$-Riesz bases.

**Corollary 5.2.** *Let $\Lambda \equiv \{\Lambda_n\}_{n\in\mathbb{N}}$, and $\Omega \equiv \{\Omega_n\}_{n\in\mathbb{N}}$ be g-Riesz bases for $\mathcal{H}$ with respect to $\{\mathcal{H}_n : n \in \mathbb{N}\}$, and let $\{e_{n,m}\}_{m\in\mathbb{J}_n}$ be an orthonormal basis for $\mathcal{H}_n$, for each $n \in \mathbb{N}$. Then, $\Lambda$ and $\Omega$ are g-woven Riesz bases for $\mathcal{H}$ if and only if $\{\Lambda_n^* e_{n,m}\}_{n\in\mathbb{N}, m\in\mathbb{J}_n}$ and $\{\Omega_n^* e_{n,m}\}_{n\in\mathbb{N}, m\in\mathbb{J}_n}$ are woven Riesz bases for $\mathcal{H}$.*

*Proof.* For each $n \in \mathbb{N}$, since $\{e_{n,m}\}_{m\in\mathbb{J}_n}$ is an orthonormal basis for $\mathcal{H}_n$, every $y_n \in \mathcal{H}_n$ has an expansion of the form

$$y_n = \sum_{m\in\mathbb{J}_n} c_{n,m} e_{n,m},$$

$$\text{where } \{c_{n,m}\}_{\substack{n\in\mathbb{N}\\ m\in\mathbb{J}_n}} \in \ell^2(\mathbb{N}).$$

Let $J \subset \mathbb{N}$ be any arbitrary finite subset and $\{\sigma, \sigma^c\}$ any partition of $\mathbb{N}$. We write $\{\Gamma_n\}_{n\in\mathbb{N}} = \{\Lambda_n\}_{n\in\sigma} \cup \{\Omega_n\}_{n\in\sigma^c}$ and $v_{n,m}, w_{n,m} \in \mathcal{H}$ for vectors defined as in the proof of Corollary 3.3. Compute

$$\begin{aligned}\Big\| \sum_{n\in J} \Gamma_n^* y_n \Big\|^2 &= \Big\| \sum_{n\in J\cap\sigma} \Lambda_n^* y_n + \sum_{n\in J\cap\sigma^c} \Omega_n^* y_n \Big\|^2 \\ &= \Big\| \sum_{n\in J\cap\sigma} \sum_{m\in\mathbb{J}_n} \langle y_n, e_{n,m}\rangle v_{n,m} \\ &\qquad + \sum_{n\in J\cap\sigma^c} \sum_{m\in\mathbb{J}_n} \langle y_n, e_{n,m}\rangle w_{n,m} \Big\|^2\end{aligned}$$

$$= \left\| \sum_{n \in J \cap \sigma} \sum_{m \in \mathbb{J}_n} c_{n,m} v_{n,m} + \sum_{n \in J \cap \sigma^c} \sum_{m \in \mathbb{J}_n} c_{n,m} w_{n,m} \right\|^2,$$

and

$$\sum_{n \in J} \|y_n\|^2 = \sum_{n \in J} \left\| \sum_{m \in \mathbb{J}_n} c_{n,m} e_{n,m} \right\|^2 = \sum_{n \in J} \sum_{m \in \mathbb{J}_n} |c_{n,m}|^2.$$

Hence, it follows that

$$A \sum_{n \in J} \|y_n\|^2 \leqslant \left\| \sum_{n \in J} \Gamma_n^* y_n \right\|^2 \leqslant B \sum_{n \in J} \|y_n\|^2$$

is equivalent to

$$A \sum_{n \in J} \sum_{m \in \mathbb{J}_n} |c_{n,m}|^2 \leqslant \left\| \sum_{n \in J \cap \sigma} \sum_{m \in \mathbb{J}_n} c_{n,m} v_{n,m} + \sum_{n \in J \cap \sigma^c} \sum_{m \in \mathbb{J}_n} c_{n,m} w_{n,m} \right\|^2$$
$$\leqslant B \sum_{n \in J} \sum_{m \in \mathbb{J}_n} |c_{n,m}|^2,$$

that is, $\{\Lambda_n\}_{n \in \sigma} \cup \{\Omega_n\}_{n \in \sigma^c}$ is a $g$-Riesz sequence if and only if

$$\{\Lambda_n^* e_{n,m}\}_{\substack{n \in \sigma \\ m \in \mathbb{J}_n}} \cup \{\Omega_n^* e_{n,m}\}_{\substack{n \in \sigma^c \\ m \in \mathbb{J}_n}}$$

is a Riesz sequence.

