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KRULL DIMENSION AND UNIQUE FACTORIZATION IN HURWITZ POLYNOMIAL RINGS

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ABSTRACT. Let R be a commutative ring with identity, and let $R[x]$ be the collection of polynomials with coefficients in R. We observe that there are many multiplications in $R[x]$ such that, together with the usual addition, $R[x]$ becomes a ring that contains R as a subring. These multiplications belong to a class of functions λ from $\mathbb{N}_0$ to $\mathbb{N}$. The trivial case when $\lambda(i) = 1$ for all i gives the usual polynomial ring. Among nontrivial cases, there is an important one, namely, the case when $\lambda(i) = i!$ for all i. For this case, it gives the well-known Hurwitz polynomial ring $R_H[x]$. In this paper, we study Krull dimension and unique factorization in $R_H[x]$. We show in general that $\dim R \leq \dim R_H[x] \leq 2\dim R + 1$. When the ring R is Noetherian we prove that $\dim R \leq \dim R_H[x] \leq \dim R + 1$. A condition for the ring R is also given in order to determine whether $\dim R_H[x] = \dim R$ or $\dim R_H[x] = \dim R + 1$ in this case. We show that $R_H[x]$ is a unique factorization domain, respectively, a Krull domain, if and only if R is a unique factorization domain, respectively, a Krull domain, containing all of the rational numbers.

1. Introduction. In this paper, a *ring* always means a commutative ring with identity. Let R be a ring, and let

$$R[x] = \left\{ \sum_{i=0}^{n} a_i x^i \mid n \geq 0,\ a_i \in R \right\}$$

be the collection of polynomials with coefficients in R. With the usual addition '+' and multiplication '$\cdot$,' $R[x]$ becomes a ring that

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contains R as a subring. This polynomial ring is an important object in commutative algebra and has been widely studied.

While standard multiplication in $R[x]$ is usually considered, in general, many other multiplications in $R[x]$ exist such that, together with the usual addition, $R[x]$ is still a ring that contains R as a subring. For example, let $\mathbb{N}_0$, respectively $\mathbb{N}$, be the set of nonnegative, respectively positive, integers, and let $\lambda : \mathbb{N}_0 \to \mathbb{N}$ be any function such that $\lambda(0) = 1$ and $\lambda(i)\lambda(j)$ divides $\lambda(i+j)$ in $\mathbb{N}$ for each i and j. Identifying the positive integer $\alpha_{i,j} = (\lambda(i+j))/(\lambda(i)\lambda(j))$ with the element $\alpha_{i,j} \cdot 1$ in R, we define a multiplication $*$ in $R[x]$ by

$$\left(\sum_{i=0}^{n} a_i x^i\right) * \left(\sum_{j=0}^{m} b_j x^j\right) = \sum_{k=0}^{n+m} \left(\sum_{i+j=k} \alpha_{i,j} a_i b_j\right) x^k.$$

With this new multiplication, $R[x]$ is also a ring containing R as a subring, see Section 2. We denote this ring by $(R[x], \lambda)$. With this observation, the usual polynomial ring $R[x]$ is a special case of $(R[x], \lambda)$ when λ is trivial, i.e., $\lambda(i) = 1$ for all i, and hence, $\alpha_{i,j} = 1$ for all i and j.

Among nontrivial cases, there is the important case where $\lambda(i) = i!$ for all i. In this case,

$$\alpha_{i,j} = \frac{\lambda(i+j)}{\lambda(i)\lambda(j)} = \frac{(i+j)!}{i!j!} = \binom{i+j}{i}$$

is a binomial coefficient, and the corresponding ring $(R[x], \lambda)$ is the well-known Hurwitz polynimial ring which is denoted by $R_H[x]$ in this paper (the term "H" stands for "Hurwitz").

Further, a product of two power series can also be defined in the same way, giving the Hurwitz power series ring $R_H[[x]]$. This type of product was first considered by Hurwitz **[11]** and was further studied in **[6, 7, 21]**.

Closely related to the power series ring, the Hurwitz power series ring has been shown to have many interesting properties, including applications in differential algebra **[14, 15]**. Notably, considered as formal functions, Hurwitz power series provide formal solutions to homogeneous linear ordinary differential equations **[15]**, see also **[16]**. Other properties of Hurwitz polynomials and Hurwitz power series may be found in **[1, 2, 3, 4, 5, 8, 17, 18]**.

