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ON A SINE POLYNOMIAL OF TURÁN

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ABSTRACT. In 1935, Turán proved that

$$S_{n,a}(x)=\sum_{j=1}^{n}\binom{n+a-j}{n-j}\sin(jx)>0,$$

$$n,a\in\mathbf{N},\quad 0<x<\pi.$$

We present various related inequalities. Among others, we show that the refinements

$$S_{2n-1,a}(x)\ge\sin(x)\quad\text{and}\quad S_{2n,a}(x)\ge 2\sin(x)(1+\cos(x))$$

are valid for all integers $n\ge 1$ and real numbers $a\ge 1$ and $x\in(0,\pi)$. Moreover, we apply our theorems on sine sums to obtain inequalities for Chebyshev polynomials of the second kind.

1. Introduction. The sequence $\sigma_{n,k}(z)$ is recursively defined by

$$\sigma_{n,0}(z)=\sum_{j=0}^{n}z^j,\qquad \sigma_{n,k}(z)=\sum_{j=0}^{n}\sigma_{j,k-1}(z),\quad k\in\mathbf{N}.$$

Then, we have

$$\sigma_{n,k}(e^{ix})=\sum_{j=0}^{n}\binom{n+k-j}{k}\cos(jx)+i\sum_{j=1}^{n}\binom{n+k-j}{k}\sin(jx).$$

In 1935, Turán **[12]** studied the imaginary part and proved by induction on n and k the remarkable inequality

$$\sum_{j=1}^{n}\binom{n+k-j}{k}\sin(jx)>0,\quad n,k\in\mathbf{N},\ 0<x<\pi. \tag{1.1}$$

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In the same paper, Turán presented an elegant inequality for a sine sum in two variables. He demonstrated that the following companion of (1.1) is valid:

$$\sum_{j=1}^{n} \binom{n+k-j}{k} \frac{\sin(jx)\sin(jy)}{j} > 0, \quad n,k \in \mathbf{N},\ 0 < x,y < \pi. \tag{1.2}$$

Szegő [**11**] applied (1.1) with $k = 2$ to establish a theorem on univalent functions. An extension of (1.1) with $k = 2$ was given by Alzer and Kwong [**3**], whereas Alzer and Fuglede [**1**] offered a positive lower bound for the sine polynomial in (1.1) under the assumption that $n, k \geq 2$. In fact, they proved that

$$\sum_{j=1}^{n} \binom{n+k-j}{k} \sin(jx) > \frac{x(\pi - x)}{\pi}, \quad 2 \leq n,k \in \mathbf{N},\ 0 < x < \pi. \tag{1.3}$$

The definition of the sums given in (1.1) and (1.2) requires that k be a nonnegative integer. However, the identity

$$\binom{n+k-j}{k} = \binom{n+k-j}{n-j}$$

reveals that, if we use the second binomial coefficient, then k can be any real number. Therefore, it is natural to ask for all real parameters a and b such that we have, for all integers $n \geq 1$ and real numbers $x \in (0, \pi)$, $y \in (0, \pi)$,

$$S_{n,a}(x) = \sum_{j=1}^{n} \binom{n+a-j}{n-j} \sin(jx) > 0$$

and

$$\Theta_{n,b}(x,y) = \sum_{j=1}^{n} \binom{n+b-j}{n-j} \frac{\sin(jx)\sin(jy)}{j} > 0. \tag{1.4}$$

In this paper, we solve both problems. Moreover, we provide several closely related inequalities. Among others, we show that there exist functions $\lambda(x)$, $\mu(x)$ and $\lambda^*(x,y)$, $\mu^*(x,y)$ such that the estimates

$$S_{2n-1,a}(x) \geq \lambda(x) > 0, \qquad S_{2n,a}(x) \geq \mu(x) > 0$$

and

$$\Theta_{2n-1,a}(x,y) \ge \lambda^*(x,y) > 0, \qquad \Theta_{2n,a}(x,y) \ge \mu^*(x,y) > 0$$

are valid for all integers $n \ge 1$ and real numbers $a \ge 1$, $x \in (0,\pi)$, $y \in (0,\pi)$.

In the next section, we collect some lemmas which are needed to prove our main results given in Section 3. Finally, in Section 4, we apply our theorems to obtain inequalities for sums involving Chebyshev polynomials of the second kind, and we also offer new integral inequalities for these polynomials. Throughout, we maintain the notation introduced in this section.

