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RECIPROCAL RELATIONS FOR TRIGONOMETRIC SUMS

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ABSTRACT. By means of the partial fraction decomposition method, a general reciprocal theorem on trigonometric sums is established. Several trigonometric reciprocities and summation formulae are derived as consequences.

1. Outline and introduction. Over the past few decades, there has been growing interest in finite sums involving integer powers of trigonometric functions [**1, 7, 17, 25, 27, 28**]. Compared to those with positive powers [**3, 19, 26**] that are relatively easier to treat, the most interesting sums have been trigonometric power sums with inverse powers of the sine or cosine, since their evaluation invariably directly involves the zeta function [**2, 6, 13, 29**] or through classical numbers, such as the Bernoulli and Euler, and polynomials [**11, 12, 14, 15**]. For more extensive information, the reader is referred to [**5**] and the list of references therein.

In 1969, Gardner [**21**] discovered the following asymptotic relation

$$\left(\frac{\pi}{2m}\right)^{2\lambda}\sum_{k=1}^{m-1}\sec^{2\lambda}\frac{k\pi}{2m}\approx\zeta(2\lambda)\quad\text{as } m\to\infty$$

and posed the problem of evaluating the finite trigonometric series in a simpler closed form. Fisher [**18**] resolved this problem by examining the equivalent expression with the sec-function replaced by the csc-function via the generating function approach. In 1971, Williams [**29**]

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found the following analogous asymptotic formula

$$\left(\frac{\pi}{2m}\right)^{2\lambda}\sum_{k=1}^{m}\cot^{2\lambda}\frac{k\pi}{2m+1}\approx\zeta(2\lambda)\quad\text{as } m\to\infty$$

that was also obtained independently by Apostol [**2**]. Subsequently, the sums of even positive powers of $\csc(k\pi)/m$ arose in a paper of the early 1990s on string theory due to Dowker [**16**]. In quite a comprehensive paper, Chu and Marini [**10**] established systematically closed formulae for 24 different classes of trigonometric sums by employing generating functions and partial fraction decompositions. Berndt and Yeap [**5**] derived, by the contour integration, not only explicit formulae for several classes of trigonometric sums, but also numerous reciprocal relations, including those for Dedekind and Gauss sums.

Many finite trigonometric sums evidently do not have evaluations in closed form. However, they may possess beautiful reciprocity theorems [**4, 5, 20, 23**]. The primary objective of this paper is to establish several reciprocity theorems for finite trigonometric sums. A couple of examples can be anticipated as follows. Let m and n be two natural numbers with $\gcd(m,n)=1$. Then, we have the following trigonometric formulae, see Corollary 3.4 (A):

$$\sum_{i=1}^{m-1}\frac{n\sin(i\pi/m)\cot(ni\pi/m)}{\cos^2(i\pi/m)-\cos^2\theta}+\sum_{j=1}^{n-1}\frac{m\sin(j\pi/n)\cot(mj\pi/n)}{\cos^2(j\pi/n)-\cos^2\theta}=0,$$

$$\begin{aligned}\sum_{i=1}^{m-1}\frac{n\sin(2i\pi/m)\cot(ni\pi/m)}{\cos^2(i\pi/m)-\cos^2\theta}+\sum_{j=1}^{n-1}\frac{m\sin(2j\pi/n)\cot(mj\pi/n)}{\cos^2(j\pi/n)-\cos^2\theta}&\\ =4mn-\frac{2}{\sin^2\theta}+2mn\frac{\cos(m+n)\theta}{\sin m\theta\sin n\theta}&.\end{aligned}$$

These relations are said to be reciprocal since each equation contains two trigonometric sums dependent upon two integer parameters m and n which can be written as $S(m,n)+S(n,m)$, with the second sum obtained from the first under the exchange $m\rightleftharpoons n$, even though $S(m,n)$ does not admit a closed form in general. If m is an odd integer and n an even integer subject to $\gcd(m,n)=1$, we have two further pairs of reciprocal relations, see Corollary 3.4 (E):

$$\sum_{i=1}^{m-1}\frac{n\sin(i\pi/m)\csc(ni\pi/m)}{\cos^2(i\pi/m)-\cos^2\theta}+\sum_{j=1}^{n-1}(-1)^j\frac{m\sin(j\pi/n)\cot(mj\pi/n)}{\cos^2(j\pi/n)-\cos^2\theta}=0,$$

$$\sum_{i=1}^{m-1}\frac{n\sin(2i\pi/m)\csc(ni\pi/m)}{\cos^2(i\pi/m)-\cos^2\theta}+\sum_{j=1}^{n-1}(-1)^j\frac{m\sin(2j\pi/n)\cot(mj\pi/n)}{\cos^2(j\pi/n)-\cos^2\theta}$$
$$=\frac{2mn\cos m\theta}{\sin m\theta\sin n\theta}-\frac{2}{\sin^2\theta};$$

and, see Corollary 3.4 (K):

$$\sum_{i=1}^{m-1}\frac{n(-1)^i\sin^n(i\pi/m)\sin(2i\pi/m)}{\sin(ni\pi/m)(\cos^2(i\pi/m)-\cos^2\theta)}$$
$$+\sum_{j=1}^{n-1}\frac{m(-1)^j\sin^n(j\pi/n)\sin(2j\pi/n)}{\sin(mj\pi/n)(\cos^2(j\pi/n)-\cos^2\theta)}=0,$$

$$\sum_{i=1}^{m-1}\frac{n(-1)^i\sin^n(i\pi/m)\sin(i\pi/m)}{\sin(ni\pi/m)(\cos^2(i\pi/m)-\cos^2\theta)}$$
$$+\sum_{j=1}^{n-1}\frac{m(-1)^j\sin^n(j\pi/n)\sin(j\pi/n)}{\sin(mj\pi/n)(\cos^2(j\pi/n)-\cos^2\theta)}=\frac{mn\sin^n\theta}{\sin m\theta\sin n\theta\cos\theta}.$$

These are just the tip of the iceberg since, as may be seen in the sequel, there exist numerous such reciprocal sums. Moreover, considering each reciprocal relation as a functional equation in $y=\cos\theta$ and then extracting the coefficients of y^k for $k=0,1,2,\dots$, we would create an infinite number of reciprocal formulae of trigonometric sums.

