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ASYMPTOTICS OF THE EIGENVALUES OF SELF-ADJOINT FOURTH ORDER DIFFERENTIAL OPERATORS WITH SEPARATED EIGENVALUE PARAMETER DEPENDENT BOUNDARY CONDITIONS

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ABSTRACT. In this paper, an eigenvalue problem for a regular fourth order ordinary differential equation is considered, where one of the boundary conditions linearly depends upon the eigenvalue parameter. The first four terms in the asymptotic expansion of the eigenvalues are derived.

1. Introduction. Separation of variables for linear partial differential equations leads to ordinary differential equations with a spectral parameter. For standard problems, this gives the well known Sturm-Liouville problems, which have an operator realization $A - zI$ with a self-adjoint operator A, where $z = \lambda^2$ and λ is the frequency parameter in the separation of variables. However, if problems which possess derivatives with respect to time in the boundary conditions are also considered, then the operator realization may be of the form

$$L(\lambda)y = \lambda^2 My - i\alpha\lambda Ky - Ay. \tag{1.1}$$

The generalized Regge and vibrating beam problems investigated in **[3, 8]** have operator representations of the form (1.1), where M, K and A are self-adjoint, $M \geq 0$, $K \geq 0$ are bounded, $M + K \gg 0$, A is bounded below with compact resolvent and α is a positive constant.

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The generalized Regge problem is realized by a second order differential operator, whereas the problem considered in **[3]** leads to a boundary eigenvalue problem of the form

$$y^{(4)} - (gy')' + hy = \lambda^2 y, \tag{1.2a}$$

$$B_j(\lambda)y = 0, \quad j = 1, 2, 3, 4, \tag{1.2b}$$

where $g \in C^1[0, a]$ and $h \in C[0, a]$ are real valued functions, $a > 0$, and (1.2b) are separated boundary conditions. Boundary conditions (1.2b) are taken at endpoint 0 for $j = 1, 2$ and at endpoint a for $j = 3, 4$, and operators B_j are constant or linearly depend upon λ.

One particular set of boundary conditions has been considered in **[3]**. The question arises, for which sets of boundary conditions does the problem (1.2) have a representation (1.1) as a pencil with self-adjoint operators? In **[4]**, necessary and sufficient conditions have been obtained for a general class of boundary conditions. For some subclasses of such boundary conditions which lead to self-adjoint operators in the operator pencil (1.1), we have derived asymptotic expansions of the eigenvalues in **[5, 6, 7]**.

In this paper, we extend our previous work to a class of boundary conditions where only one boundary condition depends upon the eigenvalue parameter. In Section 2, we construct the operator pencil, and we introduce the class of boundary conditions to be considered. In Section 3, we investigate the eigenvalues for the case $g = h = 0$. In Section 4, we prove that the boundary value problem under investigation is Birkhoff regular. In Section 5, we derive the first four terms of the eigenvalue asymptotics, and we compare them to those obtained in our previous publications. We will also make some simple observations regarding the inverse problem, that is, the question of which of the parameters of the given class of problems can be recovered from their spectra.

2. The quadratic operator pencil L. We recall that the quasi-derivatives associated with (1.2a) are given by

$$y^{[0]} = y, \qquad y^{[1]} = y', \qquad y^{[2]} = y'', \qquad y^{[3]} = y^{(3)} - gy',$$
$$y^{[4]} = y^{(4)} - (gy')' + hy,$$

see **[9,** page 26]. In this paper, we will consider a class of boundary conditions (1.2b) determined by the following properties: $B_j(\lambda)y = y^{[p_j]}(a_j)$ for $j \in \{1,2,3\}$, while $B_4(\lambda)y = y^{[p_4]}(a_4) + i\alpha\lambda y^{[q_4]}(a_4)$, where $a_j = 0$ for $j = 1,2$, and $a_j = a$ for $j = 3,4$, $\alpha > 0$. In order to have independent boundary conditions, we will assume that the numbers p_1, p_2, as well as p_3, p_4, q_4, are mutually disjoint and $\{p_4, q_4\} = \{1,2\}$.

Recall that, in applications, using separation of variables, the parameter λ emanates from derivatives with respect to the time variable in the partial differential equation and its boundary conditions, and it is reasonable that the highest space derivative occurs in a term without time derivative. Thus, the most relevant λ-dependent boundary condition would have $q_4 < p_4$ such that $q_4 = 1$ and $p_4 = 2$. Further assumptions on p_1, p_2, p_3, p_4 and q_4 will be made later and will be justified by the requirement that the operators in the associated operator pencil are self-adjoint.

Recall that the Sobolev space $W_4^2(0,a)$ is the set of (equivalence classes of) functions $y \in L_2(0,a)$ such that, for $j = 1,2,3,4$, the weak derivatives $y^{(j)}$ belong to $L_2(0,a)$.

We denote by U the collection of the boundary conditions (1.2b) and consider the linear operators $A(U)$, K and M in the space $L_2(0,a) \oplus \mathbb{C}$ with domains:

$$\mathscr{D}(A(U)) = \left\{ \widetilde{y} = \begin{pmatrix} y \\ y'(a) \end{pmatrix} : y \in W_4^2(0,a), y^{[p_j]}(a_j) = 0 \text{ for } j = 1,2,3 \right\}$$

$$\mathscr{D}(K) = \mathscr{D}(M) = L_2(0,a) \oplus \mathbb{C},$$

given by

$$(A(U))\widetilde{y} = \begin{pmatrix} y^{[4]} \\ y''(a) \end{pmatrix} \quad \text{for } \widetilde{y} \in \mathscr{D}(A(U)),$$

$$K = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

It is easy to check that $K \geq 0$, $M \geq 0$, $M + K = I$ and $M|_{\mathscr{D}(A(U))} > 0$. We associate a quadratic operator pencil

$$L(\lambda, \alpha) = \lambda^2 M - i\alpha\lambda K - A(U), \quad \lambda \in \mathbb{C}, \tag{2.1}$$

in the space $L_2(0,a) \oplus \mathbb{C}$ with the problem (1.2). We observe that (2.1) is an operator representation of eigenvalue problem (1.2) in the sense that a function y satisfies (1.2) if and only if $L(\lambda,\alpha)\widetilde{y}=0$ with $\widetilde{y}=(y,y'(a))$.

The conditions under which the differential operator $A(U)$ is self-adjoint are given in the next theorem.

Theorem 2.1 ([**4,** Theorem 1.2])**.** *Denote by P_0 the set of p in $y^{[p]}(0)=0$ for the λ-independent boundary conditions and by P_a the corresponding set for $y^{[p]}(a)=0$. Then, the differential operator $A(U)$ associated with this boundary value problem is self-adjoint if and only if $p+q=3$ for all boundary conditions of the form $y^{[p]}(a_j)+i\alpha\varepsilon_j\lambda y^{[q]}(a_j)=0$ and $\varepsilon_j=1$ if q is even in the case $a_j=0$ or odd in the case $a_j=a$. Otherwise, $\varepsilon_j=-1$, $\{0,3\}\not\subset P_0$, $\{1,2\}\not\subset P_0$, $\{0,3\}\not\subset P_a$ and $\{1,2\}\not\subset P_a$.*

Here, we observe that $h=0$ in [**4,** Theorem 1.2]. However, it is easy to see that [**4,** Theorem 1.2] also holds in the case $h\neq 0$ since the proof can easily be extended to that case as $(hy,z)=(y,hz)$ for all $\widetilde{y}$, $\widetilde{z}\in\mathscr{D}(A(U))$. Alternatively, we observe that the multiplication operator h is bounded and self-adjoint on $L_2(0,a)$ so that we may appeal to [**1,** Theorem V.4.3].

Proposition 2.2. *The operator pencil $L(\cdot,\alpha)$ is a Fredholm-valued operator function with index 0. The spectrum of the Fredholm operator $L(\cdot,\alpha)$ consists of discrete eigenvalues of finite multiplicities, and all eigenvalues of $L(\cdot,\alpha)$, $\alpha\geq 0$, lie in the closed upper half-plane and on the imaginary axis and are symmetric with respect to the imaginary axis.*

Proof. As in [**3,** Section 3], we can argue that, for all $\lambda\in\mathbb{C}$, $L(\lambda,\alpha)$ is a relatively compact perturbation of $L(0,0)$, where $L(0,0)$ is well known as a Fredholm operator. The statement on the location of the spectrum now follows as in [**3,** Lemma 3.1]. $\square$

It follows from Theorem 2.1 that we have eight different cases of boundary conditions $B_j(\lambda)y=0$. These boundary conditions are determined by the values of p_1, p_2 and p_3. Hence, we will consider

Case 1: $p_1 = 0$, $p_2 = 1$, $p_3 = 0$;

Case 2: $p_1 = 0$, $p_2 = 1$, $p_3 = 3$;

Case 3: $p_1 = 0$, $p_2 = 2$, $p_3 = 0$;

Case 4: $p_1 = 0$, $p_2 = 2$, $p_3 = 3$;

Case 5: $p_1 = 1$, $p_2 = 3$, $p_3 = 0$;

Case 6: $p_1 = 1$, $p_2 = 3$, $p_3 = 3$;

Case 7: $p_1 = 2$, $p_2 = 3$, $p_3 = 0$;

Case 8: $p_1 = 2$, $p_2 = 3$, $p_3 = 3$.