In this paper, we study the Krull dimension and unique factorization properties in the Hurwitz polynomial ring $R_H[x]$, a very important subring of the Hurwitz power series ring $R_H[[x]]$. We show in general that

$$\dim R \le \dim R_H[x] \le 2\dim R + 1$$

is similar to the result for usual polynomial rings, see **[20]**:

$$\dim R + 1 \le \dim R[x] \le 2\dim R + 1.$$

If R is a Noetherian ring, then so is $R[x]$. In this case, by using Krull's principal ideal theorem, it can be shown that $\dim R[x] = \dim R+1$, see, for example, **[13]**. Unfortunately, $R_H[x]$ is not necessarily a Noetherian ring if R is ([**5**]). Therefore, Krull's principal ideal theorem cannot be applied to determine $\dim R_H[x]$ as in the usual polynomial ring case when R is a Noetherian ring. However, we show that a similar result still holds for $\dim R_H[x]$: the upper bound $2\dim R + 1$ is reduced to $\dim R + 1$. This means that, if R is a Noetherian ring, then

$$\dim R_H[x] = \dim R \quad \text{or} \quad \dim R_H[x] = \dim R + 1.$$

In this case, a condition on R is also given in order to determine whether $\dim R_H[x] = \dim R$ or $\dim R_H[x] = \dim R + 1$.

It is well known that, if R is a unique factorization domain (UFD), then so is the polynomial ring $R[x]$. For the Hurwitz polynomial ring $R_H[x]$, we show that $R_H[x]$ is a UFD if and only if R is a UFD containing $\mathbb{Q}$ if and only if R is a UFD and $R_H[x] \cong R[x]$. The Krull domain is a generalization of UFDs. With a more technical proof we can show that the same result holds for a Krull domain R, that is, $R_H[x]$ is a Krull domain if and only if R is a Krull domain containing $\mathbb{Q}$ if and only if R is a Krull domain and $R_H[x] \cong R[x]$.

2. Multiplications in $R[x]$. In this section, we show that, in general, there are many multiplications in $R[x]$ such that, together with the usual addition, $R[x]$ becomes a ring containing R as a subring.

Let $\lambda : \mathbb{N}_0 \to \mathbb{N}$ be any function such that $\lambda(0) = 1$ and $\lambda(i)\lambda(j)$ divides $\lambda(i + j)$ in $\mathbb{N}$ for each i and j. Let

$$\alpha_{i,j} = \frac{\lambda(i + j)}{\lambda(i)\lambda(j)}.$$

Then, $\alpha_{i,j}$ is a positive integer. Note that $\alpha_{i,j}\alpha_{i+j,k} = \alpha_{i,j+k}\alpha_{j,k}$ for each i, j, and k. Let $\mathcal{F}$ be the collection of such functions λ. For each $\lambda \in \mathcal{F}$, we define a multiplication $*$ in R by

$$\left(\sum_{i=0}^{n} a_i x^i\right) * \left(\sum_{j=0}^{m} b_j x^j\right) = \sum_{k=0}^{n+m} \left(\sum_{i+j=k} \alpha_{i,j} a_i b_j\right) x^k. \tag{2.1}$$

In order to show that this multiplication is associative, we only need to show that

$$(x^i * x^j) * x^k = x^i * (x^j * x^k)$$

for each i, j, and k. However, this follows from the fact that $\alpha_{i,j}\alpha_{i+j,k} = \alpha_{i,j+k}\alpha_{j,k}$ for each i, j, and k. With this new multiplication (and the usual addition), $R[x]$ is a ring. This ring is denoted by $(R[x], \lambda)$. Assumption $\lambda(0) = 1$ guarantees that $\alpha_{i,j} = 1$ if either $i = 0$ or $j = 0$. It follows that $1 \in R$ is also the identity of $(R[x], \lambda)$. Furthermore, taking the product of two elements in R is identical to taking their product in $(R[x], \lambda)$, which implies that $(R[x], \lambda)$ contains R as a subring.

Example 2.1. Let $\lambda(i) = 1$ for all $i \in \mathbb{N}_0$. Then, $\alpha_{i,j} = 1$ for each i and j. In this case, the multiplication obtained from λ is the usual multiplication in $R[x]$, and we obtain the usual polynomial ring $R[x]$.

Example 2.2. Let $\lambda(i) = i!$ for all $i \in \mathbb{N}_0$. Then, $\lambda(0) = 1$, and $\lambda(i)\lambda(j)$ divides $\lambda(i+j)$ in $\mathbb{N}$ for each i and j since

$$\frac{\lambda(i+j)}{\lambda(i)\lambda(j)} = \frac{(i+j)!}{i!j!} = \binom{i+j}{i}$$

is a positive integer. Therefore, $\lambda \in \mathcal{F}$. The corresponding ring $(R[x], \lambda)$ is the well-known Hurwitz polynomial ring, denoted by $R_H[x]$ and studied in the following sections in this paper.

