ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 48, Number 2, 2018

SUMMABILITY OF SUBSEQUENCES OF A DIVERGENT SEQUENCE BY REGULAR MATRICES

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ABSTRACT. Stuart proved [**8,** Proposition 7] that the Cesàro matrix C_1 cannot sum almost every subsequence of a bounded divergent sequence x. At the end of the paper, he remarked, "It seems likely that this proposition could be generalized for any regular matrix, but we do not have a proof of this." In this note, we confirm Stuart's conjecture, and we extend it to the more general case of divergent sequences x.

1. Introduction. Throughout this note, we assume familiarity with summability and the standard sequence spaces, see e.g., [**2, 9**]. Thus, we denote by ω, ℓ_∞, c, c_0 and ℓ the set of all sequences in $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$), of all bounded sequences, all convergent sequences, all sequences converging to 0, and of all absolutely summable sequences, respectively.

If $A = (a_{nk})$ is an infinite matrix with scalar entries, then we consider the *application domain*:

$$\omega_A := \left\{(x_k) \in \omega \mid \sum_k a_{nk}\, x_k \text{ converges for each } n \in \mathbb{N}\right\}$$

and the *domain*:

$$c_A := \left\{(x_k) \in \omega_A \mid Ax := \left(\sum_k a_{nk}\, x_k\right)_n \in c\right\}$$

of A. The matrix (method) A is called *regular*, if $c \subset c_A$ and $\lim_A x := \lim Ax = \lim x$ ($x \in c$). The following characterization of regular matrices is contained in the theorem of Toeplitz, et al. [**2**].

2010 AMS *Mathematics subject classification.* Primary 40A05, 40C05.
Keywords and phrases. Regular matrices, summability of subsequences.
Received by the editors on December 11, 2016, and in revised form on March 10, 2017.
DOI:10.1216/RMJ-2018-48-2-413

Theorem 1.1. [**2,** Theorem 2.3.7 II]**.** *A matrix $A = (a_{nk})$ is regular if and only if:*

(a) $\sup_n \sum_k |a_{nk}| < \infty$.

(b) *For all $k \in \mathbb{N} : (a_{nk})_n \in c_0$.*

(c) $\lim_n \sum_k a_{nk} = 1$.

The *Cesàro matrix* $C_1 = (c_{nk})$ with $c_{nk} := 1/n$ if $1 \leq k \leq n$ (k, $n \in \mathbb{N}$) and $c_{nk} := 0$ otherwise is certainly the most famous example of a regular matrix.

2. Preliminary considerations. Steinhaus stated in [**7**] that a regular matrix cannot sum all sequences of 0's and 1's for which Connor gave in [**4**] a very short proof based on the Baire classification theorem. In particular, the Steinhaus theorem obviously implies that a regular matrix cannot sum all bounded sequences, which is also a corollary of the Schur theorem [**2,** Corollary 2.4.2], [**6**]. Moreover, the Hahn theorem [**2,** Theorem 2.4.5], [**5**] states that a matrix sums all bounded sequences if it sums all sequences of 0's and 1's.

The examination of the following problems may be interesting:

Problem 2.1.

(a) Determine (small) subsets Q of $\ell_\infty \setminus c$ such that a given regular matrix like C_1 cannot sum all $x \in Q$.

(b) Determine (small) subsets Q of $\ell_\infty \setminus c$ such that each regular matrix cannot sum all $x \in Q$.

In both cases, $Q \subset \ell_\infty \setminus c$ may be replaced by $Q \subset \omega \setminus c$.

A related problem is based on the question, how many subsequences of a given divergent sequence can be summed by a given regular matrix (or by any regular matrix)? This question makes sense as the following result shows.

Proposition 2.2. [**3,** Theorem], [**8,** Theorem 5]**.** *If x is any bounded divergent sequence, then each regular matrix cannot sum all subsequences of x.*

Analogously to Problem 2.1, we pose the following problem:

Problem 2.3. Let $\mathcal{I}$ be the set of all index sequences (n_i), and let $x = (x_n)$ be any bounded divergent sequence. (By definition, an index sequence is a strictly increasing sequence of natural numbers.)