The rest of the paper is organized as follows. In Section 2, we shall prove the main theorem involving a general trigonometric polynomial $P(\cos\theta)$ by means of the partial fraction decomposition. Then, in the third section, we shall derive several reciprocal relations by specifying concretely different trigonometric polynomials $P(\cos\theta)$. Observing that when one of two integers m and n is small enough, we can derive a summation formula from a reciprocal relation. Finally, we shall examine the cases $m=1,2$ in Section 4 and illustrate several trigonometric summation formulae, including those appearing previously in [**1, 10, 12, 15, 24, 25**].

2. Main theorem and proof. For two natural numbers m and n with $\gcd(m,n)=1$, it is easy to verify that $\sin m\theta \sin n\theta$ is a polynomial of $\cos\theta$ with the distinct zeros

$$\left\{\frac{i\pi}{m}\right\}_{i=0}^{m-1} \quad \text{and} \quad \left\{\frac{j\pi}{n}\right\}_{j=1}^{n}.$$

Suppose that $P(\cos\theta)$ is a polynomial in $\cos\theta$ with degree $\leq m+n$. Then, we have the following partial fraction decomposition:

$$\frac{P(\cos\theta)}{\sin m\theta \sin n\theta} = \frac{-\beta}{mn} + \sum_{i=0}^{m-1} \frac{A_i}{\cos(i\pi/m) - \cos\theta} + \sum_{j=1}^{n} \frac{B_j}{\cos(j\pi/n) - \cos\theta}.$$

The constant β can be determined by making the replacement $\theta \to iy$ throughout the equation and then evaluating the limit

$$\beta = \lim_{y\to\infty} \frac{mnP(\cosh y)}{\sinh my \sinh ny}. \tag{2.1}$$

The other connection coefficients are determined as follows:

$$\begin{aligned}
A_0 &= \lim_{\theta\to 0} \frac{(1-\cos\theta)P(\cos\theta)}{\sin m\theta \sin n\theta} = \lim_{\theta\to 0} \frac{2P(\cos\theta)\sin^2(\theta/2)}{\sin m\theta \sin n\theta} = \frac{P(1)}{2mn},\\
A_i &= \lim_{\theta\to(i\pi)/m} \frac{\cos(i\pi/m) - \cos\theta}{\sin m\theta \sin n\theta} P(\cos\theta)\\
&= (-1)^i \frac{P(\cos(i\pi/m))\sin(i\pi/m)}{m\sin(ni\pi/m)}, \quad 1 \leq i < m;\\
B_j &= \lim_{\theta\to(j\pi)/n} \frac{\cos(j\pi/n) - \cos\theta}{\sin m\theta \sin n\theta} P(\cos\theta)\\
&= (-1)^j \frac{P(\cos(j\pi/n))\sin(j\pi/n)}{n\sin(mj\pi/n)}, \quad 1 \leq j < n;\\
B_n &= \lim_{\theta\to\pi} \frac{(1+\cos\theta)P(\cos\theta)}{-\sin m\theta \sin n\theta} = (-1)^{m+n-1} \frac{P(-1)}{2mn}.
\end{aligned}$$

Therefore, the following reciprocal formula has been established:

$$\begin{aligned}\frac{mnP(\cos\theta)}{\sin m\theta\sin n\theta} &= \frac{P(1)}{2(1-\cos\theta)} - \beta + \frac{(-1)^{m+n}P(-1)}{2(1+\cos\theta)}\\ &+ \sum_{i=1}^{m-1}(-1)^i\frac{nP(\cos(i\pi/m))\sin(i\pi/m)}{\sin(ni\pi/m)(\cos(i\pi/m)-\cos\theta)}\\ &+ \sum_{j=1}^{n-1}(-1)^j\frac{mP(\cos(j\pi/n))\sin(j\pi/n)}{\sin(mj\pi/n)(\cos(j\pi/n)-\cos\theta)}.\end{aligned}\tag{2.2}$$

This formula is highlighted in the next lemma.

Lemma 2.1 (Reciprocal formula)**.** *For two natural numbers m and n subject to* $\gcd(m,n)=1$ *and a trigonometric polynomial $P(\cos\theta)$ of degree $\le m+n$, define β by* (2.1) *and the finite trigonometric sum*

$$\Omega_{m,n}[P] = \sum_{k=1}^{m-1}(-1)^k\frac{nP(\cos(k\pi/m))\sin(k\pi/m)}{\sin(nk\pi/m)(\cos(k\pi/m)-\cos\theta)}.\tag{2.3}$$

Then the following reciprocal formula holds:

$$\Omega_{m,n}[P]+\Omega_{n,m}[P] = \beta+\frac{mnP(\cos\theta)}{\sin m\theta\sin n\theta}-\frac{P(1)}{2(1-\cos\theta)}-\frac{(-1)^{m+n}P(-1)}{2(1+\cos\theta)}.\tag{2.4}$$

Replacing θ by $\pi-\theta$ in equation (2.2), we find that

$$\begin{aligned}(-1)^{m+n}\frac{mnP(-\cos\theta)}{\sin m\theta\sin n\theta} &= \frac{P(1)}{2(1+\cos\theta)} - \beta + \frac{(-1)^{m+n}P(-1)}{2(1-\cos\theta)}\\ &+ \sum_{i=1}^{m-1}(-1)^i\frac{nP(\cos(i\pi/m))\sin(i\pi/m)}{\sin(ni\pi/m)(\cos(i\pi/m)+\cos\theta)}\\ &+ \sum_{j=1}^{n-1}(-1)^j\frac{mP(\cos(j\pi/n))\sin(j\pi/n)}{\sin(mj\pi/n)(\cos(j\pi/n)+\cos\theta)}.\end{aligned}\tag{2.5}$$