Example 2.3. In general, one can construct a function λ in $\mathcal{F}$ as follows. First, define $\lambda(0) = 1$. Choose any $a_1 \in \mathbb{N}$, and let $\lambda(1) = a_1$. We can then define all $\lambda(n)$ by using induction on n. Suppose that we have defined $\lambda(0), \lambda(1), \ldots, \lambda(n)$ with $n \geq 1$ such that $\lambda(i)\lambda(j)$ divides $\lambda(i+j)$ in $\mathbb{N}$ for all $i, j \geq 0$ with $i + j \leq n$. Choose any $a_{n+1} \in \mathbb{N}$, and let

$$\lambda(n+1) = a_{n+1} \prod_{\substack{i+j=n+1 \\ 1\le i\le j\le n}} \lambda(i)\lambda(j).$$

Since $\lambda(0) = 1$, this definition guarantees that $\lambda(i)\lambda(j)$ divides $\lambda(i+j)$ for all $i, j \ge 0$ with $i + j \le n + 1$. Therefore, we obtain a function $\lambda \in \mathcal{F}$ and the corresponding ring $(R[x], \lambda)$.

Remark 2.4. More generally, whenever there is a set $\{\alpha_{i,j} \mid i, j \in \mathbb{N}_0\}$ of elements in R such that

(i) $\alpha_{i,j} = 1$ if either $i = 0$ or $j = 0$,
(ii) $\alpha_{i,j}\alpha_{i+j,k} = \alpha_{i,j+k}\alpha_{j,k}$ in R for all i, j and k,

a multiplication $*$ in $R[x]$ can be defined by (2.1) so that, together with the usual addition, $R[x]$ becomes a ring containing R as a subring.

3. Krull dimension in $R_H[x]$.

In this section, we study the Krull dimension of the Hurwitz polynomial ring $R_H[x]$ over R. Note that, if $\operatorname{char} R \neq 0$, then $\dim R_H[x] = \dim R$ [**5,** Section 7]. Hence, when studying the Krull dimension of $R_H[x]$, we may always assume that $\operatorname{char} R = 0$.

The following proposition, see [**1,** Proposition 1], is useful.

Proposition 3.1. *$R_H[x]$ is a domain if and only if R is a domain with* $\operatorname{char} R = 0$.

Theorem 3.2. *If R is a ring such that $\mathbb{Q} \subseteq R$, then $R_H[x] \cong R[x]$, and hence,* $\dim R_H[x] = \dim R[x]$.

Proof. If $\mathbb{Q} \subseteq R$, then the map $\varphi : R[x] \to R_H[x]$ defined by $\varphi(\sum_{i=0}^{n} a_i x^i) = \sum_{i=0}^{n} i! a_i x^i$ is a ring isomorphism, see, for example, [**5,** Theorem 1.4]. $\square$

Lemma 3.3. *If R is a ring, then any three different prime ideals $Q_1 \subset Q_2 \subset Q_3$ in $R_H[x]$ cannot contract to the same prime ideal in R.*

Proof. Suppose, on the contrary, that there exist prime ideals $Q_1 \subset Q_2 \subset Q_3$ in $R_H[x]$ having the same contraction to R. Let

$$P = Q_1 \cap R = Q_2 \cap R = Q_3 \cap R.$$

We have a ring epimorphism

$$R_H[x] \longrightarrow R_H[x]/P_H[x] \cong (R/P)_H[x].$$

Let $\overline{Q}_1 \subset \overline{Q}_2 \subset \overline{Q}_3$ be the images of $Q_1 \subset Q_2 \subset Q_3$ in $(R/P)_H[x]$. Then $\overline{Q}_i \cap (R/P) = (0)$, for all $i = 1, 2, 3$. If we let $(R/P)^* = (R/P) \setminus \{0\}$, then

$$(\overline{Q}_1)_{(R/P)^*} \subset (\overline{Q}_2)_{(R/P)^*} \subset (\overline{Q}_3)_{(R/P)^*}$$

is a chain of prime ideals of length 2 in $((R/P)_H[x])_{(R/P)^*} \cong K_H[x]$, where K is the quotient field of R/P. This is a contradiction since $\dim K_H[x] \leq 1$. Indeed, if $\operatorname{char} K \neq 0$, then $\dim K_H[x] = \dim K = 0$. If $\operatorname{char} K = 0$, then $\mathbb{Q} \subseteq K$, and hence, $\dim K_H[x] = \dim K[x] = 1$. $\square$

Let $\phi : R_H[x] \to R$ be the natural ring homomorphism mapping each polynomial in $R_H[x]$ to its constant term. Hence, if P is a prime ideal in R, then $\phi^{-1}(P)$ is a prime ideal in $R_H[x]$.

Theorem 3.4. *If R is a finite-dimensional ring with* $\operatorname{char} R = 0$, *then*

$$\dim R \leq \dim R_H[x] \leq 2 \dim R + 1.$$

Furthermore, if $\mathbb{Q} \subseteq R$ or R is a domain, then $\dim R + 1 \leq \dim R_H[x]$.