(a) Determine (small) subsets $\mathcal{Q}$ of $\mathcal{I}$ such that a given regular matrix like C_1 cannot sum all subsequences (x_{n_i}) of x with $(n_i) \in \mathcal{Q}$.

(b) Determine (small) subsets $\mathcal{Q}$ of $\mathcal{I}$ such that each regular matrix cannot sum all subsequences (x_{n_i}) of x with $(n_i) \in \mathcal{Q}$.

In both cases, we may assume that x is divergent and not necessarily bounded.

Following Stuart in **[8]** we consider the set of subsequences (of a bounded divergent sequence) that have index sets with positive density.

Definition 2.4 (Positive density)**.** Given a set $S \subset \mathbb{N}$, let $S_n := S \cap \mathbb{N}_n$ $(n \in \mathbb{N})$. Then the *density of* S is defined by $d(S) := \limsup_n |S_n|/n$ where $|Y|$ denotes the cardinality of any set Y. A property holds for *almost every subsequence of a given sequence* if it holds for all the subsequences that have index sets with positive density. Note that $d(S)$ is defined in **[8]** by $d(S) := (1/n) \limsup_n |S_n|$, which is, with certainty, an oversight, and that, in some papers, $d(S)$ is denoted as *upper* (*asymptotic*) *density* **[1]**.

In the following, we consider, in this sense, the set

$$\mathcal{Q} := \{(n_i) \in \mathcal{I} \mid d(\{n_i \mid i \in \mathbb{N}\}) > 0\}. \tag{2.1}$$

Stuart presented Proposition 2.5 for the case $\mathcal{Q}$ and the more general case of Proposition 2.6.

Proposition 2.5. **[8,** Proposition 6**].** *The matrix C_1 cannot sum almost every subsequence of any sequence of* 0*'s and* 1*'s.*

Proposition 2.6. **[8,** Proposition 7**].** *The matrix C_1 cannot sum almost every subsequence of any bounded divergent sequence.*

3. General results. Now, we shall prove that Stuart's proposition 2.6 remains true if we consider any regular matrix A instead of C_1 and any divergent sequence x instead of any bounded divergent sequence x.

Theorem 3.1. *Let $A = (a_{nk})$ be a regular matrix. Then, A cannot sum almost every subsequence of any divergent sequence $x = (x_k)$.*

The proof will be given in two steps, Theorem 3.3 and Theorem 3.6. In the first step, we consider exclusively *bounded divergent sequences* x. Thereby, the structure of the proof of the corresponding result is essentially based upon the proof of Proposition 2.6 [**8,** Proposition 7], whereby Stuart applied the following lemma (without proof); for the sake of completeness, we will provide a proof.

Lemma 3.2. *Let $x = (x_n) \in \ell_\infty \setminus c$. Then, for each $\varepsilon > 0$, there exists a limit point α_ε of x such that $S := \{r \in \mathbb{N} \mid |x_r - \alpha_\varepsilon| < \varepsilon\}$ has positive density.*

Proof. Let $x = (x_n) \in \ell_\infty \setminus c$ be given. Without loss of generality, we may assume that $0 < x_n \le 1$ $(n \in \mathbb{N})$ and, initially, $\varepsilon := 1/k$ for any given $k \in \mathbb{N}$. Then, we split the interval $]0, 1]$ into the intervals $I_j :=](j-1)/k,\ j/k]$ for $j \in \mathbb{N}_k$. Let $S_j := \{r \in \mathbb{N} \mid x_r \in I_j\}$. Then, there must be a subsequence of x that has the range in one of these intervals, say in I_u, and that has the support of positive density, that is, S_u has positive density. This uses the sub-additivity of the density.