The sum and difference of (2.2) and (2.5) give, respectively, the identities

$$\frac{mn\{P(\cos\theta)+(-1)^{m+n}P(-\cos\theta)\}}{\sin m\theta\sin n\theta} = \frac{P(1)}{\sin^2\theta} - 2\beta + \frac{(-1)^{m+n}P(-1)}{\sin^2\theta}$$

$$+\sum_{i=1}^{m-1}(-1)^i\frac{nP(\cos(i\pi/m))\sin(2i\pi/m)}{\sin(ni\pi/m)(\cos^2(i\pi/m)-\cos^2\theta)}$$
$$+\sum_{j=1}^{n-1}(-1)^j\frac{mP(\cos(j\pi/n))\sin(2j\pi/n)}{\sin(mj\pi/n)(\cos^2(j\pi/n)-\cos^2\theta)}$$

and

$$\frac{mn\big\{P(\cos\theta)-(-1)^{m+n}P(-\cos\theta)\big\}}{\sin m\theta\sin n\theta\ \cos\theta}=\frac{P(1)}{\sin^2\theta}-\frac{(-1)^{m+n}P(-1)}{\sin^2\theta}$$
$$+\sum_{i=1}^{m-1}(-1)^i\frac{2nP(\cos(i\pi/m))\sin(i\pi/m)}{\sin(ni\pi/m)(\cos^2(i\pi/m)-\cos^2\theta)}$$
$$+\sum_{j=1}^{n-1}(-1)^j\frac{2mP(\cos(j\pi/n))\sin(j\pi/n)}{\sin(mj\pi/n)(\cos^2(j\pi/n)-\cos^2\theta)}.$$

It is interesting that both trigonometric fractions on the left of these two equalities are expressed as sums of $\sin^2\theta$ (and also of $\cos^2\theta$ equivalently). We reformulate these reciprocal relations in the next main theorem.

Theorem 2.2 (Reciprocal formulae)**.** *For two natural numbers m and n subject to* $\gcd(m,n)=1$ *and a trigonometric polynomial $P(\cos\theta)$ of degree $\leq m+n$, define β by* (2.1) *and the finite trigonometric sums*

$$\mathcal{U}_{m,n}[P]=\sum_{k=1}^{m-1}(-1)^k\frac{nP(\cos(k\pi/m))\sin(2k\pi/m)}{\sin(nk\pi/m)(\cos^2(k\pi/m)-\cos^2\theta)},\tag{2.6}$$

$$\mathcal{V}_{m,n}[P]=\sum_{k=1}^{m-1}(-1)^k\frac{nP(\cos(k\pi/m))\sin(k\pi/m)}{\sin(nk\pi/m)(\cos^2(k\pi/m)-\cos^2\theta)}.\tag{2.7}$$

Then, the following reciprocal formulae hold:

$$\mathcal{U}_{m,n}[P]+\mathcal{U}_{n,m}[P]=\frac{mn\big\{P(\cos\theta)+(-1)^{m+n}P(-\cos\theta)\big\}}{\sin m\theta\sin n\theta}+2\beta-\frac{P(1)+(-1)^{m+n}P(-1)}{\sin^2\theta},\tag{2.8}$$

$$\mathcal{V}_{m,n}[P] + \mathcal{V}_{n,m}[P] = \frac{mn\big\{P(\cos\theta) - (-1)^{m+n}P(-\cos\theta)\big\}}{2\sin m\theta \sin n\theta\ \cos\theta} - \frac{P(1) - (-1)^{m+n}P(-1)}{2\sin^2\theta}. \tag{2.9}$$

3. Reciprocal relations. According to the parity of the integer $m+n$ and the function $P(x)$ (even or odd), the reciprocities displayed in Theorem 2.2 can further be reformulated as follows.

Theorem 3.1 (Reciprocal formulae)**.** *Under the same conditions as in Theorem* 2.2, *the following reciprocal formulae of trigonometric sums hold*:

(A) P *and* $m+n$ *have the same parity*:

$$\mathcal{U}_{m,n}[P] + \mathcal{U}_{n,m}[P] = 2\beta + \frac{2mnP(\cos\theta)}{\sin m\theta \sin n\theta} - \frac{2P(1)}{\sin^2\theta},$$
$$\mathcal{V}_{m,n}[P] + \mathcal{V}_{n,m}[P] = 0.$$

(B) P *and* $m+n$ *have the opposite parity*:

$$\mathcal{U}_{m,n}[P] + \mathcal{U}_{n,m}[P] = 2\beta,$$
$$\mathcal{V}_{m,n}[P] + \mathcal{V}_{n,m}[P] = \frac{mnP(\cos\theta)}{\sin m\theta \sin n\theta\ \cos\theta} - \frac{P(1)}{\sin^2\theta}.$$

By concretely specifying the trigonometric polynomial $P(\cos\theta)$, we can derive numerous reciprocal relations from Theorem 3.1. The following two propositions illustrate some representative examples corresponding to four classes of polynomial functions $P(\cos\theta)$. For the sake of brevity, we shall adopt the Kronecker symbol $\delta_{i,j}$ with $\delta_{i,j} = 1$ for $i = j$ and $\delta_{i,j} = 0$ for $i \neq j$, otherwise.