For more information on inequalities for trigonometric sums and polynomials, the interested reader is referred to the monograph by Milovanović, Mitrinović, Rassias [**9,** Chapter 6].

2. Lemmas. The first lemma is due to Fejér [**5**].

Lemma 2.1. *Let $x \in (0,\pi)$, and let*

$$\phi_n(x) = 2\sum_{j=1}^{n-1} \sin(jx) + \sin(nx). \tag{2.1}$$

Then, $\phi_n(x) > 0$ for $n = 1,2$ and $\phi_n(x) \ge 0$ for $n \ge 3$.

In order to prove (2.1), Fejér made use of the identity

$$\phi_n(x) = \sin(x)\left(n + 2\sum_{j=1}^{n-1}\sum_{k=1}^{j} \cos(kx)\right)$$

which he obtained by comparing the coefficients of certain power series. Here, we offer a different proof which is more elementary than Fejér's approach.

Proof. Multiplying both sides of (2.1) by $\sin(x/2)$ gives

$$\begin{aligned}\sin(x/2)\phi_n(x) &= \sum_{j=1}^{n-1} 2\sin(x/2)\sin(jx) + \sin(x/2)\sin(nx)\\ &= \sum_{j=1}^{n-1}(\cos((j-1/2)x) - \cos((j+1/2)x))\end{aligned}$$

$$\begin{aligned}
&+\frac{1}{2}(\cos((n-1/2)x)-\cos((n+1/2)x))\\
&=\cos(x/2)-\frac{1}{2}\cos((n-1/2)x)-\frac{1}{2}\cos((n+1/2)x)\\
&=\cos(x/2)-\cos(x/2)\cos(nx)\\
&=\cos(x/2)\big(1-\cos(nx)\big).
\end{aligned}$$

Since $x \in (0,\pi)$, we conclude that $\phi_n(x) > 0$ for $n = 1, 2$ and $\phi_n(x) \geq 0$ for $n \geq 3$. □

Next, an identity is presented which will be a helpful tool not only to establish our inequalities for trigonometric sums but also to cover all cases of equality.

Lemma 2.2. *Let c_k, $k = 1, \ldots, n$, be real numbers and*

$$\gamma_{k,n} = c_k + 2\sum_{j=1}^{n-k}(-1)^j c_{j+k}, \quad k = 1, \ldots, n.$$

Then,

$$\sum_{j=1}^{n} c_j \sin(jx) = \sum_{j=1}^{n} \gamma_{j,n}\phi_j(x), \tag{2.2}$$

where $\phi_j(x)$ is defined in (2.1).

Proof. We have

$$\begin{aligned}
\sum_{j=1}^{n}\gamma_{j,n}\phi_j(x) &= \sum_{j=1}^{n} c_j\phi_j(x) + 2\sum_{j=1}^{n}\left(\phi_j(x)\sum_{k=1}^{n-j}(-1)^k c_{k+j}\right)\\
&= \sum_{j=1}^{n} c_j\phi_j(x) + 2\sum_{j=2}^{n}\left(c_j\sum_{k=1}^{j-1}(-1)^{j-k}\phi_k(x)\right)\\
&= c_1\phi_1(x) + \sum_{j=2}^{n} c_j\left(\phi_j(x) + 2\sum_{k=1}^{j-1}(-1)^{j-k}\phi_k(x)\right)\\
&= c_1\sin(x) + \sum_{j=2}^{n} c_j\sin(jx)\\
&= \sum_{j=1}^{n} c_j\sin(jx).
\end{aligned}$$

□

Remark 2.3. An application of Lemmas 2.1 and 2.2 leads to a result of Steinig **[10]**, who proved that the sine polynomial in (2.2) is nonnegative on $(0, \pi)$, if $\gamma_{k,n} \geq 0$, $k = 1, \dots, n$.

Lemma 2.4. *Let a_k, $k = 1, \dots, n$, $n \geq 3$, be real numbers such that*

$$2a_k \leq a_{k-1} + a_{k+1}, \quad k = 2, \dots, n-1$$

and $0 \leq 2a_n \leq a_{n-1}$. Then, for $x \in (0, \pi)$,

$$L_n(x) = \sum_{j=1}^{n} a_j \sin(jx) - \sum_{j=1}^{n-2} a_{j+2} \sin(jx) \geq 0. \tag{2.3}$$

Proof. Let $x \in (0, \pi)$. We define

$$\widetilde{c}_j = a_j - a_{j+2}, \quad j = 1, \dots, n-2,$$
$$\widetilde{c}_{n-1} = a_{n-1}, \quad \widetilde{c}_n = a_n.$$

