ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 47, Number 2, 2017

JORDAN σ-DERIVATIONS OF PRIME RINGS

TSIU-KWEN LEE

ABSTRACT. Let R be a noncommutative prime ring with extended centroid C and with $Q_{mr}(R)$ its maximal right ring of quotients. From the viewpoint of functional identities, we give a complete characterization of Jordan σ-derivations of R with σ an epimorphism. Precisely, given such a Jordan σ-derivation $\delta\colon R \to Q_{mr}(R)$, it is proved that either δ is a σ-derivation or a derivation $d\colon R \to Q_{mr}(R)$ and a unit $u \in Q_{mr}(R)$ exist such that $\delta(x) = ud(x) + \mu(x)u$ for all $x \in R$, where $\mu\colon R \to C$ is an additive map satisfying $\mu(x^2) = 0$ for all $x \in R$. In addition, if σ is an X-outer automorphism, then δ is always a σ-derivation.

1. Introduction. Throughout this paper, R is always a prime ring with $Q_{mr}(R)$ the maximal right ring of quotients of R and with $Q_s(R)$ the symmetric Martindale ring of quotients of R. It is known that $R \subseteq Q_s(R) \subseteq Q_{mr}(R)$. The overrings $Q_s(R)$ and $Q_{mr}(R)$ of R are still prime rings with the same center, denoted by C, which is a field and is called the *extended centroid* of R. We refer the reader to **[3]** for details.

An additive map $d\colon R \to R$ is called a *derivation*, respectively *Jordan derivation*, if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$, respectively $d(x^2) = d(x)x + xd(x)$ for all $x \in R$. In 1957, Herstein proved that, if R is a prime ring of characteristic not 2, then every Jordan derivation of R is a derivation, see **[6]**. We refer the reader to the references given in **[8]** for more related results. In a recent paper **[9]** the author and Lin studied a slightly generalized definition concerning (Jordan) derivations. Let $R \subseteq S$ be rings. An additive map $\delta\colon R \to S$ is called a derivation, respectively Jordan derivation, if $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in R$, respectively $\delta(x^2) = \delta(x)x + x\delta(x)$ for all $x \in R$. It follows from **[6,**

2010 AMS *Mathematics subject classification.* Primary 16N60, 16R60, 16W25.
Keywords and phrases. Prime ring, (Jordan) σ-derivation, PI, GPI, functional identity.
This work was partly supported by the Ministry of Science and Technology of Taiwan, grant No. MOST 103-2115-M-002 -004 -MY2, and the National Center for Theoretical Sciences, Taipei Office.
Received by the editors on April 27, 2015, and in revised form on July 18, 2015.
DOI:10.1216/RMJ-2017-47-2-511

Theorem 3.1], [**4,** Corollary 6.9] and [**9,** Theorems 1.2, 2.1] that Jordan derivations of a given prime ring are completely characterized as, see [**9,** Theorem 2.2]: If an additive map $\delta\colon R \to Q_{mr}(R)$ is a Jordan derivation, then a derivation $d\colon R \to Q_{mr}(R)$ and an additive map $\mu\colon R \to C$ exist such that $\delta = d + \mu$ and $\mu(x^2) = 0$ for all $x \in R$. The converse is true if the characteristic of R is 2.

Motivated by [**5, 6, 9**], in [**8**], the author studied "Jordan σ-derivations" and "Jordan semiderivations" of prime rings from the viewpoint of functional identities. Clearly, the key point of these results is to determine the structure of Jordan σ-derivations.

Definition. Let σ be an endomorphism of R. An additive map $\delta\colon R \to Q_{mr}(R)$ is called a σ-*derivation*, respectively Jordan σ-derivation, with associated endomorphism σ if $\delta(xy) = \delta(x)y + \sigma(x)\delta(y)$ for all $x, y \in R$, respectively $\delta(x^2) = \delta(x)x + \sigma(x)\delta(x)$ for all $x \in R$.

A Jordan σ-derivation $\delta\colon R \to Q_{mr}(R)$ is called X-*inner* if there exists an element $a \in Q_{mr}(R)$ such that $\delta(x) = ax - \sigma(x)a$ for all $x \in R$. Otherwise, δ is called X-*outer*. When $\sigma = 1_R$, the identity map of R, a (Jordan) σ-derivation is merely a (Jordan) derivation. Hence, Jordan 1_R-derivations of R have been completely characterized, see [**6,** Theorem 3.1], [**9,** Theorem 1.1] if $\operatorname{char} R \neq 2$ and [**9,** Theorem 2.2] if $\operatorname{char} R = 2$. In [**8**], the author characterize Jordan σ-derivations $\delta\colon R \to Q_{mr}(R)$ with σ an epimorphism if R is not a GPI-ring. In this paper, we will obtain the same conclusion without the extra assumption that R is not a GPI-ring, see [**8,** Question 2.8]. As a consequence, if σ is an X-outer automorphism, then every Jordan σ-derivation is a σ-derivation. The key point is to solve certain functional identities of prime non-PI rings. Recall that an automorphsim σ of R is called X-inner if there exists a unit $u \in Q_s(R)$ such that $\sigma(x) = uxu^{-1}$ for all $x \in R$. Otherwise, it is called X-outer.