Next, we show that $\{\Gamma_n\}_{n \in \mathbb{N}}$ is $g$-complete if and only if

$$\{\Lambda_n^* e_{n,m}\}_{\substack{n \in \sigma \\ m \in \mathbb{J}_n}} \cup \{\Omega_n^* e_{n,m}\}_{\substack{n \in \sigma^c \\ m \in \mathbb{J}_n}}$$

is complete.

$$\begin{aligned}\{x : \Gamma_n x = 0, n \in \mathbb{N}\} &= \{x : \Lambda_n x = 0, n \in \sigma\} \cup \{x : \Omega_n x = 0, n \in \sigma^c\} \\ &= \left\{x : \sum_{m \in \mathbb{J}_n} \langle x, v_{n,m} \rangle e_{n,m} = 0, n \in \sigma \right\} \\ &\cup \left\{x : \sum_{m \in \mathbb{J}_n} \langle x, w_{n,m} \rangle e_{n,m} = 0, n \in \sigma^c \right\} \\ &= \{x : \langle x, v_{n,m} \rangle = 0, n \in \sigma, m \in \mathbb{J}_n\} \\ &\cup \{x : \langle x, w_{n,m} \rangle = 0, n \in \sigma^c, m \in \mathbb{J}_n\}.\end{aligned}$$

This completes the proof. □

**Example 5.3.** Let $\mathcal{H} = \mathbb{C}^N$, where $N > 1$ is any odd natural number, and let $\{e_n\}_{n=1}^N$ be the canonical orthonormal basis for $\mathcal{H}$, i.e.,

$$e_n = \Big(0, \ldots, 0, \underbrace{1}_{n\text{th-place}}, 0, \ldots, 0\Big).$$

Suppose that $\mathcal{H}_n = \operatorname{span}\{e_n + e_{n+1}\}$ for $n \in [N-1]$ and $\mathcal{H}_N = \operatorname{span}\{e_1 + e_N\}$. Then,

$$\{e_{n,m}\}_{m=1} = \left\{ \frac{1}{\sqrt{2}}\Big(0, \ldots, 0, \underbrace{1}_{n\text{th-place}}, 1, 0, \ldots, 0\Big)\right\}$$

is an orthonormal basis of $\mathcal{H}_n$ $(n \in [n-1])$ and

$$\{e_{N,m}\}_{m=1} = \left\{ \frac{1}{\sqrt{2}}(1, 0, \ldots, 0, 1)\right\}$$

is an orthonormal basis of $\mathcal{H}_N$.

Let $\Lambda \equiv \{\Lambda_n\}_{n=1}^N$ and $\Omega \equiv \{\Omega_n\}_{n=1}^N$, where $\Lambda_n$ is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_n$, and $\Omega_n$ is the orthogonal projection of $\mathcal{H}$ onto $\operatorname{span}\{e_n\}$ for each $n$, $1 \le n \le N$. Clearly,

$$\Lambda_n^* e_{n,1} = e_{n,1} \quad \text{and} \quad \Omega_n^* e_{n,1} = \frac{1}{\sqrt{2}} e_n.$$

It is easy to verify that $\{\Lambda_n^* e_{n,m}\}_{n\in[N],m=1}$ and $\{\Omega_n^* e_{n,m}\}_{n\in[N],m=1}$ are Riesz bases for $\mathcal{H}$. Furthermore, for any $\sigma \subset \mathbb{N}$,

$$\{\Lambda_n^* e_{n,m}\}_{\substack{n\in\sigma\\ m=1}} \bigcup \{\Omega_n^* e_{n,m}\}_{\substack{n\in\sigma^c\\ m=1}}$$

is a Riesz basis for $\mathcal{H}$. Hence, by Corollary 5.2, $\Lambda$ and $\Omega$ are $g$-woven.

The next theorem provides sufficient conditions for weaving $g$-Riesz bases in terms of $g$-Riesz sequences. This generalizes [**1,** Theorem 5.2].