Then, we have

$$L_n(x) = \sum_{j=1}^{n} \widetilde{c}_j \sin(jx). \tag{2.4}$$

Let

$$\widetilde{\gamma}_k = a_k - 2a_{k+1} + a_{k+2}, \quad k = 1, \dots, n-2,$$
$$\widetilde{\gamma}_{n-1} = a_{n-1} - 2a_n, \quad \widetilde{\gamma}_n = a_n.$$

By assumption,

$$\widetilde{\gamma}_k \geq 0, \quad k = 1, \dots, n. \tag{2.5}$$

We have

$$\widetilde{\gamma}_k = \widetilde{c}_k + 2 \sum_{j=1}^{n-k} (-1)^j \widetilde{c}_{j+k}, \quad k = 1, \dots, n,$$

so that Lemma 2.2 implies

$$\sum_{j=1}^{n} \widetilde{c}_j \sin(jx) = \sum_{j=1}^{n} \widetilde{\gamma}_j \phi_j(x). \tag{2.6}$$

Applying Lemma 2.1 and (2.5) reveals that the sum on the right-hand side of (2.6) is nonnegative. From (2.4) and (2.6), we obtain (2.3). $\square$

Remark 2.5. We assume that $L_n(x_0) = 0$ with $x_0 \in (0, \pi)$. The proof of Lemma 2.4 shows that

(i) if $n \geq 3$ and $a_n > 0$, then $\phi_n(x_0) = 0$.

Moreover, since $\phi_2(x_0) > 0$, we obtain that

(ii) if $n = 3$, then $a_2 - 2a_3 = 0$;
(iii) if $n \geq 4$, then $a_2 - 2a_3 + a_4 = 0$.

The next lemma plays an important role in the proofs of Theorems 3.1 and 3.2 given in the next section.

Lemma 2.6. *For all integers $n \geq 3$ and real numbers $a \geq 1$, $x \in (0, \pi)$, we have*

$$S_{n,a}(x) \geq S_{n-2,a}(x). \tag{2.7}$$

Proof. Let $a \geq 1$. We define

$$\widetilde{a}_j = \binom{n+a-j}{n-j}, \quad j = 1, \ldots, n.$$

Then,

$$\widetilde{a}_{j-1} - 2\widetilde{a}_j + \widetilde{a}_{j+1} = \frac{a(a-1)}{(n-j+1)!} \prod_{\nu=1}^{n-j-1} (n+a-j-\nu) \geq 0,$$

$j = 2, \ldots, n-1$,

$$\widetilde{a}_{n-1} - 2\widetilde{a}_n = a - 1, \quad \widetilde{a}_n = 1.$$

Since

$$S_{n,a}(x) - S_{n-2,a}(x) = \sum_{j=1}^{n} \widetilde{a}_j \sin(jx) - \sum_{j=1}^{n-2} \widetilde{a}_{j+2} \sin(jx),$$

we conclude from Lemma 2.4 that (2.7) holds for $x \in (0, \pi)$. $\square$

Remark 2.7. Let $n \geq 3$, $a \geq 1$, $\widetilde{x}_0 \in (0, \pi)$ and $S_{n,a}(\widetilde{x}_0) = S_{n-2,a}(\widetilde{x}_0)$. Remark 2.5 implies that $a = 1$ and $\phi_n(\widetilde{x}_0) = 0$.

3. Trigonometric sums. Our first two theorems show that Turán's inequality (1.1) can be refined if we assume that either n is odd or n is even.

Theorem 3.1. *For all odd integers $n \geq 1$ and real numbers $a \geq 1$, $x \in (0, \pi)$, we have*

$$S_{n,a}(x) \geq \sin(x). \tag{3.1}$$

Equality holds if and only if $n = 1$ or $n = 3$, $a = 1$, $x = 2\pi/3$.

Proof. Let $n \geq 3$ be odd, $a \geq 1$ and $x \in (0, \pi)$. Applying Lemma 2.6 gives

$$S_{n,a}(x) \geq S_{1,a}(x) = \sin(x).$$

Now, we discuss the cases of equality. A short calculation yields that

$$S_{3,1}(2\pi/3) = \sin(2\pi/3) = \frac{1}{2}\sqrt{3}.$$

We assume that

$$S_{n,a}(x) = \sin(x) = S_{1,a}(x).$$

Case 1. $n = 3$. Applying Remark 2.7 leads to $a = 1$ and

$$\phi_3(x) = \sin(x)(1 + 2\cos(x))^2 = 0.$$

This gives $x = 2\pi/3$.