2. Main results. Our goal of the paper is to characterize Jordan σ-derivations of prime rings. The main result is the following.

Theorem 2.1. *Let R be a noncommutative prime ring with an epimorphism σ, and let $\delta\colon R \to Q_{mr}(R)$ be a Jordan σ-derivation. Then, either δ is a σ-derivation or a derivation $d\colon R \to Q_{mr}(R)$ and a unit*

$u \in Q_s(R)$ exist such that $\delta(x) = ud(x) + \mu(x)u$ for all $x \in R$, where $\mu\colon R \to C$ is an additive map satisfying $\mu(x^2) = 0$ for all $x \in R$.

First, we deal with the case of prime PI-rings. For a prime PI-ring R, it is known that $Q_{mr}(R) = RC$.

Theorem 2.2. *Let R be a noncommutative prime* PI*-ring with an epimorphism σ, and let $\delta\colon R \to RC$ be a Jordan σ-derivation. Then, either δ is a σ-derivation or a derivation $d\colon R \to RC$ and a unit $u \in RC$ exist such that $\delta(x) = ud(x) + \mu(x)u$ for all $x \in R$, where $\mu\colon R \to C$ is an additive map satisfying $\mu(x^2) = 0$ for all $x \in R$.*

Proof. By [**8,** Corollary 2.3], if $\operatorname{char} R \neq 2$ then δ is a σ-derivation. Suppose that $\operatorname{char} R = 2$. Let $x, y \in R$. Then $xy + yx = [x, y]$. Linearizing $\delta(x^2) = \delta(x)x + \sigma(x)\delta(x)$, we see that

$$\delta([x, y]) = \delta(x)y + \delta(y)x + \sigma(x)\delta(y) + \sigma(y)\delta(x). \tag{2.1}$$

Since R is a prime PI-ring, it follows from [**13,** Theorem 2] that $Z(R)$, the center of R, is nonzero. Let $0 \neq \beta \in Z(R)$. Replacing y by β in (2.1), we see that

$$(\beta + \sigma(\beta))\delta(x) = \delta(\beta)x + \sigma(x)\delta(\beta).$$

Case 1. There is a $\beta \in Z(R)$ such that $\sigma(\beta) \neq \beta$. Since σ is an epimorphism, $\sigma(\beta) \in Z(R)$. Set $a := (\beta + \sigma(\beta))^{-1}\delta(\beta) \in RC$. Then $\delta(x) = ax - \sigma(x)a$ for all $x \in R$, that is, δ is an X-inner σ-derivation.

Case 2. $\sigma(\beta) = \beta$ for all $\beta \in Z(R)$. Then σ can be uniquely extended to an epimorphism of RC, denoted by $\widetilde{\sigma}$, defined by $\widetilde{\sigma}(x/\beta) = (\sigma(x))/\beta$ for $x \in R$ and $0 \neq \beta \in Z(R)$. Since RC is a finite-dimensional central simple C-algebra, see [**13**], and $\widetilde{\sigma}(\alpha) = \alpha$ for all $\alpha \in C$, $\widetilde{\sigma}$ is a C-linear automorphsim of RC. The Noether-Skolem theorem asserts that there exists a unit $u \in RC$ such that $\widetilde{\sigma}(x) = uxu^{-1}$ for $x \in RC$. Hence, $\delta(x^2) = \delta(x)x + uxu^{-1}\delta(x)$ for all $x \in R$. Clearly, the map $x \mapsto u^{-1}\delta(x)$ for $x \in R$ is a Jordan derivation of R into RC. In view of [**9,** Theorem 2.2], a derivation $d\colon R \to RC$ and an additive map $\mu\colon R \to C$ exist such that $u^{-1}\delta(x) = d(x) + \mu(x)$ for all $x \in R$, where $\mu(x^2) = 0$ for all $x \in R$. So $\delta(x) = ud(x) + \mu(x)u$ for all $x \in R$, as asserted. $\square$

By Theorem 2.2 together with [**8,** Theorem 2.7], in order to prove Theorem 2.1, we have to handle the case where R is a prime GPI-ring but is not a PI-ring. To prove the case, we need a result concerning functional identities.