Proposition 3.2 (Reciprocal relations: P and $m+n$ have the same parity)**.** *Under the same conditions as in Theorem* 2.2, *the following reciprocal formulae of trigonometric sums hold*:

(A) $P(\cos\theta) = \cos\lambda\theta$ *with* $\lambda \leq m+n$ *and* $\lambda = m+n$ (mod 2):

$$\mathcal{U}_{m,n}[P] + \mathcal{U}_{n,m}[P] = 4mn\delta_{\lambda,m+n} + \frac{2mn\cos\lambda\theta}{\sin m\theta \sin n\theta} - \frac{2}{\sin^2\theta},$$
$$\mathcal{V}_{m,n}[P] + \mathcal{V}_{n,m}[P] = 0.$$

(B) $P(\cos\theta) = \cos^\lambda\theta$ *with* $\lambda \le m+n$ *and* $\lambda = m+n \pmod 2$:

$$\mathcal{U}_{m,n}[P] + \mathcal{U}_{n,m}[P] = \frac{mn\delta_{\lambda,m+n}}{2^{m+n-3}} + \frac{2mn\cos^\lambda\theta}{\sin m\theta\sin n\theta} - \frac{2}{\sin^2\theta},$$
$$\mathcal{V}_{m,n}[P] + \mathcal{V}_{n,m}[P] = 0.$$

(C) $P(\cos\theta) = \sin\lambda\theta/\sin\theta$ *with* $\lambda \le 1+m+n$ *and* $\lambda \ne m+n \pmod 2$:

$$\mathcal{U}_{m,n}[P] + \mathcal{U}_{n,m}[P] = 8mn\delta_{\lambda,m+n+1} + \frac{2mn\sin\lambda\theta}{\sin\theta\sin m\theta\sin n\theta} - \frac{2\lambda}{\sin^2\theta},$$
$$\mathcal{V}_{m,n}[P] + \mathcal{V}_{n,m}[P] = 0.$$

(D) $P(\cos\theta) = \cos\lambda\theta\cos\mu\theta$ *with* $\lambda+\mu \le m+n$ *and* $\lambda+\mu = m+n \pmod 2$:

$$\mathcal{U}_{m,n}[P] + \mathcal{U}_{n,m}[P] = 2mn\delta_{\lambda+\mu,m+n} + \frac{2mn\cos\lambda\theta\cos\mu\theta}{\sin m\theta\sin n\theta} - \frac{2}{\sin^2\theta},$$
$$\mathcal{V}_{m,n}[P] + \mathcal{V}_{n,m}[P] = 0.$$

Proposition 3.3 (Reciprocal relations: P and $m+n$ have opposite parity)**.** *Under the same conditions as in Theorem* 2.2*, the following reciprocal formulae of trigonometric sums hold*:

(A) $P(\cos\theta) = \cos\lambda\theta$ *with* $\lambda < m+n$ *and* $\lambda \ne m+n \pmod 2$:

$$\mathcal{U}_{m,n}[P] + \mathcal{U}_{n,m}[P] = 0,$$
$$\mathcal{V}_{m,n}[P] + \mathcal{V}_{n,m}[P] = \frac{mn\cos\lambda\theta}{\sin m\theta\sin n\theta\ \cos\theta} - \frac{1}{\sin^2\theta}.$$

(B) $P(\cos\theta) = \cos^\lambda\theta$ *with* $\lambda < m+n$ *and* $\lambda \ne m+n \pmod 2$:

$$\mathcal{U}_{m,n}[P] + \mathcal{U}_{n,m}[P] = 0,$$
$$\mathcal{V}_{m,n}[P] + \mathcal{V}_{n,m}[P] = \frac{mn\cos^\lambda\theta}{\sin m\theta\sin n\theta\ \cos\theta} - \frac{1}{\sin^2\theta}.$$

(C) $P(\cos\theta) = \sin\lambda\theta/\sin\theta$ *with* $\lambda \le m+n$ *and* $\lambda = m+n \pmod 2$:

$$\mathcal{U}_{m,n}[P] + \mathcal{U}_{n,m}[P] = 0,$$
$$\mathcal{V}_{m,n}[P] + \mathcal{V}_{n,m}[P] = \frac{2mn\sin\lambda\theta}{\sin 2\theta\sin m\theta\sin n\theta} - \frac{\lambda}{\sin^2\theta}.$$

(D) $P(\cos\theta) = \cos\lambda\theta\cos\mu\theta$ *with* $\lambda+\mu < m+n$ *and* $\lambda+\mu \neq m+n$ (mod 2):

$$\mathcal{U}_{m,n}[P] + \mathcal{U}_{n,m}[P] = 0,$$
$$\mathcal{V}_{m,n}[P] + \mathcal{V}_{n,m}[P] = \frac{mn\cos\lambda\theta\cos\mu\theta}{\sin m\theta\sin n\theta\ \cos\theta} - \frac{1}{\sin^2\theta}.$$

In particular, we have the following remarkable reciprocal relations with the degree of polynomial $P(\cos\theta)$ related to m and/or n.

Corollary 3.4 (Reciprocal relations)**.** *Under the same conditions as in Theorem* 2.2*, the following reciprocal formulae of trigonometric sums hold*:

(A) $m+n$ *and* $P(\cos\theta) = \cos(m+n)\theta$ *have the same parity*:

$$\mathcal{U}_{m,n}[P] + \mathcal{U}_{n,m}[P] = 4mn + \frac{2mn\cos(m+n)\theta}{\sin m\theta\sin n\theta} - \frac{2}{\sin^2\theta},$$
$$\mathcal{V}_{m,n}[P] + \mathcal{V}_{n,m}[P] = 0.$$

(B) $m+n$ *and* $P(\cos\theta) = \cos m\theta\cos n\theta$ *have the same parity*:

$$\mathcal{U}_{m,n}[P] + \mathcal{U}_{n,m}[P] = 2mn + \frac{2mn\cos m\theta\cos n\theta}{\sin m\theta\sin n\theta} - \frac{2}{\sin^2\theta},$$
$$\mathcal{V}_{m,n}[P] + \mathcal{V}_{n,m}[P] = 0.$$

(C) $m+n$ *and* $P(\cos\theta) = \cos^{m+n}\theta$ *have the same parity*:

$$\mathcal{U}_{m,n}[P] + \mathcal{U}_{n,m}[P] = \frac{mn}{2^{m+n-3}} + \frac{2mn\cos^{m+n}\theta}{\sin m\theta\sin n\theta} - \frac{2}{\sin^2\theta},$$
$$\mathcal{V}_{m,n}[P] + \mathcal{V}_{n,m}[P] = 0.$$

(D) $m+n$ *and* $P(\cos\theta) = \sin(m+n)\theta/\sin\theta$ *have the opposite parity*:

$$\mathcal{U}_{m,n}[P] + \mathcal{U}_{n,m}[P] = 0,$$
$$\mathcal{V}_{m,n}[P] + \mathcal{V}_{n,m}[P] = \frac{2mn\sin(m+n)\theta}{\sin m\theta\sin n\theta\sin 2\theta} - \frac{m+n}{\sin^2\theta}.$$

For the next seven examples, we must remember the fact that $P(\cos\theta)$ should remain invariant inside $\mathcal{U}_{m,n}[P]$ and $\mathcal{V}_{m,n}[P]$ under the exchange $m \rightleftharpoons n$, even though $P(\cos\theta)$ may depend upon m and/or n.