Case 2. $n \geq 5$. From Lemma 2.6, we conclude that

$$S_{1,a}(x) = S_{n,a}(x) \geq S_{5,a}(x) \geq S_{3,a}(x) \geq S_{1,a}(x).$$

Thus,

$$S_{3,a}(x) = S_{1,a}(x) \quad \text{and} \quad S_{5,a}(x) = S_{3,a}(x).$$

Applying Remark 2.7 yields

$$\phi_3(x) = 0 \quad \text{and} \quad \phi_5(x) = 0.$$

The first equation yields $x = 2\pi/3$. However, $\phi_5(2\pi/3) = \sqrt{3}/2$, a contradiction. It follows that equality holds in (3.1) if and only if $n = 1$ or $n = 3$, $a = 1$, $x = 2\pi/3$. $\square$

Theorem 3.2. *For all even integers $n \geq 1$ and real numbers $a \geq 1$, $x \in (0, \pi)$, we have*

$$S_{n,a}(x) \geq 2\sin(x)(1+\cos(x)). \tag{3.2}$$

Equality holds if and only if $n = 2$, $a = 1$ or $n = 4$, $a = 1$, $x = \pi/2$.

Proof. Let $a \geq 1$ and $x \in (0, \pi)$. Using (2.7) gives, for even n,

$$S_{n,a}(x) \geq S_{2,a}(x) = (1+a)\sin(x) + \sin(2x) \geq 2\sin(x)(1+\cos(x)). \tag{3.3}$$

We have

$$S_{2,1}(x) = 2\sin(x)(1+\cos(x))$$

and

$$S_{4,1}(\pi/2) = 2\sin(\pi/2)(1+\cos(\pi/2)) = 2.$$

Next, we assume that

$$S_{n,a}(x) = 2\sin(x)(1+\cos(x)). \tag{3.4}$$

Case 1. $n = 2$. We obtain

$$0 = S_{2,a}(x) - 2\sin(x)(1+\cos(x)) = (a-1)\sin(x).$$

Thus, $a = 1$.

Case 2. $n = 4$. Applying (3.3) and (3.4) yields

$$S_{4,a}(x) = S_{2,1}(x).$$

It follows from Remark 2.7 that $a = 1$ and

$$\phi_4(x) = 8\sin(x)(1+\cos(x))\cos^2(x) = 0.$$

Thus, $x = \pi/2$.

Case 3. $n \geq 6$. From (2.7), (3.3) and (3.4) we conclude that

$$S_{4,a}(x) = S_{2,a}(x) \quad \text{and} \quad S_{6,a}(x) = S_{4,a}(x),$$

so that Remark 2.7 gives

$$\phi_4(x) = 0 \quad \text{and} \quad \phi_6(x) = 0.$$

The first equation yields $x = \pi/2$, but $\phi_6(\pi/2) = 2$. Hence, equality holds in (3.2) if and only if $n = 2$, $a = 1$ or $n = 4$, $a = 1$, $x = \pi/2$. $\square$

Remark 3.3. From Theorems 3.1 and 3.2, we conclude that the estimate

$$S_{n,a}(x) > \sin(x) \min\{1, 2\,(1 + \cos(x))\} \tag{3.5}$$

is valid for $n \geq 1$, $a \geq 1$ and $x \in (0, \pi)$. The lower bounds given in (1.3) and (3.5) cannot be compared. Indeed, the function

$$x \longmapsto \frac{x(\pi - x)}{\pi} - \sin(x) \min\{1, 2\,(1 + \cos(x))\}$$

is negative on $(0, x_1)$ and positive on (x_1, π), where $x_1 = 2.204\ldots$.

The following extension of inequality (1.1) is valid.

Theorem 3.4. *Let a be a real number. The inequality*

$$S_{n,a}(x) > 0 \tag{3.6}$$

holds for all integers $n \geq 1$ and real numbers $x \in (0, \pi)$ if and only if $a \geq 1$.