Now, let $\varepsilon > 0$ be given and $k \in \mathbb{N}$ chosen such that $1/k < \varepsilon$. By the previous considerations, there exists a $u \in \mathbb{N}_k$ such that S_u has positive density and the interval $[(u-1)/k, u/k]$ contains a limit point α of x. Since $S := \{r \in \mathbb{N} \mid |x_r - \alpha| < \varepsilon\} \supset S_u$, the set S has positive density since so does S_u. $\square$

Theorem 3.3. *Let $A = (a_{nk})$ be a regular matrix. Then, A cannot sum almost every subsequence of any bounded divergent sequence $x = (x_k)$.*

Proof. By [**2,** Remark 10.4.3] there exists a *normal* regular matrix that is b-equivalent to A, so that we can assume that A has already this property. (A lower triangular matrix $A = (a_{nk})$ with $a_{nn} \ne 0$, $n \in \mathbb{N}$, is called a *triangle* or *normal* matrix, cf., [**2,** 2.2.8]. Moreover, we can obviously assume that the row sums of A are equal to 1. We set

$$M := \sup_n \sum_k |a_{nk}| < \infty$$

and note that $M \ge 1$ since A is regular.

Let $x = (x_n) \in \ell_\infty \setminus c$ be given. Without loss of generality, we may assume $x_n \geq 1$, $n \in \mathbb{N}$; otherwise, we consider $y := x + (\|x\|_\infty + 1)e$ instead of x. Then, for any limit point a of x, there exists another limit point $b \in \mathbb{R}$ with positive distance $0 < \delta = |a - b| < \infty$ (otherwise, $x \in c$). In particular, we consider a to be a limit point such that

$$S := \left\{ r \in \mathbb{N} \mid |x_r - a| < \frac{\delta}{3M} =: \frac{\varepsilon}{M} \right\}$$

has positive density (cf., Lemma 3.2). We assume $a > b$; in the case of $a < b$, the proof runs analogously. Let (m_k) be the index sequence corresponding to S.

Now, we can choose a subsequence (x_{r_k}) of x with $|x_{r_k} - b| < \varepsilon/M$, $k \in \mathbb{N}$, and set $T := \{r_k \mid k \in \mathbb{N}\}$. Consequently, the distance between the values of (x_{m_k}) and (x_{r_k}) is at least ε.

Next, we construct a subsequence $y = (y_i)$ of x, that is not A-summable and has an index set with positive density. First, we choose an $n_1 \in \mathbb{N}$ such that

$$\frac{1}{n_1} |S \cap \mathbb{N}_{n_1}| \geq \frac{d(S)}{2}.$$

Let $F_1 := S \cap \mathbb{N}_{n_1}$, $\beta_1 := |F_1|$ and $y_1, y_2, \ldots, y_{\beta_1}$ be the set

$$\{x_{m_i} \mid m_i \in F_1\}$$

in its order as a subsequence of x. Obviously, we have

$$\begin{aligned}
\alpha_1 := \sum_{i=1}^{\beta_1} a_{\beta_1 i} y_i &= \sum_{\substack{i=1 \\ a_{\beta_1 i} \geq 0}}^{\beta_1} a_{\beta_1 i} y_i + \sum_{\substack{i=1 \\ a_{\beta_1 i} < 0}}^{\beta_1} a_{\beta_1 i} y_i \\
&> \left(a - \frac{\varepsilon}{M}\right) \sum_{\substack{i=1 \\ a_{\beta_1 i} \geq 0}}^{\beta_1} a_{\beta_1 i} + \left(a + \frac{\varepsilon}{M}\right) \sum_{\substack{i=1 \\ a_{\beta_1 i} < 0}}^{\beta_1} a_{\beta_1 i} \\
&= a \sum_{i=1}^{\beta_1} a_{\beta_1 i} - \frac{\varepsilon}{M} \sum_{i=1}^{\beta_1} |a_{\beta_1 i}| \\
&\geq a - \frac{\varepsilon}{M} \cdot M = a - \varepsilon
\end{aligned}$$