The corresponding boundary operators are then

$$B_1 y = y(0) \quad \text{and} \quad B_2 y = y'(0) \qquad \text{(Cases 1 and 2)}, \tag{2.2}$$

$$B_1 y = y(0) \quad \text{and} \quad B_2 y = y''(0) \qquad \text{(Cases 3 and 4)}, \tag{2.3}$$

$$B_1 y = y'(0) \quad \text{and} \quad B_2 y = y^{[3]}(0) \qquad \text{(Cases 5 and 6)}, \tag{2.4}$$

$$B_1 y = y''(0) \quad \text{and} \quad B_2 y = y^{[3]}(0) \qquad \text{(Cases 7 and 8)}, \tag{2.5}$$

$$B_3 y = y(a) \qquad \text{(Cases 1, 3, 5, 7)}, \tag{2.6}$$

$$B_3 y = y^{[3]}(a) \qquad \text{(Cases 2, 4, 6, 8)}, \tag{2.7}$$

$$B_4(\lambda) y = y''(a) + i\alpha\lambda y'(a). \tag{2.8}$$

3. Asymptotics of eigenvalues for $g = 0$ and $h = 0$. In this section, we consider the boundary value problem (1.2) with $g = h = 0$. We count all eigenvalues with their proper multiplicities and develop a formula for the asymptotic distribution of the eigenvalues, which is used to obtain the corresponding formula for general g. Observe that, for $g = h = 0$, the quasi-derivatives $y^{[j]}$ coincide with the standard derivatives $y^{(j)}$. We take the canonical fundamental system $y_j(\cdot, \lambda)$, $j = 1, \ldots, 4$, of (1.2a) with $y_j^{(m)}(0, \lambda) = \delta_{j,m+1}$ for $m = 0, \ldots, 3$. It is well known that the functions $y_j(\cdot, \lambda)$ are analytic on $\mathbb{C}$ with respect to λ. Setting

$$M(\lambda) = (B_i(\lambda) y_j(\cdot, \lambda))_{i,j=1}^4,$$

the eigenvalues of the boundary value problem (1.2) are the eigenvalues of the analytic matrix function M, where the corresponding geometric and algebraic multiplicities coincide, see [**2,** Theorem 6.3.2].

Setting $\lambda = \mu^2$ and

$$y(x,\mu) = \frac{1}{2\mu^3}\sinh(\mu x) - \frac{1}{2\mu^3}\sin(\mu x),$$

it is easy to see that

$$y_j(x,\lambda) =: \widetilde{y}_j(x,\mu) = y^{(4-j)}(x,\mu), \quad j = 1,\dots,4.$$

Since the first two rows of $M(\lambda)$ have exactly one entry 1 and all the other entries 0, an expansion of $M(\lambda)$ shows that $\det M(\lambda) = \pm\phi(\mu)$, where

$$\phi(\mu) = \det\begin{pmatrix} B_3\widetilde{y}_{\sigma(1)}(\cdot,\mu) & B_3\widetilde{y}_{\sigma(2)}(\cdot,\mu) \\ B_4(\mu^2)\widetilde{y}_{\sigma(1)}(\cdot,\mu) & B_4(\mu^2)\widetilde{y}_{\sigma(2)}(\cdot,\mu)\end{pmatrix},$$

with

$$(\sigma(1),\sigma(2)) = \begin{cases} (3,4) \text{ in Cases 1 and 2,} \\ (2,4) \text{ in Cases 3 and 4,} \\ (1,3) \text{ in Cases 5 and 6,} \\ (1,2) \text{ in Cases 7 and 8.}\end{cases}$$

In view of (2.6), (2.7) and (2.8), this gives

$$\begin{aligned}\phi(\mu) = i\alpha\mu^2&\Big(\widetilde{y}'_{\sigma(2)}(a,\mu)B_3\widetilde{y}_{\sigma(1)}(a,\mu) - \widetilde{y}'_{\sigma(1)}(a,\mu)B_3\widetilde{y}_{\sigma(2)}(a,\mu)\Big) \\ &+ \widetilde{y}''_{\sigma(2)}(a,\mu)B_3\widetilde{y}_{\sigma(1)}(a,\mu) - \widetilde{y}''_{\sigma(1)}(a,\mu)B_3\widetilde{y}_{\sigma(2)}(a,\mu).\end{aligned}$$

Each of the summands in ϕ is a product of a power in μ and a product of two sums of a trigonometric and a hyperbolic function. The term with the highest μ-power in $\phi(\mu)$ occurs with

$$i\alpha\mu^2\Big[\widetilde{y}'_{\sigma(2)}(a,\mu)B_3\widetilde{y}_{\sigma(1)}(a,\mu) - \widetilde{y}'_{\sigma(1)}(a,\mu)B_3\widetilde{y}_{\sigma(2)}(a,\mu)\Big].$$

Hence, we shall investigate the zeros of

$$\phi_0(\mu) = 2\mu^2\Big[\widetilde{y}'_{\sigma(2)}(a,\mu)B_3\widetilde{y}_{\sigma(1)}(a,\mu) - \widetilde{y}'_{\sigma(1)}(a,\mu)B_3\widetilde{y}_{\sigma(2)}(a,\mu)\Big].$$

In the above eight cases we obtain:

Case 1. $p_1 = 0$, $p_2 = 1$, $p_3 = 0$:

$$\begin{aligned}\phi_0(\mu) = \frac{1}{2\mu^2}&[(\cosh(\mu a) - \cos(\mu a))^2 \\ &- (\sinh(\mu a) + \sin(\mu a))(\sinh(\mu a) - \sin(\mu a))]\end{aligned}$$

$$= \frac{1}{\mu^2}[1 - \cos(\mu a)\cosh(\mu a)].$$

Case 2. $p_1 = 0$, $p_2 = 1$, $p_3 = 3$:

$$\begin{aligned}\phi_0(\mu) &= \frac{1}{2}\mu[(\sinh(\mu a) - \sin(\mu a))(\cosh(\mu a) - \cos(\mu a)) \\ &\qquad\qquad - (\sinh(\mu a) + \sin(\mu a))(\cosh(\mu a) + \cos(\mu a))] \\ &= -\mu[\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)].\end{aligned}$$

Case 3. $p_1 = 0$, $p_2 = 2$, $p_3 = 0$:

$$\begin{aligned}\phi_0(\mu) &= \frac{1}{2\mu}[(\sinh(\mu a) + \sin(\mu a))(\cosh(\mu a) - \cos(\mu a)) \\ &\qquad\qquad - (\sinh(\mu a) - \sin(\mu a))(\cosh(\mu a) + \cos(\mu a))] \\ &= \frac{1}{\mu}[\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)].\end{aligned}$$

Case 4. $p_1 = 0$, $p_2 = 2$, $p_3 = 3$:

$$\begin{aligned}\phi_0(\mu) &= \frac{1}{2}\mu^2[(\cosh(\mu a) - \cos(\mu a))^2 - (\cosh(\mu a) + \cos(\mu a))^2] \\ &= -2\mu^2\cos(\mu a)\cosh(\mu a).\end{aligned}$$

Case 5. $p_1 = 1$, $p_2 = 3$, $p_3 = 0$:

$$\begin{aligned}\phi_0(\mu) &= \frac{1}{2}\mu[(\sinh(\mu a) + \sin(\mu a))(\cosh(\mu a) + \cos(\mu a)) \\ &\qquad\qquad - (\sinh(\mu a) - \sin(\mu a))(\cosh(\mu a) - \cos(\mu a))] \\ &= \mu(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)).\end{aligned}$$

Case 6. $p_1 = 1$, $p_2 = 3$, $p_3 = 3$:

$$\begin{aligned}\phi_0(\mu) &= \frac{1}{2}\mu^4[(\sinh(\mu a) + \sin(\mu a))^2 - (\sinh(\mu a) - \sin(\mu a))^2] \\ &= 2\mu^4\sin(\mu a)\sinh(\mu a).\end{aligned}$$

Case 7. $p_1 = 2$, $p_2 = 3$, $p_3 = 0$:

$$\begin{aligned}\phi_0(\mu) &= \frac{1}{2}\mu^2[(\cosh(\mu a) + \cos(\mu a))^2 \\ &\qquad\qquad - (\sinh(\mu a) + \sin(\mu a))(\sinh(\mu a) - \sin(\mu a))]\end{aligned}$$

$$= \mu^2(\cos(\mu a)\cosh(\mu a) + 1).$$

Case 8. $p_1 = 2$, $p_2 = 3$, $p_3 = 3$:

$$\begin{aligned}\phi_0(\mu) &= \frac{1}{2}\mu^5[(\sinh(\mu a) + \sin(\mu a))(\cosh(\mu a) + \cos(\mu a)) \\ &\qquad - (\sinh(\mu a) - \sin(\mu a))(\cosh(\mu a) - \cos(\mu a))] \\ &= \mu^5(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)).\end{aligned}$$

Next, we give the asymptotic distributions of the zeros of $\phi_0(\mu)$ with their proper count.

Lemma 3.1.

Case 1. $p_1 = 0$, $p_2 = 1$, $p_3 = 0$: *ϕ_0 has a zero of multiplicity 2 at 0, exactly one simple zero in each interval $[2m(\pi/a), (2m + 1/2)(\pi/a)]$, $[(2m+3/2)(\pi/a), (2m+2)(\pi/a)]$, respectively, for nonnegative integer m with asymptotics*

$$\widetilde{\mu}_k = (2k - 1)\frac{\pi}{2a} + o(1), \quad k = 1, 2, \ldots,$$

simple zeros at $-\widetilde{\mu}_k$, $\widetilde{\mu}_{-k} = i\widetilde{\mu}_k$, $-i\widetilde{\mu}_k$, for $k = 1, 2, \ldots$, and no other zeros.

Case 2. $p_1 = 0$, $p_2 = 1$, $p_3 = 3$: *ϕ_0 has a zero of multiplicity 2 at 0 exactly one simple zero in each interval $((k-1/2)(\pi/a), (k+1/2)(\pi/a))$ for positive integer k with asymptotics*

$$\widetilde{\mu}_k = (4k - 1)\frac{\pi}{4a} + o(1), \quad k = 1, 2, \ldots,$$

simple zeros at $-\widetilde{\mu}_k$, $\widetilde{\mu}_{-k} = i\widetilde{\mu}_k$, $-i\widetilde{\mu}_k$, for $k = 1, 2, \ldots$, and no other zeros.

Case 3. $p_1 = 0$, $p_2 = 2$, $p_3 = 0$: *ϕ_0 has a zero of multiplicity 2 at 0, exactly one simple zero in each interval $((k-1/2)(\pi/a), (k+1/2)(\pi/a))$ for positive integer k with asymptotics*

$$\widetilde{\mu}_k = (4k + 1)\frac{\pi}{4a} + o(1), \quad k = 1, 2, \ldots,$$

simple zeros at $-\widetilde{\mu}_k$, $\widetilde{\mu}_{-k} = i\widetilde{\mu}_k$, $-i\widetilde{\mu}_k$, for $k = 1, 2, \ldots$, and no other zeros.