Proof. From (3.5), we conclude that, if $a \geq 1$, then (3.6) holds for all $n \geq 1$ and $x \in (0, \pi)$. Conversely, let (3.6) be valid for all $n \geq 1$ and $x \in (0, \pi)$. Since $S_{2,a}(\pi) = 0$, we obtain

$$\frac{d}{dx} S_{2,a}(x) \Big|_{x=\pi} = 1 - a \leq 0.$$

Thus, $a \geq 1$. $\square$

Next, we present inequalities for the sine polynomial

$$S^*_{n,a}(x) = \sum_{\substack{j=1 \\ j \text{ odd}}}^{n} \binom{n + a - j}{n - j} \sin(jx).$$

Applications of Theorems 3.1 and 3.2 lead to counterparts of (3.1) and (3.2).

Theorem 3.5. *For all odd integers $n \geq 1$ and real numbers $a \geq 1$, $x \in (0, \pi)$, we have*

$$S^*_{n,a}(x) \geq \sin(x). \tag{3.7}$$

Equality holds if and only if $n = 1$.

Proof. Let $n \geq 1$ be odd and $a \geq 1$, $x \in (0, \pi)$. Inequality (3.1) leads to

$$2S^*_{n,a}(x) = S_{n,a}(x) + S_{n,a}(\pi - x) \geq \sin(x) + \sin(\pi - x) = 2\sin(x).$$

If equality holds in (3.7), then

$$S_{n,a}(x) = \sin(x) \quad \text{and} \quad S_{n,a}(\pi - x) = \sin(\pi - x).$$

From Theorem 3.1, we conclude that $n = 1$. □

Theorem 3.6. *For all even integers $n \geq 2$ and real numbers $a \geq 1$, $x \in (0, \pi)$, we have*

$$S^*_{n,a}(x) \geq 2\sin(x). \tag{3.8}$$

Equality holds if and only if $n = 2$, $a = 1$ or $n = 4$, $a = 1$, $x = \pi/2$.

Proof. Let $n \geq 2$ be even and $a \geq 1$, $x \in (0, \pi)$. We apply (3.2) and obtain

$$\begin{aligned} 2S^*_{n,a}(x) &= S_{n,a}(x) + S_{n,a}(\pi - x) \\ &\geq 2\sin(x)(1 + \cos(x)) + 2\sin(\pi - x)(1 + \cos(\pi - x)) \\ &= 4\sin(x). \end{aligned}$$

If $n = 2$, $a = 1$ or $n = 4$, $a = 1$, $x = \pi/2$, then equality is valid in (3.8). Conversely, if equality holds in (3.8), then

$$S_{n,a}(x) = 2\sin(x)(1 + \cos(x))$$

and

$$S_{n,a}(\pi - x) = 2\sin(\pi - x)(1 + \cos(\pi - x)).$$

Applying Theorem 3.2 leads to $n = 2$, $a = 1$ or $n = 4$, $a = 1$, $x = \pi/2$. □

Now, we study the sine sum in two variables given in (1.4). The following two theorems offer improvements of inequality (1.2).

Theorem 3.7. *For all odd integers $n \geq 1$ and real numbers $a \geq 1$, $x, y \in (0, \pi)$, we have*

$$\Theta_{n,a}(x, y) \geq \sin(x)\sin(y). \tag{3.9}$$

Equality holds if and only if $n = 1$.

Proof. Let $n \geq 1$ be odd and $a \geq 1$. Since equality holds in (3.9), if $n = 1$, we suppose that $n \geq 3$. Applying Theorem 3.1 gives for $x, y \in \mathbf{R}$ with $0 < x - y < \pi$ and $0 < x + y < \pi$:

$$S_{n,a}(x - y) + S_{n,a}(x + y) \geq \sin(x - y) + \sin(x + y).$$

This leads to

$$\sum_{j=1}^{n} \binom{n + a - j}{n - j} \sin(jx)\cos(jy) \geq \sin(x)\cos(y). \tag{3.10}$$

Equality holds in (3.10) if and only if $n = 3$, $a = 1$ and $x - y = 2\pi/3$, $x + y = 2\pi/3$, that is, $x = 2\pi/3$, $y = 0$. In order to obtain (3.9) we integrate both sides of (3.10) with respect to y.

Let $x_0, y_0 \in \mathbf{R}$ with $0 < y_0 \leq x_0 < \pi$. We consider two cases.

Case 1. $x_0 \leq \pi/2$. Let $0 < y < y_0$. Then, $0 < x_0 - y < x_0 + y < \pi$. It follows from (3.10) with $>$ instead of $\geq$ that

$$\int_0^{y_0} \sum_{j=1}^{n} \binom{n + a - j}{n - j} \sin(jx_0)\cos(jy)\,dy > \int_0^{y_0} \sin(x_0)\cos(y)\,dy. \tag{3.11}$$

This leads to (3.9) with $x = x_0$, $y = y_0$ and $>$ instead of $\geq$.