In addition, the reader is reminded that the integer $m+n$ and the polynomial $P(\cos\theta)$ have the same parity for the formulae in (E), (F), (G), and opposite parity for those in (H), (I), (J), (K), (L).

(E) $P(\cos\theta)=\cos m\theta$ *with* $m=1 \pmod 2$ *and* $n=0 \pmod 2$:

$$\begin{aligned}\mathcal{U}_{m,n}[P]+\mathcal{U}_{n,m}[P]&=\frac{2mn\cos m\theta}{\sin m\theta\sin n\theta}-\frac{2}{\sin^2\theta},\\ \mathcal{V}_{m,n}[P]+\mathcal{V}_{n,m}[P]&=0.\end{aligned}$$

(F) $P(\cos\theta)=\sin n\theta/\sin\theta$ *with* $m=1 \pmod 2$ *and* $n=0 \pmod 2$:

$$\begin{aligned}\mathcal{U}_{m,n}[P]+\mathcal{U}_{n,m}[P]&=\frac{2mn\sin n\theta}{\sin\theta\sin m\theta\sin n\theta}-\frac{2n}{\sin^2\theta},\\ \mathcal{V}_{m,n}[P]+\mathcal{V}_{n,m}[P]&=0.\end{aligned}$$

(G) $P(\cos\theta)=\cos^m\theta$ *with* $m=1 \pmod 2$ *and* $n=0 \pmod 2$:

$$\begin{aligned}\mathcal{U}_{m,n}[P]+\mathcal{U}_{n,m}[P]&=\frac{2mn\cos^m\theta}{\sin m\theta\sin n\theta}-\frac{2}{\sin^2\theta},\\ \mathcal{V}_{m,n}[P]+\mathcal{V}_{n,m}[P]&=0.\end{aligned}$$

(H) $P(\cos\theta)=\cos n\theta$ *with* $m=1 \pmod 2$ *and* $n=0 \pmod 2$:

$$\begin{aligned}\mathcal{U}_{m,n}[P]+\mathcal{U}_{n,m}[P]&=0,\\ \mathcal{V}_{m,n}[P]+\mathcal{V}_{n,m}[P]&=\frac{mn\cos n\theta}{\sin m\theta\sin n\theta\ \cos\theta}-\frac{1}{\sin^2\theta}.\end{aligned}$$

(I) $P(\cos\theta)=\sin m\theta/\sin\theta$ *with* $m=1 \pmod 2$ *and* $n=0 \pmod 2$:

$$\begin{aligned}\mathcal{U}_{m,n}[P]+\mathcal{U}_{n,m}[P]&=0,\\ \mathcal{V}_{m,n}[P]+\mathcal{V}_{n,m}[P]&=\frac{2mn\sin m\theta}{\sin 2\theta\sin m\theta\sin n\theta}-\frac{m}{\sin^2\theta}.\end{aligned}$$

(J) $P(\cos\theta)=\cos m\theta/\cos\theta$ *with* $m=1 \pmod 2$ *and* $n=0 \pmod 2$:

$$\begin{aligned}\mathcal{U}_{m,n}[P]+\mathcal{U}_{n,m}[P]&=0,\\ \mathcal{V}_{m,n}[P]+\mathcal{V}_{n,m}[P]&=\frac{mn\cos m\theta}{\sin m\theta\sin n\theta\cos^2\theta}-\frac{1}{\sin^2\theta}.\end{aligned}$$

(K) $P(\cos\theta)=\sin^n\theta$ *with* $m=1 \pmod 2$ *and* $n=0 \pmod 2$:

$$\mathcal{U}_{m,n}[P]+\mathcal{U}_{n,m}[P]=0,$$

$$\mathcal{V}_{m,n}[P] + \mathcal{V}_{n,m}[P] = \frac{mn \sin^n \theta}{\sin m\theta \sin n\theta \cos \theta}.$$

(L) $P(\cos\theta) = \cos^n \theta$ *with* $m = 1 \pmod 2$ *and* $n = 0 \pmod 2$:

$$\mathcal{U}_{m,n}[P] + \mathcal{U}_{n,m}[P] = 0,$$
$$\mathcal{V}_{m,n}[P] + \mathcal{V}_{n,m}[P] = \frac{mn \cos^{n-1} \theta}{\sin m\theta \sin n\theta} - \frac{1}{\sin^2 \theta}.$$

4. Summation formulae. For $m = 1$ in Theorem 3.1, it is trivial to see that $\mathcal{U}_{1,n} = \mathcal{V}_{1,n} = 0$. Replacing the summation index k by $n-k$, we further verify $\mathcal{U}_{n,1} = 0$ for $P(x)$ and n having the same parity and $\mathcal{V}_{n,1} = 0$ for $P(x)$ and n having the opposite parity. Then, the other two reciprocal relations concerning $\mathcal{U}_{n,1}$ and $\mathcal{V}_{n,1}$ can be simplified in the summation formulae below.