Proof. It follows from Lemma 3.3 that $\dim R_H[x] \leq 2 \dim R + 1$. Now, let $n = \dim R$, and let

$$P_0 \subset P_1 \subset \cdots \subset P_n$$

be a chain of prime ideals of length n in R. Then

$$\phi^{-1}(P_0) \subset \phi^{-1}(P_1) \subset \cdots \subset \phi^{-1}(P_n)$$

is a chain of prime ideals of the same length in $R_H[x]$. This shows that $\dim R_H[x] \geq n$. If $\mathbb{Q} \subseteq R$, then $R_H[x] \cong R[x]$, and hence, $\dim R_H[x] = \dim R[x] \geq \dim R + 1$. If R is a domain, then $R_H[x]$ is also a domain, by Proposition 3.1. This means that (0) is a prime ideal in $R_H[x]$, and hence,

$$(0) \subset \phi^{-1}(P_0) \subset \phi^{-1}(P_1) \subset \cdots \subset \phi^{-1}(P_n)$$

is a chain of prime ideals of length $n+1$ in $R_H[x]$. Therefore, $\dim R_H[x] \geq n+1$. $\square$

If $\operatorname{char} R \neq 0$, then $\dim R_H[x] = \dim R$. Combining this with Theorem 3.4, we obtain the next general theorem.

Theorem 3.5. *If R is a finite-dimensional ring, then*

$$\dim R \leq \dim R_H[x] \leq 2 \dim R + 1.$$

Furthermore, if $\mathbb{Q} \subseteq R$ or R is a domain with $\operatorname{char} R = 0$, then $\dim R + 1 \leq \dim R_H[x]$.

We now study $\dim R_H[x]$ when R is a Noetherian ring. Our purpose is to reduce the upper bound $2 \dim R + 1$ in Theorem 3.5 to $\dim R + 1$. Since $R_H[x]$ may not be a Noetherian ring in this case, Krull's principal ideal theorem cannot be applied.

The next lemma plays an important role in proving the desired result.

Lemma 3.6. *Let R be a Noetherian ring. If P is a prime ideal of R such that $\operatorname{ht} P = 1$, i.e., P is a height 1 prime ideal, and $\operatorname{char} R/P = 0$, then $\operatorname{ht} P_H[x] = 1$.*

Proof. Let P_0 be a (minimal) prime ideal contained in P. Note that $P_H[x]$ is a prime ideal in $R_H[x]$. Indeed, $R_H[x]/P_H[x] \cong (R/P)_H[x]$ is a domain since $\operatorname{char} R/P = 0$. By the same reasoning, $(P_0)_H[x]$ is also a prime ideal in $R_H[x]$ ($\operatorname{char} R/P = 0$ implies $\operatorname{char} R/P_0 = 0$). Thus, $\operatorname{ht} P_H[x] \geq 1$.

Now, suppose, on the contrary, that $\operatorname{ht} P_H[x] \geq 2$. Then, there exists a chain $Q_0 \subset Q_1 \subset P_H[x]$ of prime ideals in $R_H[x]$. Let $P_1 = Q_1 \cap R$. Then $P_1 \subset P$. Since $\operatorname{ht} P = 1$, P_1 is a minimal prime ideal. Thus, $P_1 = Q_0 \cap R = Q_1 \cap R$. We have the following.

(i) R/P_1 is a Noetherian domain.
(ii) $\operatorname{ht} P/P_1 = 1$.
(iii) $\operatorname{char}(R/P_1)/(P/P_1) = \operatorname{char} R/P = 0$.

Hence, by passing to R/P_1, we may assume that R is a domain. It follows that the ring homomorphism

$$\varphi : R[x] \longrightarrow R_H[x]$$

defined by

$$\varphi\biggl(\sum_{i=0}^{k} a_i x^i\biggr) = \sum_{i=0}^{k} i!a_i x^i$$

is a ring monomorphism.

Claim 1. $P_H[x] \cap \varphi(R[x]) = \varphi(P[x])$. It is clear that $\varphi(P[x]) \subseteq P_H[x] \cap \varphi(R[x])$. For the other containment, let $f = \sum_{i=0}^{k} b_i x^i \in P_H[x]$, $b_i \in P$. If $f \in \varphi(R[x])$, then $f = \sum_{i=0}^{k} i!a_i x^i$ for some $a_i \in R$. Thus, $i!a_i = b_i \in P$ for all i. Since $\operatorname{char} R/P = 0$, $P \cap \mathbb{Z} = (0)$. It follows that $i! \notin P$, and hence, $a_i \in P$ for all i.