Case 4. $p_1 = 0$, $p_2 = 2$, $p_3 = 3$: *ϕ_0 has a zero of multiplicity 2 at 0, simple zeros*

$$\widetilde{\mu}_k = (2k-1)\frac{\pi}{2a}, \quad k = 1, 2, \ldots,$$

simple zeros at $-\widetilde{\mu}_k$, $\widetilde{\mu}_{-k} = i\widetilde{\mu}_k$, $-i\widetilde{\mu}_k$, for $k = 1, 2, \ldots$, and no other zeros.

Case 5. $p_1 = 1$, $p_2 = 3$, $p_3 = 0$: *ϕ_0 has a zero of multiplicity 2 at 0, exactly one simple zero in each interval $((k-1/2)(\pi/a), (k+1/2)(\pi/a))$ for positive integer k with asymptotics*

$$\widetilde{\mu}_k = (4k-1)\frac{\pi}{4a} + o(1), \quad k = 1, 2, \ldots,$$

simple zeros at $-\widetilde{\mu}_k$, $\widetilde{\mu}_{-k} = i\widetilde{\mu}_k$, $-i\widetilde{\mu}_k$, for $k = 1, 2, \ldots$, and no other zeros.

Case 6. $p_1 = 1$, $p_2 = 3$, $p_3 = 3$: *ϕ_0 has a zero of multiplicity 6 at 0, simple zeros*

$$\widetilde{\mu}_k = (k-1)\frac{\pi}{a}, \quad k = 2, 3, \ldots,$$

simple zeros at $-\widetilde{\mu}_k$, $\widetilde{\mu}_{-k} = i\widetilde{\mu}_k$, $-i\widetilde{\mu}_k$, for $k = 2, 3, \ldots$, and no other zeros.

Case 7. $p_1 = 2$, $p_2 = 3$, $p_3 = 0$: *ϕ_0 has a zero of multiplicity 2 at 0, exactly one simple zero in each interval $[(2k+1/2)(\pi/a), (2k+1)(\pi/a)]$ and $[(2k+1)(\pi/a), (2k+3/2)(\pi/a)]$, respectively, for nonnegative integer m with asymptotics*

$$\widetilde{\mu}_k = (2k-1)\frac{\pi}{2a} + o(1), \quad k = 1, 2, \ldots,$$

simple zeros at $-\widetilde{\mu}_k$, $\widetilde{\mu}_{-k} = i\widetilde{\mu}_k$, $-i\widetilde{\mu}_k$, for $k = 1, 2, \ldots$, and no other zeros.

Case 8. $p_1 = 2$, $p_2 = 3$, $p_3 = 3$: *ϕ_0 has a zero of multiplicity 6 at 0, exactly one simple zero in each interval $((k-1/2)(\pi/a), (k+1/2)(\pi/a))$ for positive integer k with asymptotics*

$$\widetilde{\mu}_k = (4k-5)\frac{\pi}{4a} + o(1), \quad k = 2, 3, \ldots,$$

simple zeros at $-\widetilde{\mu}_k$, $\widetilde{\mu}_{-k} = i\widetilde{\mu}_k$, $-i\widetilde{\mu}_k$, for $k = 2, 3, \ldots$, and no other zeros.

Proof. The result is obvious in Cases 4 and 6.

For Case 2, it is easy to see that ϕ_0 has a zero of multiplicity 2 at 0. Next we are going to find the zeros of ϕ_0 on the positive real axis. It is easily observed that, for $\mu \neq 0$, $\phi_0(\mu) = 0$ implies $\cosh(\mu a) \neq 0$ and $\cos(\mu a) \neq 0$, whence the positive zeros of ϕ_0 are those $\mu > 0$ for which $\tan(\mu a) + \tanh(\mu a) = 0$. Since $\tan'(\mu a) \geq 1$ and $\tanh'(\mu a) \geq 0$ for $\mu \in \mathbb{R}$, the function

$$\mu \longmapsto \tan(\mu a) + \tanh(\mu a)$$

is increasing with positive derivative on each interval

$$\left(\left(k - \frac{1}{2}\right)\frac{\pi}{a}, \left(k + \frac{1}{2}\right)\frac{\pi}{a}\right), \quad k \in \mathbb{Z}.$$

On each of these intervals, the function moves from $-\infty$ to ∞; thus, we have exactly one simple zero $\widetilde{\mu}_k$ of $\tan(\mu a) + \tanh(\mu a)$ in each interval

$$\left(\left(k - \frac{1}{2}\right)\frac{\pi}{a}, \left(k + \frac{1}{2}\right)\frac{\pi}{a}\right),$$

where k is a positive integer, and no zero in $(0, \pi/2a)$. Since $\tanh(\mu a) \to 1$ as $\mu \to \infty$, we have

$$\widetilde{\mu}_k = (4k-1)\frac{\pi}{4a} + o(1), \quad k = 1, 2, \ldots.$$

The location of the zeros on the other three half-axes follows by repeated application of $\phi_0(i\mu) = -\phi_0(\mu)$.

The proof will be complete if we show that all zeros of ϕ_0 lie on the real or the imaginary axis. To this end, we observe that the product-to-sum formula for trigonometric functions gives

$$\begin{aligned} \phi_0(\mu) &= -\mu[\cosh(\mu a)\sin(\mu a) + \sinh(\mu a)\cos(\mu a)] \\ &= -\tfrac{1}{2}\mu[\sin((1+i)\mu a) + \sin((1-i)\mu a) \\ &\qquad - i\sin((1+i)\mu a) + i\sin((1-i)\mu a)] \\ &= -\tfrac{1}{2}\mu[(1-i)\sin((1+i)\mu a) + (1+i)\sin((1-i)\mu a))] \end{aligned} \tag{3.1}$$

Setting $(1+i)\mu a = x + iy$, $x, y \in \mathbb{R}$, it follows for $\mu \neq 0$ that

$$\begin{aligned} \phi_0(\mu) = 0 &\Longrightarrow |\sin((1+i)\mu a)| = |\sin((1-i)\mu a)| \\ &\Longleftrightarrow |\sin(x+iy)| = |\sin(y-ix)| \end{aligned} \tag{3.2}$$

$$\begin{aligned}&\Longleftrightarrow \cosh^2 y - \cos^2 x = \cosh^2 x - \cos^2 y\\&\Longleftrightarrow \cosh^2(|y|) + \cos^2(|y|) = \cosh^2(|x|) + \cos^2(|x|)\end{aligned}$$

Since $\cosh^2 x + \cos^2 x = 1/2\cosh(2x) + 1/2\cos(2x) + 1$ has a positive derivative on $(0,\infty)$, this function is strictly increasing, and $\phi_0(\mu) = 0$ therefore implies by (3.2) that $|y| = |x|$, and thus, $y = \pm x$. Then,

$$\mu = \frac{x+iy}{(1+i)a} = \frac{1 \pm i}{1+i}\,\frac{x}{a}$$

is either real or pure imaginary.

Cases 5 and 8 easily follow from the result for Case 2.

For Case 3, a power series expansion shows that ϕ_0 has a zero of multiplicity 2 at 0. For the zeros on the positive real axis we merely need to replace the function

$$\mu \longmapsto \tan(\mu a) + \tanh(\mu a)$$

in the proof of Case 2 by

$$\mu \longmapsto \tan(\mu a) - \tanh(\mu a)$$

and observe that $\tanh'(\mu a) < 1$. Furthermore, in this case, we have a representation of ϕ_0 similar to (3.1), except that, on the right hand side, the factors $1-i$ and $1+i$ in front of the sine functions are interchanged. Hence, (3.2) also holds in this case, and all zeros must be real or pure imaginary.

In Case 7, it is easy to see that ϕ_0 has a zero of multiplicity 2 at 0. Next, we shall find the zeros of ϕ_0 on the positive real axis. Let $f(\mu) = \cos(\mu a)\cosh(\mu a) + 1$ and

$$I_{m,j} = \left[\left(2m + \frac{j}{2}\right)\frac{\pi}{a}, \left(2m + \frac{j+1}{2}\right)\frac{\pi}{a}\right],$$

$m = 0,1,\ldots$, $j = 0,1,2,3$. The zeros in $\mathbb{C}\setminus\{0\}$ of ϕ_0 are the zeros of f. It is obvious that, for all m and $\mu \in I_{m,0} \cup I_{m,3}$, $f(\mu) \geq 1$. On $I_{m,1}$, $\mu \mapsto \cos(\mu a)$ decreases and is negative, while $\mu \mapsto \cosh(\mu a)$ increases and is positive so that f decreases. At the endpoints of this interval f has the values $f((2m+1/2)(\pi/a)) = 1$ and $f((2m+1)(\pi/a)) = -\cosh((2m+1)\pi) + 1 < 0$. Hence, f has exactly one simple 0 on $I_{m,1}$. From $f''(\mu) = -2a^2\sin(\mu a)\sinh(\mu a)$, we see that f is strictly convex on $I_{m,2}$ with $f((2m+1)(\pi/a)) = -\cosh((2m+1)\pi) + 1 < 0$ and

$f((2m+3/2)(\pi/a))=1$. Hence, f has exactly one simple zero on $I_{m,2}$. Since

$$-\frac{1}{\cosh(\mu a)} \longrightarrow 0 \quad \text{as } \mu \to \infty,$$

we have

$$\widetilde{\mu}_m^1 = \left(2m + \frac{1}{2}\right)\frac{\pi}{a} + o(1)$$

and

$$\widetilde{\mu}_m^2 = \left(2m + \frac{3}{2}\right)\frac{\pi}{a} + o(1),$$

$m = 0, 1, \ldots$. The location of the zeros on the other three half-axes follows by repeated application of $\phi_0(i\mu) = -\phi_0(\mu)$.