since the row sums of A are assumed to be 1. Second, we choose an

$n_2^* > n_1$ such that

$$\sum_{i=1}^{\beta_1} a_{ni} y_i < \frac{\varepsilon}{6}$$

and

$$\sum_{i=1}^{\beta_1} |a_{ni}| < \frac{\varepsilon}{6b},$$

$n \geq n_2^*$, and then an $n_2 \geq n_2^*$ such that $\beta_2 := |F_1| + |F_2| = \beta_1 + |F_2| \geq n_2^*$, where $F_2 := T \cap (\mathbb{N}_{n_2} \setminus \mathbb{N}_{n_1})$. Setting $y_{\beta_1+1}, \ldots, y_{\beta_2}$ for the set $\{x_{r_i} \mid r_i \in F_2\}$ in its order as a subsequence of x, we obtain

$$\begin{aligned}
\alpha_2 &:= \sum_{i=1}^{\beta_1} a_{\beta_2 i} y_i + \sum_{\substack{i=\beta_1+1 \\ a_{\beta_2 i} \geq 0}}^{\beta_2} a_{\beta_2 i} y_i + \sum_{\substack{i=\beta_1+1 \\ a_{\beta_2 i} < 0}}^{\beta_2} a_{\beta_2 i} y_i \\
&< \frac{\varepsilon}{6} + \left(b + \frac{\varepsilon}{M}\right) \sum_{\substack{i=\beta_1+1 \\ a_{\beta_2 i} \geq 0}}^{\beta_2} a_{\beta_2 i} + \left(b - \frac{\varepsilon}{M}\right) \sum_{\substack{i=\beta_1+1 \\ a_{\beta_2 i} < 0}}^{\beta_2} a_{\beta_2 i} \\
&= \frac{\varepsilon}{6} + b \sum_{i=\beta_1+1}^{\beta_2} a_{\beta_2 i} + \frac{\varepsilon}{M} \sum_{i=\beta_1+1}^{\beta_2} |a_{\beta_2 i}| \\
&\leq \frac{\varepsilon}{6} + b\left(\sum_{i=1}^{\beta_2} a_{\beta_2 i} + \sum_{i=1}^{\beta_1} |a_{\beta_2 i}|\right) + \frac{\varepsilon}{M} M \\
&< \frac{\varepsilon}{6} + b + b\frac{\varepsilon}{6b} + \varepsilon = b + \frac{4\varepsilon}{3}.
\end{aligned}$$

Now, we choose an $n_3^* > n_2$ such that

$$\sum_{i=1}^{\beta_2} |a_{\nu i}| < \frac{\varepsilon}{6a} \quad \text{and} \quad \left| \sum_{i=1}^{\beta_2} a_{\nu i} y_i \right| < \frac{\varepsilon}{6}, \quad \nu \geq n_3^*, \tag{3.1}$$

and then an $n_3 > n_3^*$ such that

$$\frac{1}{n_3} |S \cap (\mathbb{N}_{n_3} \setminus \mathbb{N}_{n_2})| \geq \frac{d(S)}{2} \tag{3.2}$$

and

$$\beta_3 := \sum_{j=1}^{3} |F_j| = \beta_2 + |F_3| \geq n_3^*$$

where $F_3 := S \cap (\mathbb{N}_{n_3} \setminus \mathbb{N}_{n_2})$, and, noting the regularity of A and $0 < \varepsilon/(6a) < 1$,

$$\sum_{i=\beta_2+1}^{\beta_3} a_{\beta_3 i} > 1 - \frac{\varepsilon}{6a}. \tag{3.3}$$

Setting $y_{\beta_2+1}, \ldots, y_{\beta_3}$ for the members of the set $\{x_{m_i} \mid m_i \in F_3\}$ in its order as a subsequence of x, we get by (3.1) and (3.3):