**Theorem 5.4.** *Let $\Lambda \equiv \{\Lambda_i\}_{i\in\mathbb{N}}$ and $\Omega \equiv \{\Omega_i\}_{i\in\mathbb{N}}$ be g-Riesz bases for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in \mathbb{N}\}$, for which there are uniform constants $0 < A \leqslant B < \infty$ so that, for every $\sigma \subset \mathbb{N}$, the family*

$$\{\Lambda_i\}_{i\in\sigma} \cup \{\Omega_i\}_{i\in\sigma^c}$$

*is a g-Riesz sequence with g-Riesz bounds $A$ and $B$. Then, for every $\sigma \subset \mathbb{N}$, the family $\{\Lambda_i\}_{i\in\sigma} \cup \{\Omega_i\}_{i\in\sigma^c}$ is a g-Riesz basis.*

*Proof.* We prove the result in the following steps.

*Step* 1. First, we discuss the case $|\sigma| < \infty$. We prove the result by induction on the cardinality of $\sigma$. The case $|\sigma| = 0$ is trivial. Suppose that the result is true for every $\sigma$ with $|\sigma| = n$.

Now, let $\sigma \subset \mathbb{N}$ with $|\sigma| = n+1$, and choose $i_0 \in \sigma$. Let $\sigma_1 = \sigma \setminus \{i_0\}$. Then,

$$\{\Lambda_i\}_{i\in\sigma_1} \cup \{\Omega_i\}_{i\in\sigma_1^c}$$

is a $g$-Riesz basis by induction hypothesis. Assume that

$$\{\Lambda_i\}_{i\in\sigma} \cup \{\Omega_i\}_{i\in\sigma^c}$$

is not a $g$-Riesz basis, that is,

$$\{\Lambda_i^* e_{i,k}\}_{\substack{i\in\sigma\\ k\in\mathbb{N}}} \cup \{\Omega_i^* e_{i,k}\}_{\substack{i\in\sigma^c\\ k\in\mathbb{N}}}$$

is not complete in $\mathcal{H}$. Then,

$$\Omega_{i_0}^* e_{i_0,k} \notin \operatorname{span}\left( \{\Lambda_i^* e_{i,k}\}_{\substack{i\in\sigma\\ k\in\mathbb{N}}} \cup \{\Omega_i^* e_{i,k}\}_{\substack{i\in\sigma^c\\ k\in\mathbb{N}}}\right).$$

Indeed, if

$$\Omega_{i_0}^* e_{i_0,k} \in \operatorname{span}\left( \{\Lambda_i^* e_{i,k}\}_{\substack{i\in\sigma\\ k\in\mathbb{N}}} \cup \{\Omega_i^* e_{i,k}\}_{\substack{i\in\sigma^c\\ k\in\mathbb{N}}}\right),$$

then

$$\begin{aligned}\overline{\operatorname{span}}&\left( \{\Lambda_i^* e_{i,k}\}_{\substack{i\in\sigma\\ k\in\mathbb{N}}} \cup \{\Omega_i^* e_{i,k}\}_{\substack{i\in\sigma^c\\ k\in\mathbb{N}}}\right)\\ &\qquad\supset \overline{\operatorname{span}}\left( \{\Lambda_i^* e_{i,k}\}_{\substack{i\in\sigma_1\\ k\in\mathbb{N}}} \cup \{\Omega_i^* e_{i,k}\}_{\substack{i\in\sigma_1^c\\ k\in\mathbb{N}}}\right) = \mathcal{H},\end{aligned}$$

that is,

$$\{\Lambda_i^* e_{i,k}\}_{\substack{i\in\sigma\\ k\in\mathbb{N}}} \cup \{\Omega_i^* e_{i,k}\}_{\substack{i\in\sigma^c\\ k\in\mathbb{N}}}$$

is complete in $\mathcal{H}$, which is a contradiction. Hence,

$$\{\Gamma_i\}_{i\in\mathbb{N}} \equiv \{\Lambda_i^* e_{i,k}\}_{\substack{i\in\sigma\\ k\in\mathbb{N}}} \cup \{\Omega_i^* e_{i,k}\}_{\substack{i\in\sigma^c\\ k\in\mathbb{N}}} \cup \{\Omega_{i_0}^* e_{i_0,k}\}$$

is a Riesz sequence in $\mathcal{H}$.