Claim 2. $Q_1 \cap \varphi(R[x]) = P_H[x] \cap \varphi(R[x])$. Consider the chain

$$Q_0 \cap \varphi(R[x]) \subseteq Q_1 \cap \varphi(R[x]) \subseteq P_H[x] \cap \varphi(R[x])$$

of prime ideals in $\varphi(R[x])$. Note that $Q_1 \cap \varphi(R[x]) \neq (0)$. Indeed, taking any $0 \neq f = \sum_{i=0}^{k} b_i x^i \in Q_1$, we have $0 \neq k!f \in Q_1 \cap \varphi(R[x])$. Since R is Noetherian, $P[x]$ is a height 1 prime ideal in $R[x]$. By Claim 1, $P_H[x] \cap \varphi(R[x])$ is a height 1 prime ideal in $\varphi(R[x])$. Since $\varphi(R[x])$ is a domain and $Q_1 \cap \varphi(R[x]) \neq (0)$,

$$Q_1 \cap \varphi(R[x]) = P_H[x] \cap \varphi(R[x]).$$

Claim 3. $Q_1 = P_H[x]$. Let $f = \sum_{i=0}^{k} b_i x^i \in P_H[x]$. Then $k!f \in P_H[x] \cap \varphi(R[x]) = Q_1 \cap \varphi(R[x]) \subseteq Q_1$. We have

$$Q_1 \cap \mathbb{Z} = (Q_1 \cap R) \cap \mathbb{Z} = P_1 \cap \mathbb{Z} \subseteq P \cap \mathbb{Z} = (0).$$

Therefore, $k! \notin Q_1$ and $f \in Q_1$.

Claim 3 contradicts the assumption that $Q_1 \subset P_H[x]$. Therefore, $\operatorname{ht} P_H[x] = 1$. $\square$

Remark 3.7. If P is a prime ideal of a Noetherian ring R such that $\operatorname{ht} P = 1$, then $\operatorname{ht} P[x] = 1$. Indeed, $\operatorname{ht} P[x] \geq 1$ is obvious. If P is minimal over aR, then $P[x]$ is minimal over $aR[x]$. Krull's principal ideal theorem, [**13,** Theorem 142], shows that $\operatorname{ht} P[x] \leq 1$. The same argument cannot be applied in order to show that $\operatorname{ht} P_H[x] \leq 1$ in

Lemma 3.6 since $R_H[x]$ may not be a Noetherian ring. In fact, $R_H[x]$ is a Noetherian ring if and only if R is a Noetherian ring and $\mathbb{Q} \subseteq R$, see [**5,** Corollary 7.7].

Theorem 3.8. *If R is a finite-dimensional Noetherian ring with* $\operatorname{char} R = 0$*, then*

$$\dim R \leq \dim R_H[x] \leq \dim R + 1.$$

Furthermore, $\dim R_H[x] = \dim R + 1$ *if one of the following holds.*

(i) $\mathbb{Q} \subseteq R$.
(ii) *R is a domain.*
(iii) $\dim R = 0$*, i.e., R is an Artinian ring.*

Proof. We show $\dim R_H[x] \leq \dim R + 1$ by using induction on $\dim R$. If $\dim R = 0$, then $\dim R_H[x] \leq 1$ by Theorem 3.5. Suppose that $\dim R = n \geq 1$ and that the result holds for any ring with dimension $< n$. We show that a chain of prime ideals of length $n+2$ in $R_H[x]$ does not exist. Suppose, on the contrary, that such a chain exists, say,

$$Q_0 \subset Q_1 \subset Q_2 \subset \cdots \subset Q_{n+2}.$$

Let $P = Q_2 \cap R$. Since $Q_0 \subset Q_1 \subset Q_2$ cannot contract to the same prime ideal in R, P is not a minimal prime ideal of R, i.e., $\operatorname{ht} P \geq 1$. We have a ring epimorphism $R_H[x] \to R_H[x]/P_H[x] \cong (R/P)_H[x]$. Let

$$\overline{Q}_2 \subset \cdots \subset \overline{Q}_{n+2}$$

be the images of $Q_2 \subset \cdots \subset Q_{n+2}$ in $(R/P)_H[x]$.

Case 1. $\operatorname{char} R/P \neq 0$. In this case,

$$\dim(R/P)_H[x] = \dim(R/P) \leq \dim R - \operatorname{ht} P \leq n - 1.$$

This is a contradiction since the chain $\overline{Q}_2 \subset \cdots \subset \overline{Q}_{n+2}$ has length n.

Case 2. $\operatorname{char} R/P = 0$. By the induction hypothesis,

$$\dim(R/P)_H[x] \leq \dim(R/P) + 1 \leq \dim R - \operatorname{ht} P + 1 \leq \dim R = n.$$

Since the chain $\overline{Q}_2 \subset \cdots \subset \overline{Q}_{n+2}$ has length n and $(R/P)_H[x]$ is a domain, we must have $\operatorname{ht} P = 1$ and $\overline{Q}_2 = (0)$. The latter equality means $P_H[x] = Q_2$, and hence, $\operatorname{ht} P_H[x] \geq 2$. However, this is impossible by Lemma 3.6.