Case 2. $\pi/2 < x_0$.

Case 2.1. $y_0 \leq \pi - x_0$. Let $0 < y < y_0$. Then, $0 < x_0 - y < x_0 + y < \pi$, so that we obtain (3.11) and (3.9) with $x = x_0$, $y = y_0$ and $>$ instead of $\geq$.

Case 2.2. $\pi - x_0 < y_0$. We set $x_1 = \pi - x_0$ and $y_1 = \pi - y_0$. Let $0 < y < x_1$. Then, $0 < y_1 - y < y_1 + y < \pi$. This leads to (3.11) with

x_1 instead of y_0 and y_1 instead of x_0. Using

$$\sin(jy_1) = (-1)^{j-1}\sin(jy_0), \quad j \in \mathbf{N},$$

and

$$\sin(jx_1) = (-1)^{j-1}\sin(jx_0), \quad j \in \mathbf{N},$$

we obtain (3.9) with $x = x_0$, $y = y_0$ and $>$ instead of $\geq$. $\square$

Theorem 3.8. *For all even integers $n \geq 2$ and real numbers $a \geq 1$, $x, y \in (0, \pi)$, we have*

$$\Theta_{n,a}(x,y) \geq 2\sin(x)\sin(y)(1+\cos(x)\cos(y)). \tag{3.12}$$

Equality holds if and only if $n = 2$, $a = 1$.

Proof. The proof is similar to that of Theorem 3.7. Therefore, we only offer a proof sketch. Since

$$\Theta_{2,a}(x,y) = 2\sin(x)\sin(y)\left(\frac{a+1}{2} + \cos(x)\cos(y)\right),$$

we conclude that, if $n = 2$, then (3.12) is valid with equality if and only if $a = 1$. Let $n \geq 4$. Using Theorem 3.2, we obtain for $x, y \in \mathbf{R}$ with $0 < x - y < \pi$ and $0 < x + y < \pi$:

$$\sum_{j=1}^{n} \binom{n+a-j}{n-j} \sin(jx)\cos(jy) \geq 2\sin(x)\cos(y) + \sin(2x)\cos(2y). \tag{3.13}$$

Equality holds in (3.13) if and only if $n = 4$, $a = 1$, $x = \pi/2$, $y = 0$.

Let $x_0, y_0 \in \mathbf{R}$ with $0 < y_0 \leq x_0 < \pi$. We assume that $x_0 \leq \pi/2$. If $0 < y < y_0$, then $0 < x_0 - y < x_0 - y < \pi$. Next, we integrate both sides of (3.13) (with $x = x_0$ and $>$ instead of $\geq$) from $y = 0$ to $y = y_0$. This gives

$$\Theta_{n,a}(x_0,y_0) > 2\sin(x_0)\sin(y_0)(1+\cos(x_0)\cos(y_0)). \tag{3.14}$$

By similar arguments, we conclude that (3.14) also holds if $x_0 > \pi/2$. $\square$

Let

$$\Theta^*_{n,a}(x,y) = \sum_{\substack{j=1 \\ j \text{ odd}}}^{n} \binom{n+a-j}{n-j} \frac{\sin(jx)\sin(jy)}{j}.$$

Applying Theorems 3.7 and 3.8 we obtain the following companions of (3.7) and (3.8).

Theorem 3.9. *For all real numbers $a \ge 1$ and $x, y \in (0,\pi)$, we have*

$$\Theta^*_{n,a}(x,y) \ge \sin(x)\sin(y), \quad \text{if } n \text{ is odd} \tag{3.15}$$

and

$$\Theta^*_{n,a}(x,y) \ge 2\sin(x)\sin(y), \quad \text{if } n \text{ is even.} \tag{3.16}$$

Equality holds in (3.15) *if and only if $n = 1$ and in* (3.16) *if and only if $n = 2$, $a = 1$.*

The proof of this theorem is quite similar to the proofs of Theorems 3.5 and 3.6; therefore, we omit the details. We conclude this section with a generalization of inequality (1.2).

Theorem 3.10. *Let a be a real number. The inequality*

$$\Theta_{n,a}(x,y) > 0 \tag{3.17}$$

holds for all integers $n \ge 1$ and real numbers $x, y \in (0,\pi)$ if and only if $a \ge 1$.