The proof for Case 7 will be complete if we show that all zeros of ϕ_0 lie on the real or the imaginary axis. To this end, we observe that the operator associated with the eigenvalue problem

$$y^{(4)} = \tau y, \qquad y''(0) = 0, \qquad y^{(3)}(0) = 0, \qquad y(a) = 0, \qquad y'(a) = 0, \tag{3.3}$$

is self-adjoint, see Theorem 2.1. It is easy to see that this operator is non-negative. The substitution $\tau = \mu^4$ shows that f, as a function of μ, is the characteristic function of problem (3.3). Hence, the zeros of f are fourth roots of nonnegative real numbers, which means that all zeros of f are real or pure imaginary.

For Case 1, it is easy to see that 0 is a zero of ϕ_0 of multiplicity 2. Let $g(\mu) = \cos(\mu a)\cosh(\mu a) - 1$. The zeros of ϕ_0 are the zeros of g. An obvious modification of the proof of Case 7 shows that g has no zeros in the interval $I_{m,1}$ and $I_{m,2}$, whereas g has simple zeros in the intervals $I_{m,0}$ and $I_{m,3}$. The reasoning for the asymptotics and the zeros on the other three semiaxes is the same as in the proof of Case 7. In order to complete the proof, observe that, as for (3.3), the eigenvalues of

$$y^{(4)} = \tau y, \qquad y(0) = 0, \qquad y'(0) = 0, \qquad y(a) = 0, \qquad y'(a) = 0, \tag{3.4}$$

are nonnegative real numbers. Hence, the eigenvalues of problem (3.4) are real and nonnegative. The substitution $\tau = \mu^4$ shows that

$$\mu \longmapsto \mu^{-4} g(\mu)$$

is the characteristic function of problem (3.4) so that all zeros of g are real or pure imaginary. $\square$

Proposition 3.2. *For $g=0$ and $h=0$, there exists a positive integer k_0 such that the eigenvalues $\widehat{\lambda}_k$, $k\in\mathbb{Z}$, counted with multiplicity, of problem (1.2), where $B_1(\lambda)y=y^{[p_1]}(0)$, $B_2(\lambda)y=y^{[p_2]}(0)$, $B_3(\lambda)y=y^{[p_3]}(a)$ and $B_4(\lambda)y=y''(a)+i\alpha\lambda y'(a)$, $\alpha>0$, can be indexed in such a way that the eigenvalues $\widehat{\lambda}_k$ are pure imaginary for $|k|<k_0$ and satisfy $\widehat{\lambda}_{-k}=-\overline{\widehat{\lambda}_k}$ and $\Im\lambda_k\ge 0$ for $k\ge k_0$. For $k>0$, we can write $\widehat{\lambda}_k=\widehat{\mu}_k^2$, where the $\widehat{\mu}_k$ have the following asymptotic representation as $k\to\infty$:*

Case 1. $p_1=0$, $p_2=1$, $p_3=0$:

$$\widehat{\mu}_k=(2k-1)\frac{\pi}{2a}+o(1);$$

Case 2. $p_1=0$, $p_2=1$, $p_3=3$:

$$\widehat{\mu}_k=(4k-1)\frac{\pi}{4a}+o(1);$$

Case 3. $p_1=0$, $p_2=2$, $p_3=0$:

$$\widehat{\mu}_k=(4k+1)\frac{\pi}{4a}+o(1);$$

Case 4. $p_1=0$, $p_2=2$, $p_3=3$:

$$\widehat{\mu}_k=(2k-1)\frac{\pi}{2a}+o(1);$$

Case 5. $p_1=1$, $p_2=3$, $p_3=0$:

$$\widehat{\mu}_k=(4k-1)\frac{\pi}{4a}+o(1);$$

Case 6. $p_1=1$, $p_2=3$, $p_3=3$:

$$\widehat{\mu}_k=(k-1)\frac{\pi}{a}+o(1);$$

Case 7. $p_1=2$, $p_2=3$, $p_3=0$:

$$\widehat{\mu}_k=(2k-1)\frac{\pi}{2a}+o(1);$$

Case 8. $p_1 = 2$, $p_2 = 3$, $p_3 = 3$:

$$\widehat{\mu}_k = (4k-5)\frac{\pi}{4a} + o(1).$$

In particular, the number of pure imaginary eigenvalues is odd in each case.

Proof. In each case, we will show that the zeros of ϕ are asymptotically close to the zeros of ϕ_0. We begin with Case 8. Since the remaining cases are very similar, we will only indicate the changes needed in the other seven cases.

Case 8. A straightforward calculation gives

$$\begin{aligned}\phi(\mu) = {} & \tfrac{1}{2} i\alpha\mu^5(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)) \\ & - \tfrac{1}{2}\mu^4(1 - \cos(\mu a)\cosh(\mu a)).\end{aligned} \tag{3.5}$$

Up to the constant factor $(i\alpha)/2$, the first term equals $\phi_0(\mu)$. It follows that, for μ with $\phi_0(\mu) \neq 0$, $\sin(\mu a) \neq 0$ and $\sinh(\mu a) \neq 0$, we have

$$\begin{aligned}\phi_1(\mu) = \frac{2\phi(\mu) - i\alpha\phi_0(\mu)}{\phi_0(\mu)} &= \frac{1}{\mu}\,\frac{1}{\tan(\mu a) + \tanh(\mu a)} \\ &\quad - \frac{1}{\mu}\,\frac{1}{\tan(\mu a) + \tanh(\mu a)}\,\frac{1}{\cos(\mu a)\cosh(\mu a)}.\end{aligned} \tag{3.6}$$

Fix $\varepsilon \in (0, \pi/4a)$, for $k = 2, 3, \ldots$. Let $R_{k,\varepsilon}$ be the boundaries of the squares determined by the vertices $(4k-5)(\pi/4a) \pm \varepsilon \pm i\varepsilon$, $k \in \mathbb{Z}$. These squares do not intersect due to $\varepsilon < \pi/2a$. Since $\tan z = -1$ if and only if $z = j\pi - \pi/4$ and $j \in \mathbb{Z}$, it follows from the periodicity of tan that the number

$$C_1(\varepsilon) = 2\min\{|\tan(\mu a) + 1| : \mu \in R_{k,\varepsilon}\}$$

is positive and independent of ε. Since

$$\tanh(\mu a) \longrightarrow 1$$

uniformly in the strip

$$\left\{\mu \in \mathbb{C} : \mathrm{Re}\ \mu \geq 1, |\mathrm{Im}\ \mu| \leq \frac{\pi}{4a}\right\} \quad \text{as } |\mu| \to \infty,$$

there is an integer $k_1(\varepsilon)$ such that

$$|\tan(\mu a) + \tanh(\mu a)| \geq C_1(\varepsilon) \quad \text{for all } \mu \in R_{k,\varepsilon} \text{ with } k > k_1(\varepsilon).$$

By periodicity, there is a number $C_2(\varepsilon) > 0$ such that $|\cos(\mu a)| > C_2(\varepsilon)$ for all $\mu \in R_{k,\varepsilon}$ and all k. Observing that $|\cosh(\mu a)| \geq |\sinh(\Re\mu a)|$, it follows that there exists $k_2(\varepsilon) \geq k_1(\varepsilon)$ such that, for all μ on the squares $R_{k,\varepsilon}$ with $k \geq k_2(\varepsilon)$, the estimate $|\phi_1(\mu)| < \alpha$ holds. We can assume from Lemma 3.1 that $\widetilde{\mu}_k$ is inside of $R_{k,\varepsilon}$ and no other zero of ϕ_0 has this property. By definition of ϕ_1 in (3.6) we have

$$\phi(\mu) = \tfrac{1}{2}(\phi_1(\mu) + i\alpha)\phi_0(\mu) \quad \text{for } \mu \in R_{k,\varepsilon}. \tag{3.7}$$

Hence, it follows from Rouché's theorem that there is exactly one (simple) zero $\widehat{\mu}_k$ of ϕ in each $R_{k,\varepsilon}$ for $k \geq k_2(\varepsilon)$. In view of $\phi_0(i\mu) = -\phi_0(\mu)$ and $\phi_1(i\mu) = -\phi_1(\mu)$ for all $\mu \in \mathbb{C}$, the same reasoning applies to the corresponding squares along the positive imaginary semiaxis. Observing that ϕ is an even function, it follows that the same estimate applies to the corresponding squares along the other two remaining semiaxes. Therefore, ϕ has zeros $\pm\widehat{\mu}_k$, $\pm\widehat{\mu}_{-k}$ for $k > k_2(\varepsilon)$ with the same asymptotic behavior as the zeros $\pm\widetilde{\mu}_k$, $\pm i\widetilde{\mu}_k$ of ϕ_0, stated in Lemma 3.1.

Next, we shall estimate ϕ_1 on the squares S_k, $k \in \mathbb{N}$, whose vertices are $\pm k(\pi/a) \pm ik(\pi/a)$. For $k \in \mathbb{Z}$ and $\gamma \in \mathbb{R}$,

$$\tan\left(\left(\frac{k\pi}{a} + i\gamma\right)a\right) = \tan(i\gamma a) = i\tanh(\gamma a) \in i\mathbb{R}. \tag{3.8}$$

Therefore, we have, for $\mu = k\pi/a + i\gamma$ where $k \in \mathbb{Z}$ and $\gamma \in \mathbb{R}$, that

$$|\tan(\mu a)| < 1 \quad \text{and} \quad |\tan(\mu a) \pm 1| \geq 1. \tag{3.9}$$

For $\mu = x + iy$, $x, y \in \mathbb{R}$ and $x \neq 0$, we have

$$\tanh(\mu a) = \frac{e^{(ax+iay)} - e^{-(ax+iay)}}{e^{(ax+iay)} + e^{-(ax+iay)}} \longrightarrow \pm 1 \tag{3.10}$$

uniformly in y as $x \to \pm\infty$. Hence, there is a $\widetilde{k}_1 > 0$ such that, for all $k \in \mathbb{Z}$, $|k| \geq \widetilde{k}_1$ and $\gamma \in \mathbb{R}$,

$$\left|\tanh\left(\left(\frac{k\pi}{a} + i\gamma\right)a\right) - \operatorname{sgn}(k)\right| < \frac{1}{2}. \tag{3.11}$$

It follows from (3.9) and (3.11) for $\mu = k\pi/a + i\gamma$, $k \in \mathbb{Z}$, $|k| \ge \widetilde{k}_1$ and $\gamma \in \mathbb{R}$ that

$$|\tan(\mu a) + \tanh(\mu a)| \ge \frac{1}{2}. \tag{3.12}$$

Furthermore, we shall use the estimates

$$\left|\cosh\left(\left(\frac{k\pi}{a} + i\gamma\right)a\right)\right| \ge |\sinh(k\pi)|, \tag{3.13}$$

$$\left|\cos\left(\left(\frac{k\pi}{a} + i\gamma\right)a\right)\right| = \cosh(\gamma a) \ge 1, \tag{3.14}$$

which hold for all $k \in \mathbb{Z}$ and all $\gamma \in \mathbb{R}$. Therefore, it follows from (3.12)–(3.14) and the corresponding estimates with μ replaced by $i\mu$ that there is a $\widehat{k}_1 \ge \widetilde{k}_1$ such that $|\phi_1(\mu)| < \alpha$ for all $\mu \in S_k$ with $k > \widehat{k}_1$. Again, from (3.7) and Rouché's theorem we conclude that the functions ϕ_0 and ϕ have the same number of zeros in the square S_k, for $k \in \mathbb{N}$ with $k \ge \widehat{k}_1$.