Theorem 4.1 (Summation formulae)**.** *For each natural number n and a trigonometric polynomial $P(\cos\theta)$ of degree $\leq n+1$, the following summation formulae hold*:

(A) *$P(x)$ and n have the same parity*:

$$\sum_{k=0}^{n-1} (-1)^k \frac{P(\cos(k\pi/n))}{\cos^2(k\pi/n) - \cos^2\theta} = \frac{2nP(\cos\theta)}{\sin 2\theta \sin n\theta}. \tag{4.1}$$

(B) *$P(x)$ and n have the opposite parity*:

$$\sum_{k=0}^{n-1} (-1)^k \frac{P(\cos(k\pi/n))\cos(k\pi/n)}{\cos^2(k\pi/n) - \cos^2\theta} = \frac{nP(\cos\theta)}{\sin\theta \sin n\theta} + \lim_{x\to\infty} \frac{nP(\cosh x)}{\sinh x \sinh nx}. \tag{4.2}$$

Then, letting $m = 2$ in Theorem 3.1, we see that n is odd. By computing

$$\mathcal{U}_{2,n} = 0 \quad \text{and} \quad \mathcal{V}_{2,n} = \frac{nP(0)}{\cos^2\theta}(-1)^{(n-1)/2}$$

as well as $\mathcal{U}_{n,2} = 0$ for the even polynomials $P(x)$, and $\mathcal{V}_{n,2} = 0$ for the odd polynomials $P(x)$, in view of the involution $k \to n-k$ on the summation index, the remaining two reciprocal relations for $\mathcal{U}_{n,2}$ and $\mathcal{V}_{n,2}$ can be reformulated as the following summation formulae.

Theorem 4.2 (Summation formulae)**.** *For each odd integer n and a trigonometric polynomial $P(\cos\theta)$ of degree $\le n+2$, the following summation formulae hold*:

(A) *$P(x)$ is an odd function*:

(4.3)
$$\sum_{k=0}^{n-1}(-1)^k\frac{P(\cos(k\pi/n))}{\cos^2(k\pi/n)-\cos^2\theta}=\frac{2nP(\cos\theta)}{\sin 2\theta\sin n\theta}+\lim_{x\to\infty}\frac{2nP(\cosh x)}{\sinh 2x\sinh nx}.$$

(B) *$P(x)$ is an even function*:

$$\begin{aligned}(4.4)\quad \sum_{k=0}^{n-1}(-1)^k&\frac{P(\cos(k\pi/n))\sec(k\pi/n)}{\cos^2(k\pi/n)-\cos^2\theta}\\&=\frac{2nP(\cos\theta)\sec\theta}{\sin 2\theta\sin n\theta}+\frac{nP(0)}{\cos^2\theta}(-1)^{(n+1)/2}.\end{aligned}$$

The four formulae displayed in the last two theorems are particularly useful for evaluating finite trigonometric sums. Some of them are briefly covered below.

4.1. Letting $P(x)=\cos n\theta$ in (4.1), we find that the following trigonometric identity holds for each natural number n (both odd and even):

$$(4.5)\qquad \sum_{k=0}^{n-1}\frac{1}{\cos^2(k\pi/n)-\cos^2\theta}=\frac{2n\cos n\theta}{\sin 2\theta\sin n\theta}.$$

This identity was used by Chu and Marini **[10]** to evaluate the positive trigonometric sums below as polynomials of n:

$$A_{2\lambda}(n):=\sum_{1\le k<n/2}\sec^{2\lambda}\frac{k\pi}{n},$$
$$B_{2\lambda}(n):=\sum_{1\le k<n/2}\csc^{2\lambda}\frac{k\pi}{n},$$
$$C_{2\lambda}(n):=\sum_{1\le k<n/2}\tan^{2\lambda}\frac{k\pi}{n},$$

$$D_{2\lambda}(n) := \sum_{1 \leq k < n/2} \cot^{2\lambda} \frac{k\pi}{n}.$$

Note that the first two sums were also evaluated by Grabner and Prodinger [**24**] through the contour integral method and Gauthier and Bruckman [**22**] by applying derivative operators to $\sec^2\theta$ and $\csc^2\theta$. The closed formulae for these four sums are exemplified for $\lambda = 1, 2, 3$ as follows.

(A) $n = 0 \pmod 2$: Chu and Marini [**10,** B1, B3]

$$\begin{aligned} A_2(n) = B_2(n) &= \frac{(n+2)(n-2)}{6}, \\ A_4(n) = B_4(n) &= \frac{(n+2)(n-2)}{90}(14+n^2), \\ A_6(n) = B_6(n) &= \frac{(n+2)(n-2)}{1890}(284+29n^2+2n^4); \end{aligned}$$

(B) $n = 0 \pmod 2$: Chu and Marini [**10,** B5, B7]

$$\begin{aligned} C_2(n) = D_2(n) &= \frac{(n-1)(n-2)}{6}, \\ C_4(n) = D_4(n) &= \frac{(n-1)(n-2)}{90}\{n^2+3n-13\}, \\ C_6(n) = D_6(n) &= \frac{(n-1)(n-2)}{1890}\{2n^4+6n^3-28n^2-96n+251\}. \end{aligned}$$

(C) $n = 1 \pmod 2$: Chu and Marini [**10,** A1]

$$\begin{aligned} A_2(n) &= \frac{n^2}{2}, \\ A_4(n) &= \frac{n^2}{6}\{2+n^2\}, \\ A_6(n) &= \frac{n^2}{30}\{8+5n^2+2n^4\}. \end{aligned}$$

(D) $n = 1 \pmod 2$: Chu and Marini [**10,** A3]

$$\begin{aligned} B_2(n) &= \frac{(n+1)(n-1)}{6}, \\ B_4(n) &= \frac{(n+1)(n-1)}{90}(11+n^2), \end{aligned}$$

$$B_6(n) = \frac{(n+1)(n-1)}{1890}\{191 + 23n^2 + 2n^4\}.$$

(E) $n = 1 \pmod 2$: Chu and Marini [**10,** A5]

$$C_2(n) = \frac{n(n-1)}{2}, \quad \text{cf., Berndt and Yeap } [\mathbf{5}];$$
$$C_4(n) = \frac{n(n-1)}{6}\{n^2 + n - 3\},$$
$$C_6(n) = \frac{n(n-1)}{30}\{2n^4 + 2n^3 - 8n^2 - 8n + 15\}.$$