Now, $\sigma_1^c = \sigma^c \cup \{i_0\}$. We obtained $\{\Lambda_i^* e_{i,k}\}_{i\in\sigma_1,k\in\mathbb{N}} \cup \{\Omega_i^* e_{i,k}\}_{i\in\sigma_1^c,k\in\mathbb{N}}$ by deleting the element $\Lambda_{i_0}^* e_{i_0,k}$ from the Riesz sequence $\{\Gamma_i\}_{i\in\mathbb{N}}$.

Therefore, $\{\Lambda_i^* e_{i,k}\}_{i\in\sigma_1,k\in\mathbb{N}} \cup \{\Omega_i^* e_{i,k}\}_{i\in\sigma_1^c,k\in\mathbb{N}}$ cannot be a Riesz basis for $\mathcal{H}$, i.e., $\{\Lambda_i\}_{i\in\sigma_1} \cup \{\Omega_i\}_{i\in\sigma_1^c}$ cannot be a $g$-Riesz basis, which is a contradiction. Hence,

$$\{\Lambda_i\}_{i\in\sigma} \cup \{\Omega_i\}_{i\in\sigma^c}$$

is a $g$-Riesz basis.

*Step* 2. Consider $|\sigma| = \infty$. Suppose that there exists a $\sigma \in \mathbb{N}$ with both $\sigma$ and $\sigma^c$ infinite, such that $\{\Lambda_i\}_{i\in\sigma} \cup \{\Omega_i\}_{i\in\sigma^c}$ is not $g$-complete, i.e., $\{\Lambda_i^* e_{i,k}\}_{i\in\sigma,k\in\mathbb{N}} \cup \{\Omega_i^* e_{i,k}\}_{i\in\sigma^c,k\in\mathbb{N}}$ is not complete in $\mathcal{H}$. Then,

$$M = \overline{\operatorname{span}}\left( \{\Lambda_i^* e_{i,k}\}_{\substack{i\in\sigma\\ k\in\mathbb{N}}} \cup \{\Omega_i^* e_{i,k}\}_{\substack{i\in\sigma^c\\ k\in\mathbb{N}}} \right) \neq \mathcal{H}.$$

Thus, there exists a non-zero vector $x_0 \in \mathcal{H}$ such that $x_0 \perp M$. Since $\{\Omega_i^* e_{i,k}\}_{i,k\in\mathbb{N}}$ is a Bessel sequence, we can find $\sigma_1 \subset \sigma$ with $|\sigma| < \infty$ such that

$$\sum_{i\in\sigma\setminus\sigma_1} \sum_{k\in\mathbb{N}} |\langle x_0, \Omega_i^* e_{i,k}\rangle|^2 < \frac{A}{2}\|x_0\|^2.$$

From Step 1, the family

$$\{\Lambda_i^* e_{i,k}\}_{\substack{i\in\sigma_1\\ k\in\mathbb{N}}} \cup \{\Omega_i^* e_{i,k}\}_{\substack{i\in\sigma\setminus\sigma_1\\ k\in\mathbb{N}}} \cup \{\Omega_i^* e_{i,k}\}_{\substack{i\in\sigma^c\\ k\in\mathbb{N}}}$$

is a Riesz basis with Riesz bounds $A$ and $B$. Using $x_0 \perp M$, we compute

$$\begin{aligned} A\|x_0\|^2 \leqslant{} & \sum_{i\in\sigma_1} \sum_{k\in\mathbb{N}} |\langle x_0, \Lambda_i^* e_{i,k}\rangle|^2 \\ & + \sum_{i\in\sigma\setminus\sigma_1} \sum_{k\in\mathbb{N}} |\langle x_0, \Omega_i^* e_{i,k}\rangle|^2 \\ & + \sum_{i\in\sigma^c} \sum_{k\in\mathbb{N}} |\langle x_0, \Omega_i^* e_{i,k}\rangle|^2 \\ ={} & \sum_{i\in\sigma\setminus\sigma_1} \sum_{k\in\mathbb{N}} |\langle x_0, \Omega_i^* e_{i,k}\rangle|^2 < \frac{A}{2}\|x_0\|^2, \end{aligned}$$

which is absurd. Thus, $\{\Lambda_i\}_{i\in\sigma} \cup \{\Omega_i\}_{i\in\sigma^c}$ is $g$-complete, and hence, a $g$-Riesz basis. $\square$

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