We first introduce some notation. For any maps $f\colon R^{r-1} \to Q_{mr}(R)$ and $g\colon R^{r-2} \to Q_{mr}(R)$ we write

$$f^i(\overline{x}_r) = f(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_r)$$

and

$$g^{ij}(\overline{x}_r) = g^{ji}(\overline{x}_r) = g(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_r),$$

where $\overline{x}_r = (x_1, \ldots, x_r) \in R^r$ and $1 \le i < j \le r$.

We are now ready to state the key result, which will be used in the proof of Theorem 2.1 and is also interesting in itself. Although it has a more general form, we prove only the following for our purpose.

Theorem 2.3. *Let R be a prime ring, which is not a* PI-*ring, and let σ be an* X-*outer automorphism of R. Further, suppose that $E_{i1}, F_{\ell s}\colon R^{r-1} \to Q_{mr}(R)$ are $(r-1)$-additive maps, where $1 \le i$, $\ell \le r$ and $s = 1, 2$. Suppose that*

$$\sum_{i=1}^{r} E_{i1}^{i}(\overline{x}_r)x_i + \sum_{\ell=1}^{r} x_\ell F_{\ell 1}^{\ell}(\overline{x}_r) + \sum_{\ell=1}^{r} x_\ell^{\sigma} F_{\ell 2}^{\ell}(\overline{x}_r) \in C \tag{2.2}$$

for all $\overline{x}_r \in R^r$. Then there exist a nonzero ideal I of R, $(r-2)$-additive maps $p_{i1\ell s}\colon I^{r-2} \to Q_{mr}(R)$ and $(r-1)$-additive maps $\lambda_{i1}\colon I^{r-1} \to C$, where $1 \le i$, $\ell \le r$ and $s = 1, 2$, such that

$$E_{i1}^{i}(\overline{x}_r) = \sum_{\substack{1 \le \ell \le r \\ \ell \ne i}} x_\ell p_{i1\ell 1}^{i\ell}(\overline{x}_r) + \sum_{\substack{1 \le \ell \le r \\ \ell \ne i}} x_\ell^{\sigma} p_{i1\ell 2}^{i\ell}(\overline{x}_r) + \lambda_{i1}^{i}(\overline{x}_r),$$

$$F_{\ell 1}^{\ell}(\overline{x}_r) = -\sum_{\substack{1 \le i \le r \\ i \ne \ell}} p_{i1\ell 1}^{i\ell}(\overline{x}_r)x_i - \lambda_{\ell 1}^{\ell}(\overline{x}_r)$$

and

$$F_{\ell 2}^{\ell}(\overline{x}_r) = -\sum_{\substack{1\le i\le r\\ i\neq \ell}} p_{i1\ell 2}^{i\ell}(\overline{x}_r)x_i$$

for all $\overline{x}_r \in I^r$, where $1 \le i,\ \ell \le r$.

The proof of Theorem 2.3 will be given in the next section. A right ideal ρ of R is called dense if ρ is a dense submodule of R_R, that is, given $x, y \in R$ with $y \neq 0$, there exists an element $r \in R$ such that $xr \in \rho$ and $yr \neq 0$.

Lemma 2.4. *Let σ be an automorphism of R. Suppose that $\delta\colon R \to Q_{mr}(R)$ is a σ-derivation. Then δ can be uniquely extended to a σ-derivation from $Q_{mr}(R)$ to itself.*

Proof. It is known that σ can be uniquely extended to an automorphism of $Q_{mr}(R)$, denoted by σ also. Let $q \in Q_{mr}(R)$. Choose a dense right ideal ρ of R such that $q\rho \subseteq R$. Let $f\colon \rho \to Q_{mr}(R)$ be the map defined by $f(x) = \delta(qx) - \sigma(q)\delta(x)$ for $x \in \rho$. We claim that f is a right R-module map. Indeed, let $x \in \rho$ and $r \in R$. Then

$$\begin{aligned} f(xr) &= \delta(qxr) - \sigma(q)\delta(xr)\\ &= \big(\delta(qx)r + \sigma(qx)\delta(r)\big) - \sigma(q)\big(\delta(x)r + \sigma(x)\delta(r)\big)\\ &= \big(\delta(qx) - \sigma(q)\delta(x)\big)r\\ &= f(x)r, \end{aligned}$$

as claimed. Note that $\rho Q_{mr}(R)$ is a dense right ideal of $Q_{mr}(R)$ by the fact that ρ is a dense right ideal of R. Moreover, R is also a dense submodule of $Q_{mr}(R)_R$. Thus, f can be uniquely extended to a right $Q_{mr}(R)$-module map from $\rho Q_{mr}(R)$ into $Q_{mr}(R)$, denoted by $\widetilde{f}$. Since $Q_{mr}(Q_{mr}(R)) = Q_{mr}(R)$, $\widetilde{f}\colon \rho Q_{mr}(R) \to Q_{mr}(R)$ can be realized as an element of $Q_{mr}(R)$. We define such an element as $\widetilde{\delta}(q)$, that is, $\widetilde{f}(y) = \widetilde{\delta}(q)y$ for $y \in \rho Q_{mr}(R)$. Thus, $\widetilde{\delta}\colon Q_{mr}(R) \to Q_{mr}(R)$ and $\widetilde{\delta}(x) = \delta(x)$ for $x \in R$. It is routine to check that $\widetilde{\delta}$ is a σ-derivation. Clearly, such an extension is unique. $\square$