Since ϕ_0 has $4k+2$ zeros inside S_k, and thus $4k+2+4$ zeros inside of S_{k+1}, it follows that ϕ has no large zeros other than the zeros $\pm\widehat{\mu}_k$ found above for $|k|$ sufficiently large, and that there are $\widehat{\mu}_k$ for small $|k|$ such that the $\widehat{\lambda}_k = \widehat{\mu}_k^2$ account for all eigenvalues of problem (1.2) since each of these eigenvalues gives rise to two zeros of ϕ, counted with multiplicity. By Proposition 2.2, all eigenvalues with nonzero real part occur in pairs

$$\widehat{\lambda}_k,\ -\overline{\widehat{\lambda}_k} \quad \text{with } \Re\widehat{\lambda}_k \ge 0,$$

which shows that we can index all such eigenvalues as $\widehat{\lambda}_{-k} = -\overline{\widehat{\lambda}_k}$. Since an odd number of indices remain, the number of pure imaginary eigenvalues must be odd.

The functions ϕ in Cases 2, 3 and 5 are, respectively, the following:

Case 2.

$$\begin{aligned}\phi(\mu) = &-\tfrac{1}{2}i\alpha\mu(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)) \\ &- \tfrac{1}{2}(1 + \cos(\mu a)\cosh(\mu a)).\end{aligned} \tag{3.15}$$

Case 3.

$$\phi(\mu) = \frac{i\alpha}{2\mu}(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)) \tag{3.16}$$

$$+\frac{1}{\mu^2}\sin(\mu a)\sinh(\mu a).$$

Case 5.

$$\phi(\mu) = \tfrac{1}{2}i\alpha\mu(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)) + \cos(\mu a)\cosh(\mu a). \tag{3.17}$$

Then, all of the estimates are as in Case 8, and the results, respectively, in Cases 2, 3 and 5 immediately follow from that in Case 8.

Case 6. A straightforward calculation gives

$$\phi(\mu) = i\alpha\mu^4\sin(\mu a)\sinh(\mu a) + \tfrac{1}{2}\mu^3(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)). \tag{3.18}$$

Then,

$$\phi_1(\mu) = \frac{2\phi(\mu) - i\alpha\phi_0(\mu)}{\phi_0(\mu)} = \frac{1}{2\mu}(\coth(\mu a) + \cot(\mu a)).$$

The result follows similarly as in the proof of Case 8, replacing μ by $\mu \pm (\pi/2)$ and $\mu \pm i(\pi/2)$, respectively.

Case 4. The function ϕ in this case is

$$\phi(\mu) = -i\alpha\mu^2\cos(\mu a)\cosh(\mu a) + \tfrac{1}{2}\mu(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)). \tag{3.19}$$

The result is similar to Case 6 with each trigonometric and hyperbolic function replaced by its derivative.

Case 7. A straightforward calculation gives

$$\phi(\mu) = \tfrac{1}{2}i\alpha\mu^2(\cos(\mu a)\cosh(\mu a) + 1) - \tfrac{1}{2}\mu(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)). \tag{3.20}$$

Then,

$$\begin{aligned}\phi_1(\mu) &= \frac{2\phi(\mu) - i\alpha\phi_0(\mu)}{\phi_0(\mu)} \\ &= \frac{1}{\mu}(\tanh(\mu a) - \tan(\mu a)) \\ &\quad - \frac{1}{\mu}\frac{\tanh(\mu a) - \tan(\mu a)}{\cos(\mu a)\cosh(\mu a) + 1}\end{aligned}$$

Observing that, in the strip $\{\mu = x + iy : x \geq 1, |y| \leq \pi/2a\}$, we have $|\tan(\mu a)| < C_1$ and $|\tanh(\mu a)| < C_2$ with suitable constants C_j, the result follows with the proof similar to that for Case 8.

Case 1. The function ϕ in this case is

$$\begin{aligned} \phi(\mu) = {} & \frac{1}{2\mu^2} i\alpha(1 - \cos(\mu a)\cosh(\mu a)) \\ & + \frac{1}{2\mu^3}(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)), \end{aligned} \tag{3.21}$$

and reasoning as in Case 7 completes the proof. □

4. Birkhoff regularity. We refer to [**2,** Definition 7.3.1] for the definition of Birkhoff regularity.

Proposition 4.1. *The boundary value problem* (1.2a), (2.2)–(2.8) *is Birkhoff regular for* $\alpha > 0$ *with respect to the eigenvalue parameter* μ *given by* $\lambda = \mu^2$.

Proof. The characteristic function of (1.2a) as defined in [**2,** (7.1.4)] is $\pi(\rho) = \rho^4 - 1$, and its zeros are i^{k-1}, $k = 1, \ldots, 4$. We can choose

$$C(x, \mu) = \operatorname{diag}(1, \mu, \mu^2, \mu^3)(i^{(k-1)(j-1)})_{k,j=1}^4$$

according to [**2,** Theorem 7.2.4.A]. The boundary conditions (2.2)–(2.8) can be written in the form

$$B_j(\lambda)y = \widehat{B}_j(\mu)(y(a_j), y'(a_j), y''(a_j), y^{(3)}(a_j)), \quad j = 1, 2, 3, 4,$$

where

$$\widehat{B}_3(\mu) = \begin{cases} \varepsilon_1^\top & \text{for Cases } 1, 3, 5, 7, \\ (0, -g(a), 0, 1) & \text{for Cases } 2, 4, 6, 8, \end{cases}$$

$$\begin{aligned} \widehat{B}_j(\mu) &= \varepsilon_{p_j+1}^\top \quad \text{for } j = 1, 2, \\ \widehat{B}_4(\mu) &= (0, i\alpha\mu^2, 1, 0). \end{aligned}$$

Thus, the boundary matrices defined in [**2,** (7.3.1)] are given by

$$W^{(0)}(\mu) = \begin{pmatrix} \widehat{B}_1(\mu) \\ \widehat{B}_2(\mu) \\ 0 \\ 0 \end{pmatrix} C(0,\mu) = \begin{pmatrix} \mu^{p_1} & \mu^{p_1} i^{p_1} & \mu^{p_1} i^{2p_1} & \mu^{p_1} i^{3p_1} \\ \mu^{p_1} & \mu^{p_2} i^{p_1} & \mu^{p_2} i^{2p_2} & \mu^{p_2} i^{3p_2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$W^{(1)}(\mu) = \begin{pmatrix} 0 \\ 0 \\ \widehat{B}_3(\mu) \\ \widehat{B}_4(\mu) \end{pmatrix} C(a,\mu) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix},$$

where $\gamma_j = 1$ for Cases 1, 3, 5 and 7 and $\gamma_j = (-i)^{j-1}\mu^3 + i^{j+1}g(a)\mu$ for Cases 2, 4, 6 and 8; furthermore, $\beta_j = i^j\alpha\mu^3 + (-1)^{j-1}\mu^2$. Choosing $C_2(\mu) = \operatorname{diag}(\mu^{p_1}, \mu^{p_2}, \mu^{p_3}, \mu^3)$, it follows that $C_2(\mu)^{-1}W^{(j)}(\mu) = W_0^{(j)} + O(\mu^{-1})$, where

$$W_0^{(0)} = \begin{pmatrix} 1 & i^{p_1} & i^{2p_1} & i^{3p_1} \\ 1 & i^{p_2} i^{2p_2} & i^{3p_2} & \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad W_0^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & i^{p_3} & i^{2p_3} & i^{3p_3} \\ i\alpha & -\alpha & -i\alpha & \alpha \end{pmatrix}.$$

The Birkhoff matrices are

$$W_0^{(0)}\Delta_j + W_0^{(1)}(I - \Delta_j), \tag{4.1}$$

where Δ_j, $j = 1,2,3,4$, are the 4×4 diagonal matrices with two consecutive ones and two consecutive zeros in the diagonal in a cyclic arrangement, see [**2,** Proposition 4.1.7, Definition 7.3.1]. It is easy to see that, after a permutation of columns, the matrices (4.1) are block diagonal consisting of 2×2 blocks taken from two consecutive columns (in the sense of cyclic arrangement) of the first two rows of $W_0^{(0)}$ and the last two rows of $W_0^{(1)}$, respectively. Hence, the determinants of the Birkhoff matrices (4.1) are

$$\begin{vmatrix} i^{(j-1)p_1} & i^{jp_1} \\ i^{(j-1)p_2} & i^{jp_2} \end{vmatrix} \begin{vmatrix} i^{(j+1)p_3} & i^{(j+2)p_3} \\ i^{j+2}\alpha & i^{j+3}\alpha \end{vmatrix}$$
$$= -i^{j(p_1+p_2+p_3+1)+p_3}(i - i^{p_3})(i^{-p_1} - i^{-p_2})\alpha \neq 0.$$