Proof. Let $a \ge 1$. From Theorems 3.7 and 3.8 we conclude that $\Theta_{n,a}(x,y)$ is positive for all $n \ge 1$ and $x, y \in (0,\pi)$. Conversely, if (3.17) holds for all $n \ge 1$ and $x, y \in (0,\pi)$, then we have

$$\Theta_{2,a}(0,y) = 0$$

and

$$\frac{\partial}{\partial x}\Theta_{2,a}(x,y)\Big|_{x=0} = \sin(y)(1+a+2\cos(y)) \ge 0.$$

It follows that

$$1 + a + 2\cos(y) \ge 0.$$

We let $y \to \pi$ and obtain $a - 1 \ge 0$. $\square$

4. Chebyshev polynomials. The Chebyshev polynomials of the first and second kind, $T_n(x)$ and $U_n(x)$, $n = 0, 1, 2, \dots$, are polynomials in x of degree n, defined by

$$T_n(x) = \cos(nt)$$

and

$$U_n(x) = \frac{\sin((n+1)t)}{\sin(t)}, \quad x = \cos(t), \ t \in [0, \pi]. \tag{4.1}$$

They have remarkable applications in numerical analysis, approximation theory and other branches of mathematics. The main properties of these functions may be found, for instance, in Mason and Handscomb [**8**].

Using the notation (4.1), we obtain from (1.1) the inequality

$$\Lambda_{n,k}(x) = \sum_{j=0}^{n} \binom{n+k-j}{n-j} U_j(x) > 0, \tag{4.2}$$
$$0 \le n \in \mathbf{Z}, \quad k \in \mathbf{N}, \quad -1 < x < 1.$$

The results presented in Section 3 lead to inequalities for Chebyshev polynomials of the second kind. As examples, we offer counterparts of Theorems 3.1 and 3.2 which provide refinements of (4.2).

Theorem 4.1. *For all even integers $n \ge 0$ and real numbers $a \ge 1$, $x \in (-1, 1)$, we have*

$$\Lambda_{n,a}(x) \ge 1.$$

Equality holds if and only if $n = 0$ or $n = 2$, $a = 1$, $x = -1/2$.

Theorem 4.2. *For all odd integers $n \ge 1$ and real numbers $a \ge 1$, $x \in (-1, 1)$, we have*

$$\Lambda_{n,a}(x) \ge 2(1 + x).$$

Equality holds if and only if $n = 1$, $a = 1$ or $n = 3$, $a = 1$, $x = 0$.

The inequalities

$$0 < \sum_{j=1}^{n} \frac{\sin(jx)}{j} < \pi - x, \quad n \in \mathbf{N}, \ 0 < x < \pi, \tag{4.3}$$

are classical results in the theory of trigonometric polynomials. In 1910, Fejér conjectured the validity of the left-hand side. The first proofs were published by Jackson [**7**] in 1911 and Gronwall [**6**] in 1912. The right-hand side is due to Turán [**13**], who proved his inequality in 1938. The following interesting companion of (4.3) was given by Carslaw [**4**] in 1917:

$$0 < \sum_{j=0}^{n} \frac{\sin((2j+1)x)}{2j+1} \leq 1, \quad 0 \leq n \in \mathbf{Z},\ 0 < x < \pi. \tag{4.4}$$

Equality holds if and only if $n = 0$, $x = \pi/2$ (also see [**2**]). We show that applications of (4.3) and (4.4) lead to integral inequalities for the Chebyshev polynomials.

Theorem 4.3. *For all integers $n \geq 1$ and real numbers $x \in (0,1)$, we have*

$$\arccos(x) < \int_x^1 \frac{U_{2n}(t)}{\sqrt{1-t^2}}\, dt < \pi - \arccos(x). \tag{4.5}$$

If $x \in (-1,0)$, then (4.5) holds with $>$ instead of $<$.

Proof. Let $t \in (0,\pi)$ and

$$F_n(t) = \sum_{j=1}^{n} \frac{\sin(2jt)}{j}.$$

Then,

$$F_n'(t) = 2\sum_{j=1}^{n} \cos(2jt) = \frac{\sin((2n+1)t)}{\sin(t)} - 1 = U_{2n}(\cos(t)) - 1.$$

This gives, for $z \in (0,\pi)$:

$$F_n(z) = \int_0^z F_n'(t)\, dt = \int_0^z U_{2n}(\cos(t))\, dt - z = \int_{\cos(z)}^1 \frac{U_{2n}(t)}{\sqrt{1-t^2}}\, dt - z.$$

If $x \in (0,1)$, then $\arccos(x) \in (0,\pi/2)$. Using (4.3) leads to

$$0 < F_n(\arccos(x)) = \int_x^1 \frac{U_{2n}(t)}{\sqrt{1-t^2}}\, dt - \arccos(x) < \pi - 2\arccos(x).$$

This implies (4.5).