Therefore, every chain of prime ideals in $R_H[x]$ must have length $\leq n+1$. This concludes the proof of $\dim R_H[x] \leq \dim R + 1$.

If $\mathbb{Q} \subseteq R$ or R is a domain, then Theorem 3.4 shows that $\dim R+1 \leq \dim R_H[x]$. Thus, $\dim R_H[x] = \dim R + 1$. This proves (i) and (ii).

If R is an Artinian ring, then it is a finite product of local Artinian rings, say,

$$R = R_1 \times R_2 \times \cdots \times R_t.$$

Since $\operatorname{char} R = 0$, $\operatorname{char} R_i = 0$ for some i. Hence, if M_i is the prime ideal of R_i, then $\operatorname{char} R_i/M_i = 0$ (since M_i is the nilradical of R_i). Since $\mathbb{Q} \subseteq R_i/M_i$ (note that R_i/M_i is a field),

$$\dim(R_i/M_i)_H[x] = \dim R_i/M_i + 1 = 1.$$

We have

$$\dim R_H[x] \geq \dim(R_i)_H[x] \geq \dim(R_i/M_i)_H[x] = 1.$$

Hence, (iii) is proved. □

If $\operatorname{char} R \neq 0$, then $\dim R_H[x] = \dim R$. Adding this to Theorem 3.8, we obtain the following.

Theorem 3.9. *If R is a finite-dimensional Noetherian ring, then*

$$\dim R \leq \dim R_H[x] \leq \dim R + 1.$$

Furthermore, $\dim R_H[x] = \dim R + 1$ *if one of the following holds.*

(i) $\mathbb{Q} \subseteq R$.
(ii) *R is a domain with* $\operatorname{char} R = 0$.
(iii) $\dim R = 0$*, i.e., R is an Artinian ring, and* $\operatorname{char} R = 0$.

By Theorem 3.8, for a finite-dimensional Noetherian ring R with $\operatorname{char} R = 0$, $\dim R_H[x]$ is either $\dim R$ or $\dim R + 1$. If $\dim R = 0$, i.e., R is Artinian, then $\dim R_H[x] = \dim R + 1$.

We now show that, if $\dim R \geq 1$, then $\dim R_H[x]$ can be either $\dim R$ or $\dim R + 1$. Of course, if $\mathbb{Q} \subseteq R$ or R is a domain, then $\dim R_H[x] = \dim R + 1$.

The next example illustrates the case where $\dim R_H[x] = \dim R$.

Example 3.10. For any $n \geq 1$, there exists a Noetherian ring R with $\operatorname{char} R = 0$ such that $\dim R_H[x] = \dim R = n$.

Proof. Let R_1 be a Noetherian ring with $\operatorname{char} R_1 = 0$ and $\dim R_1 \leq n-1$, and let R_2 be a Noetherian ring with $\operatorname{char} R_2 \neq 0$ and $\dim R_2 = n$. Let $R = R_1 \times R_2$. Then, R is a Noetherian ring with $\operatorname{char} R = 0$ and $\dim R = n$. We have

$$\dim R_H[x] = \max\{\dim(R_1)_H[x], \dim(R_2)_H[x]\}.$$

From $\dim(R_1)_H[x] \leq \dim R_1 + 1 \leq n$ and $\dim(R_2)_H[x] = \dim R_2 = n$, we obtain $\dim R_H[x] = n$. $\square$

In general, for a Noetherian ring R with $\dim R = n \geq 1$, we can determine when $\dim R_H[x] = \dim R$ and when $\dim R_H[x] = \dim R + 1$ by the next theorem.

Theorem 3.11. *Let R be a Noetherian ring with $\dim R = n \geq 1$. Then the following are equivalent*:

(i) $\dim R_H[x] = \dim R = n$.
(ii) *For a minimal prime ideal P of R,* $\operatorname{char} R/P = 0$ *implies* $\dim R/P \leq n - 1$.

Proof. If $\operatorname{char} R \neq 0$, then (i) and (ii) are always true. Hence, we assume that $\operatorname{char} R = 0$.

(i) $\Rightarrow$ (ii). Suppose that P is a minimal ideal of R such that char $R/P = 0$. Since R/P is a domain,

$$n = \dim R_H[x] \geq \dim(R/P)_H[x] = \dim(R/P) + 1$$

by Theorem 3.9. Hence, $n - 1 \geq \dim R/P$.