Thus, the problem (1.2a), (2.2)–(2.8) is Birkhoff regular. $\square$

5. Asymptotic expansions of eigenvalues. Let D, as a function of μ with $\lambda = \mu^2$, be the characteristic function of problem (1.2a), (2.2)–(2.8) with respect to the fundamental system y_j, $j = 1, 2, 3, 4$, with $y_j^{[m]}(0) = \delta_{j,m+1}$ for $m = 0, 1, 2, 3$, and δ is the Kronecker delta. Denote by D_0 the corresponding characteristic function for $g = 0$. Note that the characteristic functions D_0 and ϕ_0 considered in Section 3 have the same zeros, counted with multiplicity. Due to Birkhoff regularity, g influences only lower order terms in D, see [**2,** subsections 4.3, 7.3]. Therefore, it may be inferred that, outside of the interior of the small squares R_k, $-R_k$, iR_k, $-iR_{-k}$ around the zeros of D_0,

$$|D(\mu) - D_0(\mu)| < |D_0(\mu)|$$

if $|\mu|$ is sufficiently large. Since the fundamental system y_j, $j = 1, 2, 3, 4$, analytically depends upon μ, D and D_0 are also analytic functions. Hence, applying Rouché's theorem both to the large squares S_k and to the small squares, which are sufficiently far away from the origin, it follows that the eigenvalues of the boundary value problem for general g have the same asymptotic distribution as those for $g = h = 0$, whence Proposition 3.2 leads to

Proposition 5.1. *For $g \in C^1[0, a]$ and $h \in C[0, a]$, there exists a positive integer k_0 such that the eigenvalues λ_k, $k \in \mathbb{Z}$, counted with multiplicity, of problem* (1.2a), (2.2)–(2.8)*, can be enumerated in such a way that the eigenvalues λ_k are pure imaginary for $|k| < k_0$, and $\lambda_{-k} = -\overline{\lambda_k}$ for $k \geq k_0$. For $k > 0$, we can write $\lambda_k = \mu_k^2$, where the μ_k have the following asymptotic representation as $k \to \infty$:*

Case 1. $p_1 = 0$, $p_2 = 1$, $p_3 = 0$:

$$\mu_k = (2k - 1)\frac{\pi}{2a} + o(1);$$

Case 2. $p_1 = 0$, $p_2 = 1$, $p_3 = 3$:

$$\mu_k = (4k - 1)\frac{\pi}{4a} + o(1);$$

Case 3. $p_1 = 0$, $p_2 = 2$, $p_3 = 0$:

$$\mu_k = (4k + 1)\frac{\pi}{4a} + o(1);$$

Case 4. $p_1 = 0$, $p_2 = 2$, $p_3 = 3$:

$$\mu_k = (2k-1)\frac{\pi}{2a} + o(1);$$

Case 5. $p_1 = 1$, $p_2 = 3$, $p_3 = 0$:

$$\mu_k = (4k-1)\frac{\pi}{4a} + o(1);$$

Case 6. $p_1 = 1$, $p_2 = 3$, $p_3 = 3$:

$$\mu_k = (k-1)\frac{\pi}{a} + o(1);$$

Case 7. $p_1 = 2$, $p_2 = 3$, $p_3 = 0$:

$$\mu_k = (2k-1)\frac{\pi}{2a} + o(1);$$

Case 8. $p_1 = 2$, $p_2 = 3$, $p_3 = 3$:

$$\mu_k = (4k-5)\frac{\pi}{4a} + o(1).$$

In particular, the number of pure imaginary eigenvalues is odd in each case.

In the remainder of the section we shall establish more precise asymptotic expansions of the eigenvalues. According to **[2,** Theorem 8.2.1], (1.2a) has an asymptotic fundamental system $\{\eta_1, \eta_2, \eta_3, \eta_4\}$ of the form

$$\eta_\nu^{(j)}(x,\mu) = \delta_{\nu,j}(x,\mu)e^{i^{\nu-1}\mu x}, \quad \nu = 1, \dots, 4,\ j = 0, \dots, 3, \tag{5.1}$$

where

$$\delta_{\nu,j}(x,\mu) = \left[\frac{d^j}{dx^j}\right]\left\{\sum_{r=0}^{2}(\mu i^{\nu-1})^{-r}\varphi_r(x)e^{i^{\nu-1}\mu x}\right\}e^{-i^{\nu-1}\mu x} + \{o(\mu^{-2+j})\}_\infty, \tag{5.2}$$

and $[d^j/dx^j]$ means that we omit those terms of the Leibniz expansion which contain a function $\varphi_r^{(k)}$ with $k > 4-r$, where $\{o(\cdot)\}_\infty$ means that the estimate is uniform in x.

Since the coefficient of $y^{(3)}$ in (1.2a) is 0, we have $\varphi_0(x) = 1$, see **[2,** (8.2.3)].

We now determine the functions φ_1 and φ_2. In this regard, observe that $n_0 = 0$ and $l = 4$ in the notation of [**2,** (8.1.2), (8.1.3)], also see [**2,** Theorem 8.1.2]. From [**2,** (8.2.45)], we know that

$$\varphi_r = \varphi_{1,r} = \varepsilon_1^{\mathsf{T}} V Q^{[r]} \varepsilon_1, \tag{5.3}$$

where ε_ν is the νth unit vector in $\mathbb{C}^4$, $V = (i^{(j-1)(k-1)})_{j,k=1}^4$ and $Q^{[r]}$ are 4×4 matrices given by [**2,** (8.2.28), (8.2.33), (8.2.34)], that is, $Q^{[0]} = I_4$,

$$\Omega_4 Q^{[1]} - Q^{[1]}\Omega_4 = Q^{[0]\prime} = 0, \tag{5.4}$$

$$\Omega_4 Q^{[2]} - Q^{[2]}\Omega_4 = Q^{[1]\prime} - \frac{1}{4} g \Omega_4 \varepsilon \varepsilon^\top \Omega_4^{-2} Q^{[0]}, \tag{5.5}$$

$$0 = \varepsilon_\nu^{\mathsf{T}} \left(Q^{[2]\prime} + \frac{1}{4} \sum_{j=1}^{2} k_{3-j} \Omega_4 \varepsilon \varepsilon^{\mathsf{T}} \Omega_4^{-1-j} Q^{[2-j]} \right) \varepsilon_\nu, \tag{5.6}$$

$\nu = 1, 2, 3, 4$, where $k_2 = -g$, $k_1 = -g'$, $\Omega_4 = \operatorname{diag}(1, i, -1, -i)$ and $\varepsilon^{\mathsf{T}} = (1, 1, 1, 1)$. Let $G(x) = \int_0^x g(t)\, dt$. A lengthy, but straightforward, calculation gives

$$\varphi_1 = \frac{1}{4} G, \quad \varphi_2 = \frac{1}{32} G^2 - \frac{1}{8} g, \tag{5.7}$$

and thus,

(5.8)

$$\eta_\nu(x,\mu) = \left(1 + \frac{1}{4} i^{-\nu+1} G(x) \mu^{-1} + (-1)^{\nu-1} \left(\frac{1}{32} G(x)^2 - \frac{1}{8} g(x) \right) \mu^{-2} \right) e^{i^{\nu-1}\mu x} + \{o(\mu^{-2})\}_\infty e^{i^{\nu-1}\mu x} \quad \text{for } \nu = 1, 2, 3, 4.$$

The characteristic function of (1.2a), (2.2)–(2.8) is

$$D(\mu) = \det(\gamma_{jk} \exp(\varepsilon_{jk}))_{j,k=1}^4,$$

where

$$\begin{aligned}
&\varepsilon_{1k} = \varepsilon_{2k} = 0, && \varepsilon_{3k} = \varepsilon_{4k} = i^{k-1}\mu a, \\
&\gamma_{1k} = \delta_{k,p_1}(0,\mu), && \gamma_{2k} = \delta_{k,p_2}(0,\mu) \quad \text{if } p_2 \le 2, \\
&\gamma_{2k} = \delta_{k,3}(0,\mu) - g(0)\delta_{k,1}(0,\mu) \quad \text{if } p_2 = 3, \\
&\gamma_{3k} = \delta_{k,0}(a,\mu) \quad \text{if } p_3 = 0, \\
&\gamma_{3k} = \delta_{k,3}(a,\mu) - g(a)\delta_{k,1}(a,\mu) \quad \text{if } p_3 = 3,
\end{aligned}$$

$$\gamma_{4k} = \delta_{k,2}(a,\mu) + i\alpha\mu^2\delta_{k,1}(a,\mu).$$

Note that

$$D(\mu) = \sum_{m=1}^{5} \psi_m(\mu) e^{\omega_m \mu a}, \tag{5.9}$$

where $\omega_1 = 1+i$, $\omega_2 = -1+i$, $\omega_3 = -1-i$, $\omega_4 = 1-i$, $\omega_5 = 0$, and each of the functions $\psi_1, \dots, \psi_5$ has asymptotic representations of the form $c_k\mu^k + c_{k-1}\mu^{k-1} + \cdots + c_{k_0}\mu^{k_0} + o(\mu^{k_0})$.