We define for $x \in (-1,1)$:

$$G_n(x) = \arccos(x) - \int_x^1 \frac{U_{2n}(t)}{\sqrt{1-t^2}}\,dt \quad \text{and} \quad w(x) = \arccos(x) - \frac{\pi}{2}.$$

Applying

$$w(x) + w(-x) = 0 \tag{4.6}$$

and

$$\int_0^1 \frac{U_{2n}(t)}{\sqrt{1-t^2}}\,dt = \frac{\pi}{2}$$

yields

$$G_n(x)+G_n(-x)=\left(2\int_0^1-\int_x^1-\int_{-x}^1\right)\frac{U_{2n}(t)}{\sqrt{1-t^2}}\,dt = \left(\int_0^x-\int_{-x}^0\right)\frac{U_{2n}(t)}{\sqrt{1-t^2}}\,dt.$$

Since U_{2n} is an even function, we obtain

$$\int_0^x \frac{U_{2n}(t)}{\sqrt{1-t^2}}\,dt = \int_0^x \frac{U_{2n}(-t)}{\sqrt{1-t^2}}\,dt = \int_{-x}^0 \frac{U_{2n}(t)}{\sqrt{1-t^2}}\,dt.$$

Thus,

$$G_n(x) + G_n(-x) = 0. \tag{4.7}$$

Using (4.6) and (4.7) gives that the function

$$H_n(x) = G_n(x) - 2w(x)$$

satisfies

$$H_n(x) + H_n(-x) = 0. \tag{4.8}$$

From (4.5), (4.7) and (4.8), we conclude that, for $x \in (-1,0)$, we have

$$-G_n(x) = G_n(-x) < 0 < H_n(-x) = -H_n(x).$$

This leads to (4.5) with $>$ instead of $<$. □

Remark 4.4. The functions

$$x \longmapsto \int_x^1 \frac{U_n(t)}{\sqrt{1-t^2}}\,dt - \arccos(x), \quad n = 3, 7, 11,$$

and

$$x \longmapsto \pi - \arccos(x) - \int_x^1 \frac{U_n(t)}{\sqrt{1-t^2}}\,dt, \quad n = 1, 5, 9,$$

attain positive and negative values on $(0,1)$. This implies that in general Theorem 4.3 is not true for Chebyshev polynomials of odd degree.

We conclude this paper with a theorem which offers sharp upper and lower bounds for an integral involving the Chebyshev polynomials of the first and second kind.

Theorem 4.5. *For all integers $n \geq 0$ and real numbers $x \in (-1,1)$, we have*

$$0 < \int_x^1 \frac{T_{n+1}(t)\,U_n(t)}{\sqrt{1-t^2}}\,dt \leq 1. \tag{4.9}$$

Equality holds on the right-hand side if and only if $n = 0$, $x = 0$.

Proof. Let $x \in (-1,1)$. Since $T_1(t) = t$ and $U_0(t) = 1$, we obtain

$$\int_x^1 \frac{T_1(t)U_0(t)}{\sqrt{1-t^2}}\,dt = \sqrt{1-x^2}.$$

This leads to (4.9) with $n = 0$.

Let $n \geq 1$. We denote the sine sum in (4.4) by $I_n(x)$. Then, for $t \in (0,\pi)$,

$$\begin{aligned} I_n'(t) &= \sum_{j=0}^{n} \cos((2j+1)t) = \frac{\cos((n+1)t)\sin((n+1)t)}{\sin(t)} \\ &= T_{n+1}(\cos(t))\,U_n(\cos(t)). \end{aligned}$$

Hence, we obtain for $z \in (0,\pi)$:

$$I_n(z) = \int_0^z I_n'(t)\,dt = \int_{\cos(z)}^1 \frac{T_{n+1}(t)\,U_n(t)}{\sqrt{1-t^2}}\,dt,$$

so that (4.4) (with $<$ instead of $\leq$) gives

$$0 < I_n(\arccos(x)) = \int_x^1 \frac{T_{n+1}(t)\,U_n(t)}{\sqrt{1-t^2}}\,dt < 1. \quad \square$$

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