(ii) $\Rightarrow$ (i). Suppose, on the contrary, that $\dim R_H[x] = n + 1$. Then, there exists a chain of prime ideals

$$Q_0 \subset Q_1 \subset \cdots \subset Q_{n+1}$$

in $R_H[x]$. Let $P = Q_0 \cap R$. Then $P_H[x] \subseteq Q_0$. Let

$$\overline{Q}_0 \subset \overline{Q}_1 \subset \cdots \overline{Q}_{n+1}$$

be the images of $Q_0 \subset Q_1 \subset \cdots \subset Q_{n+1}$ in $(R/P)_H[x]$ (through the epimorphism $R_H[x] \to R_H[x]/P_H[x] \cong (R/P)_H[x]$). Then $\overline{Q}_0 \subset \overline{Q}_1 \subset$

$\cdots \overline{Q}_{n+1}$ is a chain of prime ideals in $(R/P)_H[x]$ of length $n+1$. This means that $\dim(R/P)_H[x] \geq n+1$. However, we can see that this is impossible by considering the next two cases.

Case 1. $\operatorname{char} R/P = 0$. By the assumption, $\dim R/P \leq n-1$. We have

$$\dim(R/P)_H[x] = \dim R/P + 1 \leq (n-1)+1 = n.$$

Case 2. $\operatorname{char} R/P \neq 0$. In this case, we have

$$\dim(R/P)_H[x] = \dim R/P \leq \dim R = n. \quad \square$$

Example 3.12. Using Theorem 3.11, we conclude that $\dim R_H[x] = \dim R = n$ for the ring $R = R_1 \times R_2$ in the proof of Example 3.10. Indeed, minimal ideals of R are of the form $P_1 \times R_2$ or $R_1 \times P_2$ (where P_i is a minimal prime ideal of R_i). Since $\operatorname{char} R_2 \neq 0$, $\operatorname{char} R/(R_1 \times P_2) = \operatorname{char} R_2/P_2 \neq 0$. Thus, we only need to consider $\operatorname{char} R/(P_1 \times R_2)$. However, whether or not $\operatorname{char} R/(P_1 \times R_2) = 0$, we always have $\dim R/(P_1 \times R_2) = \dim R_1/P_1 \leq \dim R_1 \leq n-1$. By Theorem 3.11, $\dim R_H[x] = \dim R = n$.

4. Unique factorizations in $R_H[x]$. In this section, we study unique factorization properties in $R_H[x]$. We may assume that $\operatorname{char} R = 0$ since $R_H[x]$ is not a domain if $\operatorname{char} R \neq 0$.

Lemma 4.1. *If R is a domain with* $\operatorname{char} R = 0$*, then x is an irreducible element in $R_H[x]$.*

Proof. Suppose that there exist

$$f = \sum_{i=0}^{r} b_i x^i, \qquad g = \sum_{j=0}^{s} c_j x^j \quad \text{in } R_H[x]$$

such that $x = f * g$. We may assume that $r \leq s$. Since $R_H[x]$ is a domain, by comparing the degree on both sides of $x = f * g$, we see that $r = 0$ and $s = 1$. It follows that $1 = b_0 c_1$, and hence, $f = b_0$ is a unit. $\square$

Theorem 4.2. *The following are equivalent for a ring R:*

(i) *$R_H[x]$ is a* UFD.

(ii) *R is a* UFD *and* $\mathbb{Q} \subseteq R$.
(iii) *R is a* UFD *and* $R_H[x] \cong R[x]$.

Proof.

(i) $\Rightarrow$ (ii). Suppose that $R_H[x]$ is a UFD. In particular, $R_H[x]$ is a domain. Thus, R is a domain with $\operatorname{char} R = 0$ (Proposition 3.1). If we can show that $\mathbb{Q} \subseteq R$, then we are done. Indeed, if $\mathbb{Q} \subseteq R$, then $R[x] \cong R_H[x]$ is a UFD, and hence, R is a UFD. We show that $\mathbb{Q} \subseteq R$ by proving the converse. Suppose, on the contrary, that $\mathbb{Q} \not\subseteq R$. Then, there exists a prime number p that is not a unit in R. We have

$$\underbrace{x * x * \cdots * x}_{p \text{ times}} = p!x^p = (p!) * x^p.$$

By Lemma 4.1, x is a prime element in $R_H[x]$ (since $R_H[x]$ is a UFD). Thus, x divides either $p!$ or x^p in $R_H[x]$. It is easy to see that x cannot divide $p!$. Thus, x divides x^p. Therefore, there exists an element f in $R_H[x]$ such that $x * f = x^p$, and hence, f must have the form $f = bx^{p-1}$ for some $b \in R$. We have

$$pbx^p = x * (bx^{p-1}) = x * f = x^p.$$

This means that $pb = 1$ and p is a unit in R, a contradiction.

(ii) $\Rightarrow$ (iii). If $\mathbb{Q} \subseteq R$, then $R_H[x] \cong R[x]$ (Theorem 3.2).

(iii) $\Rightarrow$ (i). It follows from the well-known result that, if R is a UFD, then so is $R[x]$, see **[10]**. $\square$

Corollary 4.3. *If R is a* UFD*, then $R_H[x]$ is never a* UFD *unless it is isomorphic to $R[x]$.*

Example 4.4. By Theorem 4.2, $\mathbb{Z}_H[x]$ is not a UFD.