(F) $n = 1 \pmod 2$: Chu and Marini [**10,** A7]

$$D_2(n) = \frac{(n-1)(n-2)}{6}, \quad \text{cf., Apostol } [\mathbf{2}];$$
$$D_4(n) = \frac{(n-1)(n-2)}{90}\{n^2 + 3n - 13\},$$
$$D_6(n) = \frac{(n-1)(n-2)}{1890}\{2n^4 + 6n^3 - 28n^2 - 96n + 251\}.$$

4.2. When $n = 0 \pmod 2$ and $P(x) = 1$, we find from (4.1) that

$$\sum_{k=0}^{n-1}(-1)^k \frac{1}{\cos^2(k\pi/n) - \cos^2\theta} = \frac{2n}{\sin 2\theta \sin n\theta}. \tag{4.6}$$

This formula can be used to evaluate the following trigonometric sums as polynomials of n:

$$\mathcal{A}_{2\lambda}(n) := \sum_{1 \le k < n/2} (-1)^{k-1} \csc^{2\lambda}\frac{k\pi}{n},$$
$$\mathcal{B}_{2\lambda}(n) := \sum_{1 \le k < n/2} (-1)^{k-1} \sec^{2\lambda}\frac{k\pi}{n},$$
$$\mathcal{C}_{2\lambda}(n) := \sum_{1 \le k < n/2} (-1)^{k-1} \cot^{2\lambda}\frac{k\pi}{n},$$
$$\mathcal{D}_{2\lambda}(n) := \sum_{1 \le k < n/2} (-1)^{k-1} \tan^{2\lambda}\frac{k\pi}{n}.$$

The formulae corresponding to $\lambda = 1, 2, 3$ are given as follows.

(A) Chu and Marini [**10,** B2, B4]

$$\begin{aligned}
\mathcal{A}_2(n) &= (-1)^{n/2}\mathcal{B}_2(n) = \frac{(-1)^{n/2}}{2} + \frac{2+n^2}{12},\\
\mathcal{A}_4(n) &= (-1)^{n/2}\mathcal{B}_4(n) = \frac{(-1)^{n/2}}{2} + \frac{88+40n^2+7n^4}{720},\\
\mathcal{A}_6(n) &= (-1)^{n/2}\mathcal{B}_6(n) = \frac{(-1)^{n/2}}{2} + \frac{3056+1344n^2+294n^4+31n^6}{30240}.
\end{aligned}$$

(B) Chu and Marini [**10,** B6, B8]

$$\begin{aligned}
\mathcal{C}_2(n) &= (-1)^{n/2}\mathcal{D}_2(n) = \frac{(n+2)(n-2)}{12},\\
\mathcal{C}_4(n) &= (-1)^{n/2}\mathcal{D}_4(n) = \frac{(n+2)(n-2)}{720}\big\{7n^2-52\big\},\\
\mathcal{C}_6(n) &= (-1)^{n/2}\mathcal{D}_6(n) = \frac{(n+2)(n-2)}{30240}\big\{31n^4-464n^2+2008\big\}.
\end{aligned}$$

4.3. When $n = 1 \pmod 2$ and $P(x) = 1$, we analogously find from (4.2) that

$$\sum_{k=0}^{n-1}(-1)^k\frac{\cos(k\pi/n)}{\cos^2(k\pi/n)-\cos^2\theta} = \frac{n}{\sin\theta\sin n\theta}. \tag{4.7}$$

This formula has been used by Chu and Marini [**10,** equation (A0.3a)] to evaluate the following trigonometric sums as polynomials of n:

$$\begin{aligned}
\mathbf{A}_{2\lambda+1}(n) &:= \sum_{1\le k<n/2}(-1)^{k-1}\sec^{1+2\lambda}\frac{k\pi}{n},\\
\mathbf{B}_{2\lambda}(n) &:= \sum_{1\le k<n/2}(-1)^{k-1}\cos\frac{k\pi}{n}\csc^{2\lambda}\frac{k\pi}{n},\\
\mathbf{C}_{2\lambda}(n) &:= \sum_{1\le k<n/2}(-1)^{k-1}\sec\frac{k\pi}{n}\tan^{2\lambda}\frac{k\pi}{n},\\
\mathbf{D}_{2\lambda}(n) &:= \sum_{1\le k<n/2}(-1)^{k-1}\cos\frac{k\pi}{n}\cot^{2\lambda}\frac{k\pi}{n}.
\end{aligned}$$

The formulae corresponding to $\lambda = 1, 2, 3$ are displayed as follows.

(A) Chu and Marini [**10,** A2]

$$\begin{aligned}
\mathbf{A}_1(n) &= \frac{1}{2} + \frac{n(-1)^{(n+1)/2}}{2},\\
\mathbf{A}_3(n) &= \frac{1}{2} + \frac{n(-1)^{(n+1)/2}}{4}\big\{n^2+1\big\},\\
\mathbf{A}_5(n) &= \frac{1}{2} + \frac{n(-1)^{(n+1)/2}}{48}\big\{5n^4+10n^2+9\big\}.
\end{aligned}$$

(B) Chu and Marini [**10,** A4]

$$\begin{aligned}
\mathbf{B}_2(n) &= \frac{n^2-1}{12},\\
\mathbf{B}_4(n) &= \frac{n^2-1}{720}\big\{7n^2+17\big\},\\
\mathbf{B}_6(n) &= \frac{n^2-1}{30240}\big\{31n^4+178n^2+367\big\}.
\end{aligned}$$

(C) Chu and Marini [**10,** A6]

$$\begin{aligned}
\mathbf{C}_0(n) &= \frac{1}{2} + (-1)^{(n+1)/2}\frac{n}{2},\\
\mathbf{C}_2(n) &= (-1)^{(n+1)/2}\frac{n(n^2-1)}{4},\\
\mathbf{C}_4(n) &= (-1)^{(n+1)/2}\frac{n(n^2-1)}{48}\big\{5n^2-9\big\},\\
\mathbf{C}_6(n) &= (-1)^{(n+1)/2}\frac{n(n^2-1)}{1440}\big\{61n^4-214n^2+225\big\}.
\end{aligned}$$