It follows from (5.9) that

$$D_1(\mu) := D(\mu)e^{-\omega_1\mu a} = \psi_1(\mu) + \sum_{m=2}^{5} \psi_m(\mu) e^{(\omega_m-\omega_1)\mu a}, \tag{5.10}$$

where $\omega_2 - \omega_1 = -2$, $\omega_3 - \omega_1 = -2-2i$, $\omega_4 - \omega_1 = -2i$, $\omega_5 - \omega_1 = -1-i$. If $\arg\mu \in -3\pi/8, \pi/8$, we have $|e^{(\omega_m-\omega_1)\mu a}| \le e^{-\sin(\pi/8)|\mu|a}$ for $m = 2,3,5$, and the terms $\psi_m(\mu)e^{(\omega_m-\omega_1)\mu a}$ for $m = 2,3,5$ can be absorbed by $\psi_1(\mu)$ since they are of the form $o(\mu^{-s})$ for any integer s. Hence, for $\arg\mu \in -3\pi/8, \pi/8$,

$$D_1(\mu) = \psi_1(\mu) + \psi_4(\mu)e^{(\omega_4-\omega_3)\mu a} = \psi_1(\mu) + \psi_4(\mu)e^{-2i\mu a}, \tag{5.11}$$

where

$$\psi_1(\mu) = [\gamma_{13}\gamma_{24} - \gamma_{14}\gamma_{23}][\gamma_{31}\gamma_{42} - \gamma_{32}\gamma_{41}], \tag{5.12}$$

$$\psi_4(\mu) = [\gamma_{12}\gamma_{23} - \gamma_{13}\gamma_{22}][\gamma_{31}\gamma_{44} - \gamma_{34}\gamma_{41}]. \tag{5.13}$$

A straightforward calculation gives for $p_3 = 0$ that

(5.14)
$$\begin{aligned}\gamma_{31}\gamma_{42} - \gamma_{32}\gamma_{41} &= -(1+i)\alpha\mu^3 - 2(\alpha\varphi_1(a)+1)\mu^2 \\ &\quad - (1-i)(\alpha\varphi_1^2(a) - \tfrac{1}{4}\alpha g(a) + 2\varphi_1(a))\mu + o(\mu),\end{aligned}$$

(5.15)
$$\begin{aligned}\gamma_{31}\gamma_{44} - \gamma_{34}\gamma_{41} &= (1-i)\alpha\mu^3 + 2(\alpha\varphi_1(a)-1)\mu^2 \\ &\quad + (1+i)(\alpha\varphi^2(a) - \tfrac{1}{4}\alpha g(a) - 2\varphi_1(a))\mu + o(\mu),\end{aligned}$$

while, for $p_3 = 3$, we have

$$\begin{aligned}\gamma_{31}\gamma_{42} - \gamma_{32}\gamma_{41} &= -2\alpha\mu^6 - (1-i)(2\alpha\varphi_1(a)+1)\mu^5 \\ &\quad + 2i(\alpha\varphi_1^2(a) + \varphi_1(a))\mu^4 + o(\mu^4),\end{aligned} \tag{5.16}$$

$$\gamma_{31}\gamma_{44} - \gamma_{34}\gamma_{41} = 2\alpha\mu^6 + (1+i)(2\alpha\varphi_1(a) - 1)\mu^5 + 2i(\alpha\varphi_1^2(a) - \varphi_1(a))\mu^4 + o(\mu^4). \tag{5.17}$$

For the other two factors in (5.12) and (5.13) we must consider four different cases.

Cases 1 and 2. $p_1 = 0$, $p_2 = 1$. Here, we have

$$\gamma_{13}\gamma_{24} - \gamma_{14}\gamma_{23} = (1-i)\mu + \tfrac{1}{4}(1+i)\mu^{-1}g(0) + o(\mu^{-1}), \tag{5.18}$$

$$\gamma_{12}\gamma_{23} - \gamma_{13}\gamma_{22} = -(1+i)\mu - \tfrac{1}{4}(1-i)\mu^{-1}g(0) + o(\mu^{-1}). \tag{5.19}$$

Cases 3 and 4. $p_1 = 0$, $p_2 = 2$. Here, we have

$$\gamma_{13}\gamma_{24} - \gamma_{14}\gamma_{23} = -2\mu^2 + o(1), \tag{5.20}$$

$$\gamma_{12}\gamma_{23} - \gamma_{13}\gamma_{22} = 2\mu^2 + o(1). \tag{5.21}$$

Cases 5 and 6. $p_1 = 1$, $p_2 = 3$. Here, we have

$$\gamma_{13}\gamma_{24} - \gamma_{14}\gamma_{23} = -2i\mu^4 + o(\mu^2), \tag{5.22}$$

$$\gamma_{12}\gamma_{23} - \gamma_{13}\gamma_{22} = -2i\mu^4 + o(\mu^2). \tag{5.23}$$

Cases 7 and 8. $p_1 = 2$, $p_2 = 3$. Here, we have

$$\gamma_{13}\gamma_{24} - \gamma_{14}\gamma_{23} = -(1-i)\mu^5 + \tfrac{3}{4}(1+i)\mu^3 g(0) + o(\mu^3), \tag{5.24}$$

$$\gamma_{12}\gamma_{23} - \gamma_{13}\gamma_{22} = (1+i)\mu^5 - \tfrac{3}{4}(1-i)\mu^3 g(0) + o(\mu^3). \tag{5.25}$$

Case 1. We obtain from (5.14), (5.15), (5.18) and (5.19)

$$\psi_1(\mu) = -2\alpha\mu^4 - \tfrac{1}{2}(1-i)(4+\alpha G(a))\mu^3 + i[\tfrac{1}{8}\alpha G^2(a) + G(a) - \tfrac{1}{2}\alpha g(0) - \tfrac{1}{2}\alpha g(a)]\mu^2 + o(\mu^2), \tag{5.26}$$

$$\psi_4(\mu) = -2\alpha\mu^4 + \tfrac{1}{2}(1+i)(4-\alpha G(a))\mu^3 - i[\tfrac{1}{8}\alpha G^2(a) - G(a) - \tfrac{1}{2}\alpha g(0) - \tfrac{1}{2}\alpha g(a)]\mu^2 + o(\mu^2). \tag{5.27}$$

Case 2. We have from (5.16), (5.17), (5.18) and (5.19)

$$\psi_1(\mu) = -2(1-i)\alpha\mu^7 + i(2+\alpha G(a))\mu^6 + (1+i)[\tfrac{1}{8}\alpha G^2(a) + \tfrac{1}{2}G(a) - \tfrac{1}{2}\alpha g(0)]\mu^5 + o(\mu^5), \tag{5.28}$$

$$\psi_4(\mu) = -2(1+i)\alpha\mu^7 + i(2-\alpha G(a))\mu^6 + (1-i)[\tfrac{1}{8}\alpha G^2(a) - \tfrac{1}{2}G(a) - \tfrac{1}{2}\alpha g(0)]\mu^5 + o(\mu^5). \tag{5.29}$$

Case 3. We have from (5.14), (5.15), (5.20) and (5.21)

$$\begin{aligned}\psi_1(\mu) &= 2(1+i)\alpha\mu^5 + (4+\alpha G(a))\mu^4 \\ &\quad + (1-i)[\tfrac{1}{8}\alpha G^2(a) + G(a) - \tfrac{1}{2}\alpha g(a)]\mu^3 + o(\mu^3),\end{aligned} \tag{5.30}$$

$$\begin{aligned}\psi_4(\mu) &= 2(1-i)\alpha\mu^5 - (4-\alpha G(a))\mu^4 \\ &\quad + (1+i)[\tfrac{1}{8}\alpha G^2(a) - G(a) - \tfrac{1}{2}\alpha g(a)]\mu^3 + o(\mu^3).\end{aligned} \tag{5.31}$$

Case 4. We have from (5.16), (5.17), (5.20) and (5.21)

$$\begin{aligned}\psi_1(\mu) &= 4\alpha\mu^8 + (1-i)(2+\alpha G(a))\mu^7 \\ &\quad - i[\tfrac{1}{4}\alpha G^2(a) + G(a)]\mu^6 + o(\mu^6),\end{aligned} \tag{5.32}$$

$$\begin{aligned}\psi_4(\mu) &= 4\alpha\mu^8 - (1+i)(2-\alpha G(a))\mu^7 \\ &\quad + i[\tfrac{1}{4}\alpha G^2(a) - G(a)]\mu^6 + o(\mu^6).\end{aligned} \tag{5.33}$$

Case 5. We have from (5.14), (5.15), (5.22) and (5.23)

$$\begin{aligned}\psi_1(\mu) &= -2(1-i)\alpha\mu^7 + i(4+\alpha G(a))\mu^6 \\ &\quad + (1+i)[\tfrac{1}{8}\alpha G^2(a) + G(a) - \tfrac{1}{2}\alpha g(a)]\mu^5 + o(\mu^5),\end{aligned} \tag{5.34}$$

$$\begin{aligned}\psi_4(\mu) &= -2(1+i)\alpha\mu^7 + i(4-\alpha G(a))\mu^6 \\ &\quad + (1-i)[\tfrac{1}{8}\alpha G^2(a) - G(a) - \tfrac{1}{2}\alpha g(a)]\mu^5 + o(\mu^5).\end{aligned} \tag{5.35}$$

Case 6. We have from (5.16), (5.17), (5.22) and (5.23)

$$\begin{aligned}\psi_1(\mu) &= 4i\alpha\mu^{10} + (1+i)(2+\alpha G(a))\mu^9 \\ &\quad + [\tfrac{1}{4}\alpha G^2(a) + G(a)]\mu^8 + o(\mu^8),\end{aligned} \tag{5.36}$$

$$\begin{aligned}\psi_4(\mu) &= -4i\alpha\mu^{10} - (1-i)(2-\alpha G(a))\mu^9 \\ &\quad + [\tfrac{1}{4}\alpha G^2(a) - G(a)]\mu^8 + o(\mu^8).\end{aligned} \tag{5.37}$$

Case 7. We have from (5.14), (5.15), (5.24) and (5.25)

$$\begin{aligned}\psi_1(\mu) &= 2\alpha\mu^8 + \tfrac{1}{2}(1-i)(4+\alpha G(a))\mu^7 \\ &\quad - i[\tfrac{1}{8}\alpha G^2(a) + G(a) + \tfrac{3}{2}\alpha g(0) - \tfrac{1}{2}\alpha g(a)]\mu^6 + o(\mu^6),\end{aligned} \tag{5.38}$$

$$\begin{aligned}\psi_4(\mu) &= 2\alpha\mu^8 - \tfrac{1}{2}(1+i)(4-\alpha G(a))\mu^7 \\ &\quad + i[\tfrac{1}{8}\alpha G^2(a) - G(a) + \tfrac{3}{2}\alpha g(0) - \tfrac{1}{2}\alpha g(a)]\mu^6 + o(\mu^6).\end{aligned} \tag{5.39}$$