$$\begin{aligned}
\alpha_3 &:= \sum_{i=1}^{\beta_2} a_{\beta_3 i} y_i + \sum_{\substack{i=\beta_2+1 \\ a_{\beta_3 i} \geq 0}}^{\beta_3} a_{\beta_3 i} y_i + \sum_{\substack{i=\beta_2+1 \\ a_{\beta_3 i} < 0}}^{\beta_3} a_{\beta_3 i} y_i \\
&> -\frac{\varepsilon}{6} + \left(a - \frac{\varepsilon}{M}\right) \sum_{\substack{i=\beta_2+1 \\ a_{\beta_3 i} \geq 0}}^{\beta_3} a_{\beta_3 i} + \left(a + \frac{\varepsilon}{M}\right) \sum_{\substack{i=\beta_2+1 \\ a_{\beta_3 i} < 0}}^{\beta_3} a_{\beta_3 i} \\
&= -\frac{\varepsilon}{6} + a \sum_{i=\beta_2+1}^{\beta_3} a_{\beta_3 i} - \frac{\varepsilon}{M} \sum_{i=\beta_2+1}^{\beta_3} |a_{\beta_3 i}| \\
&> -\frac{\varepsilon}{6} + a\left(1 - \frac{\varepsilon}{6a}\right) - \frac{\varepsilon}{M} M = a - \frac{4\varepsilon}{3}.
\end{aligned}$$

Continuing inductively, we get a sequence (F_n) of finite and pairwise disjoint sets and a subsequence $y = (y_i)$ of x with the following properties: the index set

$$F := \bigcup_n F_n$$

of y has density at least $d(S)/2$ by (3.2), and the corresponding subsequence (α_ν) of Ay oscillates between values greater than $a-(4/3)\varepsilon$ and less than $b + (4/3)\varepsilon$. Thus, since $a - b = \delta = 3\varepsilon$, the constructed sequence y is not A-summable. □

In the next step, we consider exclusively *unbounded* sequences x and regular matrices A. Below, the next obvious remark is useful.

Remark 3.4. *A regular matrix cannot sum any subsequence of any fixed sequence if there exists a row-finite submatrix of it with this property.*

Proposition 3.5. *If $a = (a_k) \in \omega \setminus \varphi$ and $x = (x_k)$ is any unbounded sequence, then there exists a subsequence $y = (y_i)$ of x with positive density such that $\left(\sum_{i=1}^{m} a_i y_i\right)_m$ is unbounded.*

Proof. Let $\beta_2 > 1$ be such that $a_{\beta_2} \neq 0$. Set $F_1 := \{1, \ldots, \beta_2 - 1\}$, $\beta_1 := |F_1|$ and $y_1 := x_1, \ldots, y_{\beta_1} := x_{\beta_1}$. Then,

$$\frac{1}{\beta_1}|F_1| \geq \frac{1}{2}.$$

In view of $\sup_k |x_k| = \infty$, we can choose $k_1 > \beta_1$ such that

$$|a_{\beta_2} x_{k_1}| > \left| \sum_{i=1}^{\beta_1} a_i y_i \right| + 1.$$

Then, we set $y_{\beta_2} := x_{k_1}$ and $F_2 := \{k_1\}$, and we get

$$\alpha_2 := \left| \sum_{i=1}^{\beta_2} a_i y_i \right| \geq |a_{\beta_2} y_{\beta_2}| - \left| \sum_{i=1}^{\beta_1} a_i y_i \right| > 1.$$

Now, we choose an $s_2 > k_1$ such that

$$\frac{1}{s}|\mathbb{N}_s \setminus \mathbb{N}_{k_1}| \geq \frac{1}{2}, \quad s \geq s_2. \tag{3.4}$$

Let $r_3 > s_2$ be such that $a_{r_3} \neq 0$. We set $\beta_3 := r_3 - 1$ and $F_3 := \mathbb{N}_{\beta_3} \setminus \mathbb{N}_{k_1}$. We take

$$y_{\beta_2+1} := x_{k_1+1}, \ldots, y_{\beta_3} := x_{k_1+\beta_3-\beta_2}.$$

Now, we choose $k_2 > k_1 + \beta_3 - \beta_2$ such that

$$|a_{r_3} x_{k_2}| > \left| \sum_{i=1}^{\beta_3} a_i y_i \right| + 2.$$

Then, we set $F_4 := \{k_2\}$ and $\beta_4 := \beta_3 + 1$, $y_{\beta_4} := x_{k_2}$, and, noting $a_{\beta_4} y_{\beta_4} = a_{r_3} x_{k_2}$, we obtain

$$\alpha_4 := \left| \sum_{i=1}^{\beta_4} a_i y_i \right| \geq |a_{\beta_4} y_{\beta_4}| - \left| \sum_{i=1}^{\beta_3} a_i y_i \right| > 2.$$