(D) Chu and Marini [**10,** A8]

$$\begin{aligned}
\mathbf{D}_2(n) &= \frac{n^2-7}{12},\\
\mathbf{D}_4(n) &= \frac{7n^4-110n^2+463}{720},\\
\mathbf{D}_6(n) &= \frac{31n^6-735n^4+6489n^2-20905}{30240}.
\end{aligned}$$

4.4. For $m = n \pmod 2$ with $m \le n$, let $P(x) = \cos m\theta$ in (4.1). The corresponding formula reads

$$\sum_{k=0}^{n-1}(-1)^k \frac{\cos(mk\pi/n)}{\cos^2(k\pi/n) - \cos^2\theta} = \frac{2n\ \cos m\theta}{\sin 2\theta \sin n\theta}. \tag{4.8}$$

This formula may be utilized to evaluate the following trigonometric sums:

$$\sum_{\substack{k=1\\k\neq n/2}}^{n-1} (-1)^k \cos\frac{mk\pi}{n} \sec^{2\lambda}\frac{k\pi}{n},$$

$$\sum_{k=1}^{n-1}(-1)^k \cos\frac{mk\pi}{n} \csc^{2\lambda}\frac{k\pi}{n};$$

$$\sum_{\substack{k=1\\k\neq n/2}}^{n-1} (-1)^k \cos\frac{mk\pi}{n} \tan^{2\lambda}\frac{k\pi}{n},$$

$$\sum_{k=1}^{n-1}(-1)^k \cos\frac{mk\pi}{n} \cot^{2\lambda}\frac{k\pi}{n}.$$

Observing that $\cos(mk\pi)/n$ can be expressed as a polynomial of degree m in $\cos(k\pi)/n$, we can reduce the evaluation of these trigonometric sums to those examined in subsections 4.2 and 4.3. However, the resulting expressions will not be reproduced due to their complexity. When n is even, the second and fourth sums have been evaluated by Cvijovic, et al., **[12, 15]** in terms of higher-order Bernoulli polynomials.

4.5. For $m = n \pmod 2$ with $m \le n$, let $P(x) = \sin m\theta/\sin\theta$ in (4.2). The corresponding formula is given by

$$\sum_{k=0}^{n-1}(-1)^k \frac{\sin(mk\pi/n)\cot(k\pi/n)}{\cos^2(k\pi/n) - \cos^2\theta} = \frac{n\ \sin m\theta}{\sin^2\theta \sin n\theta}. \tag{4.9}$$

This formula may be utilized to evaluate the following trigonometric sums:

$$\sum_{\substack{k=1\\k\neq n/2}}^{n-1} (-1)^k \sin\frac{mk\pi}{n} \sec^{2\lambda}\frac{k\pi}{n} \cot\frac{k\pi}{n},$$

$$\sum_{\substack{k=1\\k\neq n/2}}^{n-1}(-1)^k\sin\frac{mk\pi}{n}\tan^{1+2\lambda}\frac{k\pi}{n};$$
$$\sum_{k=1}^{n-1}(-1)^k\sin\frac{mk\pi}{n}\csc^{2\lambda}\frac{k\pi}{n}\cot\frac{k\pi}{n},$$
$$\sum_{k=1}^{n-1}(-1)^k\sin\frac{mk\pi}{n}\cot^{1+2\lambda}\frac{k\pi}{n};$$

where the last one for even n may also be found in [**12**]. Since $\sin(mk\pi)/n$ can be written as $\sin(k\pi)/n$ times a polynomial of degree $m-1$ in $\cos(k\pi)/n$, the four sums displayed above can again be reduced to those examined in subsections 4.2 and 4.3.

4.6. There are numerous articles dedicated to the evaluation of finite trigonometric sums. For example, by utilizing the following four formulae, it is possible to evaluate some similar sums appearing in [**1, 9, 11, 14, 23**].

(A) $P(x)=\sin\theta\sin m\theta$ in (4.1) with $m\not\equiv_2 n$ and $m<n$:

$$\sum_{k=0}^{n-1}(-1)^k\frac{\sin(mk\pi/n)\sin(k\pi/n)}{\cos^2(k\pi/n)-\cos^2\theta}=\frac{n\ \sin m\theta}{\cos\theta\sin n\theta}.$$

(B) $P(x)=\cos m\theta$ in (4.2) with $m\not\equiv_2 n$ and $m<n$:

$$\sum_{k=0}^{n-1}(-1)^k\frac{\cos(mk\pi/n)\cos(k\pi/n)}{\cos^2(k\pi/n)-\cos^2\theta}=\frac{n\ \cos m\theta}{\sin\theta\sin n\theta}.$$

(C) $P(x)=\sin 2\theta\sin m\theta$ in (4.3) with $mn\equiv_2 1$ and $m\le n$:

$$\sum_{k=0}^{n-1}(-1)^k\frac{\sin(2k\pi/n)\sin(mk\pi/n)}{\cos^2(k\pi/n)-\cos^2\theta}=\frac{2n\ \sin m\theta}{\sin n\theta}-2n\chi(m=n).$$

(D) $P(x)=\sin\theta\sin m\theta$ in (4.4) with $m\equiv_2 n$ and $m\le n$:

$$\sum_{k=0}^{n-1}(-1)^k\frac{\sin(mk\pi/n)\tan(k\pi/n)}{\cos^2(k\pi/n)-\cos^2\theta}=\frac{n\sin m\theta}{\cos^2\theta\sin n\theta}+\frac{n(-1)^{(m+n)/2}}{\cos^2\theta}.$$

In addition, several variants of trigonometric sums exist that should allow for recovery by our trigonometric formulae. We limit this to stating the following examples from Berndt and Yeap [**5**] and Byrne and Smith [**6**]

$$\sum_{k=1}^{n} \cot^2 \frac{(2k-1)\pi}{4n} = 2n^2 - n, \tag{4.10}$$

$$\sum_{k=1}^{n} \sin \frac{2mk\pi}{n} \cot \frac{k\pi}{n} = n - 2m; \tag{4.11}$$

where the last one is due to Eisenstein (1844). The interested reader may refer to [**5, 6, 8, 25, 30**] for proofs and further similar identities.

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