Case 8. It follows from (5.16), (5.17), (5.24) and (5.25)

$$\begin{aligned}\psi_1(\mu) &= 2(1-i)\alpha\mu^{11} - i(2+\alpha G(a))\mu^{10} \\ &\quad - (1+i)[\tfrac{1}{8}\alpha G^2(a) + \tfrac{1}{2}G(a) + \tfrac{3}{2}\alpha g(0)]\mu^9 + o(\mu^9),\end{aligned} \tag{5.40}$$

$$\begin{aligned}\psi_4(\mu) &= 2(1+i)\alpha\mu^{11} - i(2-\alpha G(a))\mu^{10} \\ &\quad - (1-i)[\tfrac{1}{8}\alpha G^2(a) - \tfrac{1}{2}G(a) + \tfrac{3}{2}\alpha g(0)]\mu^9 + o(\mu^9).\end{aligned} \tag{5.41}$$

We already know by Proposition 5.1 that the zeros μ_k of D satisfy the asymptotic representations $\mu_k = k\pi/a + \tau_0 + o(1)$ as $k \to \infty$. In order to improve on these asymptotic representations, write

$$\begin{aligned}\mu_k &= k\frac{\pi}{a} + \tau(k), \\ \tau(k) &= \sum_{m=0}^{n} \tau_m k^{-m} + o(k^{-n}),\end{aligned} \tag{5.42}$$

$k = 1, 2, \ldots$. Due to the symmetry of the eigenvalues, we only need to find the asymptotic expansions as $k \to \infty$. We know τ_0 from Proposition 5.1, and it is our aim to find τ_1 and τ_2. To this end, we substitute (5.42) into $D_1(\mu_k) = 0$ and then compare the coefficients of k^0, k^{-1} and k^{-2}.

Observe that

$$\begin{aligned}e^{-2i\mu_k a} &= e^{-2i\tau(k)a} \\ &= e^{-2i\tau_0 a} \exp\left(-2ia\left(\frac{\tau_1}{k} + \frac{\tau_2}{k^2} + o(k^{-2})\right)\right) \\ &= e^{-2i\tau_0 a}\left(1 - 2ia\tau_1\frac{1}{k} - (2a^2\tau_1^2 + 2ia\tau_2)\frac{1}{k^2} + o(k^{-2})\right),\end{aligned} \tag{5.43}$$

while

$$\frac{1}{\mu_k} = \frac{a}{\pi k}\left(1 + \frac{a\tau(k)}{k\pi}\right)^{-1} = \frac{a}{k\pi} - \frac{a^2\tau_0}{k^2\pi^2} + o(k^{-2}). \tag{5.44}$$

Using (5.11), $D_1(\mu_k) = 0$ can be written as

$$\mu_k^{-\gamma}\psi_1(\mu_k) + \mu_k^{-\gamma}\psi_4(\mu_k)e^{-2i\tau_k a} = 0, \tag{5.45}$$

where γ is the highest μ-power in $\psi_1(\mu)$ and $\psi_4(\mu)$. Substituting (5.28), (5.29), (5.43) and (5.44) into (5.45) and comparing the coefficients of k^0, k^{-1} and k^{-2}, we obtain the next theorem.

Theorem 5.2. *For $g \in C^1[0,a]$ and $h \in C[a,b]$, there exists a positive integer k_0 such that the eigenvalues λ_k, $k \in \mathbb{Z}$, counted with multiplicity, of problem* (1.2a), (2.2)–(2.8), *can be enumerated in such a way that the eigenvalues λ_k are pure imaginary for $|k| < k_0$, and $\lambda_{-k} = -\overline{\lambda_k}$ for $k \geq k_0$, where $\lambda_k = \mu_k^2$ and μ_k have the asymptotic representations*

$$\mu_k = k\frac{\pi}{a} + \tau_0 + \frac{\tau_1}{k} + \frac{\tau_2}{k^2} + o(k^{-2}),$$

and the numbers τ_0, τ_1, τ_2 are as follows:

Case 1.

$$\tau_0 = -\frac{\pi}{2a}, \qquad \tau_1 = \frac{i}{\pi\alpha} + \frac{1}{4}\frac{G(a)}{\pi},$$
$$\tau_2 = \frac{i}{2\pi\alpha} - \frac{a}{\pi^2\alpha^2} + \frac{1}{8}\frac{G(a)}{\pi} - \frac{1}{4}\frac{a}{\pi^2}(g(0) + g(a)).$$

Case 2.

$$\tau_0 = -\frac{\pi}{4a}, \qquad \tau_1 = \frac{i}{2\pi\alpha} + \frac{1}{4}\frac{G(a)}{\pi},$$
$$\tau_2 = \frac{i}{8\pi\alpha} - \frac{a}{4\pi^2\alpha^2} + \frac{1}{16}\frac{G(a)}{\pi} - \frac{1}{4}\frac{a}{\pi^2}g(0).$$

Case 3.

$$\tau_0 = \frac{\pi}{4a}, \qquad \tau_1 = \frac{i}{\pi\alpha} + \frac{1}{4}\frac{G(a)}{\pi},$$
$$\tau_2 = -\frac{1}{4}\frac{i}{\pi\alpha} - \frac{a}{\pi^2\alpha^2} - \frac{1}{16}\frac{G(a)}{\pi} - \frac{1}{4}\frac{a}{\pi^2}g(a).$$

Case 4.

$$\tau_0 = -\frac{\pi}{2a}, \qquad \tau_1 = \frac{i}{2\pi\alpha} + \frac{1}{4}\frac{G(a)}{\pi},$$
$$\tau_2 = \frac{i}{4\pi\alpha} - \frac{a}{4\pi^2\alpha^2} + \frac{1}{8}\frac{G(a)}{\pi}.$$

Case 5.

$$\tau_0 = -\frac{\pi}{4a}, \qquad \tau_1 = \frac{i}{\pi\alpha} + \frac{1}{4}\frac{G(a)}{\pi},$$
$$\tau_2 = \frac{i}{4\pi\alpha} - \frac{a}{\pi^2\alpha^2} + \frac{1}{16}\frac{G(a)}{\pi} - \frac{1}{4}\frac{a}{\pi^2}g(a).$$

Case 6.

$$\tau_0 = -\frac{\pi}{a}, \qquad \tau_1 = \frac{i}{2\pi\alpha} + \frac{1}{4}\frac{G(a)}{\pi},$$
$$\tau_2 = \frac{i}{2\pi\alpha} - \frac{1}{4}\frac{a}{\pi^2\alpha^2} + \frac{1}{4}\frac{G(a)}{\pi}.$$

Case 7.

$$\tau_0 = -\frac{\pi}{2a}, \qquad \tau_1 = \frac{i}{\pi\alpha} + \frac{1}{4}\frac{G(a)}{\pi},$$
$$\tau_2 = \frac{i}{2\pi\alpha} - \frac{a}{\pi^2\alpha^2} + \frac{1}{8}\frac{G(a)}{\pi} + \frac{1}{4}\frac{a}{\pi^2}(3g(0) - g(a)).$$

Case 8.

$$\tau_0 = -\frac{5\pi}{4a}, \qquad \tau_1 = \frac{i}{2\pi\alpha} + \frac{1}{4}\frac{G(a)}{\pi},$$
$$\tau_2 = \frac{5i}{8\pi\alpha} - \frac{1}{4}\frac{a}{\pi^2\alpha^2} + \frac{5}{16}\frac{G(a)}{\pi} + \frac{3}{4}\frac{a}{\pi^2}g(0).$$

In particular, the number of pure imaginary eigenvalues is odd.

Remark 5.3. We have seen that the function h in differential equation (1.2a) does not influence the first four terms of the asymptotic expansion of the eigenvalues. Hence, we can easily compare our results with those in **[3, 5, 6, 7]**, where the differential equation is (1.2a) with $h = 0$. In **[5]**, we considered the case of two λ-dependent boundary conditions at one endpoint, whereas in **[6, 7]**, we investigated the cases of three and four λ-dependent boundary conditions, respectively.

The proof of the asymptotic expansions follows the same pattern. Firstly, the leading term ϕ_0 of the characteristic determinant of the boundary value problem is found, and its zeros give the leading terms. Secondly, Rouché's theorem guarantees that an asymptotic expansion exists, which has the leading terms found in the first step. And, thirdly, an asymptotic expansion of the fundamental system is used successively to find the higher order terms in the asymptotic expansions. This last step is, in principle, rather straightforward by solving linear and first order differential equations, see (5.4)–(5.6), (5.43), (5.45). Although this can be computed to arbitrary order, we have restricted our calculations to the first four terms since higher terms become more and more lengthy. We have used the computer algebra program `Sage` to double check some of these calculations.

The functions ϕ_0 in Cases 1, 3, and 7 did not appear in [**5, 6, 7**]. Case 3 in this paper was investigated in [**3**], where, however, only the first three terms of the asymptotics of the eigenvalues were provided.

Remark 5.4. Comparing the asymptotic expansions in [**5**]–[**7**] with those in this paper, we see that the leading term in all expansions is $k\pi a^{-1}$. Hence, the length a of the interval $[0,a]$ can be recovered from the leading term of the asymptotic expansion of the eigenvalues. We recall that the eigenvalues λ_k are indexed in such a way that $\lambda_{-k} = \overline{\lambda_k}$ for all eigenvalues which are not pure imaginary. Therefore, the index set is $\mathbb{Z}$ or $\mathbb{Z} \setminus \{0\}$, depending upon whether the number of pure imaginary eigenvalues is odd or even. Although it appears that there is some pattern regarding the parity of pure imaginary eigenvalues, it is not that straightforward; in the case of two λ-dependent boundary conditions, the number of pure imaginary eigenvalues can be even or odd. If we know the parity of the pure imaginary eigenvalues, we can obtain some additional information regarding the boundary conditions of the underlying spectral problem. The values of τ_0, τ_1 and τ_2 give a finer classification of the boundary conditions. However, we will not present the details here. Those are left to another publication, once we have completed the spectral asymptotics for additional classes of boundary conditions. Of course, the more detailed the asymptotic expansion, the better the chance that the boundary conditions can be uniquely recovered from the terms in the expansion. This is a major reason why we have considered the first four terms of the expansions. Further terms in the expansion will be longer and will become difficult or impossible to evaluate with computer algebra.

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