Let R be a domain, and let K be the quotient field of R. For an ideal I of R, the v-operation is defined by $I_v = (I^{-1})^{-1}$, where, for $J \subseteq K$, J^{-1} is defined by $J^{-1} = \{z \in K \mid zJ \subseteq R\}$. The t-operation is defined by $I_t = \cup J_v$, where the union is taken over all finitely generated ideals J of R such that $J \subseteq I$. An ideal I in R is called a *t-invertible* ideal if $(II^{-1})_t = R$. A domain R is called a *Krull domain* if there is a non-empty collection of prime ideals $\{P_\alpha\}$ in R such that $R = \cap R_{P_\alpha}$, each R_{P_α} is a PID, and every non-zero element of R is contained in only finitely many P_αs. A UFD is always a Krull domain [**9**]. A domain R is a Krull domain if and only if every proper principal ideal is a t-product of t-invertible prime ideals, see [**12,** Theorem 3.9].

Theorem 4.5. *The following are equivalent for a ring R:*

(i) *$R_H[x]$ is a Krull domain.*
(ii) *R is a Krull domain and $\mathbb{Q} \subseteq R$.*
(iii) *R is a Krull domain and $R_H[x] \cong R[x]$.*

Proof.

(i) $\Rightarrow$ (ii). Suppose that $R_H[x]$ is a Krull domain, in particular, $R_H[x]$ is a domain. Hence, R is a domain with $\operatorname{char} R = 0$. If we can show that $\mathbb{Q} \subseteq R$, then we are done. Indeed, if $\mathbb{Q} \subseteq R$, then $R[x] \cong R_H[x]$ is a Krull domain, and hence, R is a Krull domain.

We now show that $\mathbb{Q} \subseteq R$. Suppose, on the contrary, that $\mathbb{Q} \not\subseteq R$. Let p be the smallest prime number that is not a unit in R (so that $(p-1)!$ is a unit in R). Since $R_H[x]$ is a Krull domain, we write the principal ideal (x) as a t-product of t-invertible prime ideals, $(x) = (P_1^{e_1} P_2^{e_2} \cdots P_l^{e_l})_t$. Since

$$p!x^p = \underbrace{x * x * \cdots * x}_{p \text{ times}},$$

and $(p-1)!$ is a unit in R,

$$(p) * (x^p) = \underbrace{(x) * (x) * \cdots * (x)}_{p \text{ times}} = (P_1^{pe_1} P_2^{pe_2} \cdots P_l^{pe_l})_t.$$

It follows that $(p) = (P_1^{f_1} P_2^{f_2} \cdots P_l^{f_l})_t$, where $0 \le f_i \le pe_i$, $i = 1, 2, \ldots, l$.

Claim. $f_i \le (p-1)e_i$, $i = 1, 2, \ldots, l$. Since $p^k!x^p = \underbrace{x * x * \cdots * x}_{p^k \text{ times}}$,

$$(p^k!) * (x^p) = (P_1^{p^k e_1} P_2^{p^k e_2} \cdots P_l^{p^k e_l})_t.$$

The number of p-factors in $p^k!$ in $\mathbb{N}$ is

$$1 + p + \cdots + p^{k-1} = \frac{p^k - 1}{p - 1}.$$

This implies that $(p^k-1)/(p-1)f_i \leq p^k e_i$, and hence, $(p^k-1)/(p^k(p-1))f_i \leq e_i$. Letting k go to ∞, we obtain $f_i/(p-1) \leq e_i$, and the claim is proved.

Now, since $(p-1)!x^{p-1} = \underbrace{x * x * \cdots * x}_{p-1 \text{ times}}$ and $(p-1)!$ is a unit in R,

$$(x^{p-1}) = (P_1^{(p-1)e_1} P_2^{(p-1)e_2} \cdots P_l^{(p-1)e_l})_t \subseteq (P_1^{f_1} P_2^{f_2} \cdots P_l^{f_l})_t = (p).$$

Thus, $x^{p-1} = p * (ax^{p-1})$ for some $a \in R$, which shows that p is a unit, a contradiction.

(ii) $\Rightarrow$ (iii). If $\mathbb{Q} \subseteq R$, then $R_H[x] \cong R[x]$.

(iii) $\Rightarrow$ (i). It follows from the fact that, if R is a Krull domain, then so is $R[x]$, see, for example, **[19]**. $\square$

Corollary 4.6. *If R is a Krull domain, then $R_H[x]$ is never a Krull domain unless it is isomorphic to $R[x]$.*

Example 4.7. By Theorem 4.5, $\mathbb{Z}_H[x]$ is not a Krull domain. Therefore, $R_H[x]$ may not be a Krull domain even when R is a principal ideal domain (PID) with characteristic zero.

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