Continuing inductively, we get a sequence (F_n) of finite and pairwise disjoint sets and a subsequence $y = (y_i)$ of x with the following properties: the index set $F := \bigcup_n F_n$ of y has density at least $1/2$ by (3.4), and the sequence $(\sum_{i=1}^m a_i y_i)$ is unbounded. $\square$

Theorem 3.6. *Let $A = (a_{nk})$ be any regular matrix and $x = (x_k)$ any unbounded sequence. Then, A cannot sum almost every subsequence of x.*

Proof. We may assume that all rows of A are finite since, otherwise, by Proposition 3.5, there exists a subsequence y of x with positive density satisfying $y \notin \omega_A \supset c_A$. Moreover, since A is regular, from Remark 3.4, we may assume that

- for all $n \in \mathbb{N}$, there exists a $k \in \mathbb{N} : a_{nk} \neq 0$;
- $r = (r_n)$ with $r_n := \max\{k \mid a_{nk} \neq 0\}$ is strictly increasing and $r_1 > 1$;

otherwise, we consider a row submatrix of A with these properties.

Set $F_1 := \{1, \ldots, r_1 - 1\}$, $\beta_1 := |F_1|$, $n_1 := 1$ and $y_1 := x_1, \ldots, y_{\beta_1} := x_{\beta_1}$. Then,

$$\frac{1}{\beta_1} |F_1| \geq \frac{1}{2}.$$

Set $\beta_2 := \beta_1 + 1$. In view of $\limsup_k |x_k| = \infty$, we can choose $k_1 > \beta_1$ such that

$$|a_{n_1 r_1} x_{k_1}| > \left| \sum_{i=1}^{\beta_1} a_{n_1 i} y_i \right| + 1.$$

Then, we set $y_{\beta_2} := x_{k_1}$ and $F_2 := \{k_1\}$, and we obtain

$$\alpha_1 := \left| \sum_{i=1}^{\beta_2} a_{n_1 i} y_i \right| \geq |a_{n_1 \beta_2} y_{\beta_2}| - \left| \sum_{i=1}^{\beta_1} a_{n_1 i} y_i \right| > 1.$$

Now, we choose an $s_2 > k_1$ such that

$$\frac{1}{s}|\mathbb{N}_s \setminus \mathbb{N}_{k_1}| \geq \frac{1}{2}, \quad s \geq s_2. \tag{3.5}$$

Choose $n_2 \in \mathbb{N}$ such that $r_{n_2} > \max\{\beta_2 + 1, s_2\}$, and set $\beta_3 := r_{n_2} - 1$, $F_3 := \mathbb{N}_{\beta_3} \setminus \mathbb{N}_{k_1}$ and $\beta_4 := r_{n_2}$. We put

$$y_{\beta_2+1} := x_{k_1+1}, \ldots, y_{\beta_3} := x_{k_1+\beta_3-\beta_2}.$$

Next, we choose $k_2 > k_1 + \beta_3 - \beta_2$, such that

$$|a_{n_2\beta_4} x_{k_2}| > \left| \sum_{i=1}^{\beta_3} a_{n_2 i} y_i \right| + 2.$$

Then, we set $y_{\beta_4} := x_{k_2}$ and $F_4 := \{k_2\}$, and we get

$$\alpha_2 := \left| \sum_{i=1}^{\beta_4} a_{n_2 i} y_i \right| \geq |a_{n_2\beta_4} y_{\beta_4}| - \left| \sum_{i=1}^{\beta_3} a_{n_2 i} y_i \right| > 2.$$

Continuing inductively, we obtain a sequence (F_n) of finite and pairwise disjoint sets and a subsequence $y = (y_i)$ of x with the following properties: by (3.5), the index set $F := \cup_n F_n$ of y has density at least $1/2$, and the corresponding subsequence (α_ν) is unbounded. Thus, the constructed sequence y is not A-summable. $\square$

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