Lemma 2.5 ([**8,** Lemma 2.6]). *Suppose that R is not a* PI-*ring,* $\operatorname{char} R = 2$*, and let σ be an endomorphism of R. Let $\delta, A, B\colon R \to$*

$Q_{mr}(R)$ be additive maps satisfying

$$\delta(xy) + \sigma(x)\delta(y) = A(y)x + B(x)y$$

for all $x, y \in R$. Then B is a σ-derivation. In addition, if σ is an X-*outer automorphism, then $A = 0$.*

For the next lemma, we refer the reader to the proof of [**8,** Case 2, Theorem 2.7].

Lemma 2.6. *Let R be a noncommutative prime ring with σ an* X-*outer automorphism. If R is not a* GPI-*ring, then every Jordan σ-derivation from R into $Q_{mr}(R)$ is a σ-derivation.*

Theorem 2.7. *Let R be a noncommutative prime ring with σ an* X-*outer automorphism. Then, every Jordan σ-derivation from R into $Q_{mr}(R)$ is a σ-derivation.*

Proof. By Theorem 2.2 and Lemma 2.6, we may assume that R is a prime GPI-ring but is not a PI-ring. By [**8,** Corollary 2.3], we may assume further that $\text{char} R = 2$. Let $x, y, z \in R$. Then, by the identity $[xy, z] + [zx, y] + [yz, x] = 0$ and using (2.1) to expand $\delta([xy, z]) + \delta([zx, y]) + \delta([yz, x])$, we see that

$$\begin{aligned} &\big(\delta(yz) + \delta(y)z\big)x + \big(\delta(zx) + \delta(z)x\big)y + \big(\delta(xy) + \delta(x)y\big)z \\ &\quad = \sigma(x)\big(\delta(yz) + \sigma(y)\delta(z)\big) + \sigma(y)\big(\delta(zx) + \sigma(z)\delta(x)\big) \\ &\qquad + \sigma(z)\big(\delta(xy) + \sigma(x)\delta(y)\big) \end{aligned} \tag{2.3}$$

for all $x, y, z \in R$. In view of Theorem 2.3, a nonzero ideal I of R and additive maps $A, B \colon I \to Q_{mr}(R)$ exist such that

$$\delta(xy) + \sigma(x)\delta(y) = A(y)x + B(x)y \tag{2.4}$$

for all $x, y \in I$. Note that $Q_{mr}(I) = Q_{mr}(R)$. It follows from Lemma 2.5 that $A = 0$ on I and B is a σ-derivation on I.

Replacing y with x in (2.4) and noting that δ is a Jordan σ-derivation, we see that

$$\delta(x)x = \delta(x^2) + \sigma(x)\delta(x) = B(x)x,$$

and so,

$$\big(B(x)+\delta(x)\big)x = 0 \quad \text{for all } x \in I.$$

Set $h := B + \delta$. Then $h(x)y = h(y)x$ for all $x, y \in I$. Thus, $h(x)yz = h(y)xz = h(xz)y$ for all $x, y, z \in I$. Since R is not commutative, neither is I. Thus, a $z \in I$ exists such that 1 and z are linearly independent over C. It follows from [**11,** Theorem 2(a)] that $h(x) = 0$ for all $x \in I$, that is, $B = \delta$ on I. Since B is a σ-derivation on I, so is δ on I. Note that $Q_{mr}(I) = Q_{mr}(R)$. In view of Lemma 2.4, B can be uniquely extended to a σ-derivation $\widetilde{B}\colon R \to Q_{mr}(R)$.

We claim that $\delta = \widetilde{B}$ on R. This implies that δ is itself a σ-derivation. Let $g := \delta - \widetilde{B}$. Then $g\colon R \to Q_{mr}(R)$ is also a Jordan σ-derivation and $g(I) = 0$. Our aim is to show that $g = 0$. Let $x \in R$ and $w \in I$. Then,

$$\begin{aligned} g(xw + wx) &= g(x)w + g(w)x + \sigma(x)g(w) + \sigma(w)g(x) \\ &= g(x)w + \sigma(w)g(x), \end{aligned}$$

implying that $g(x)w = \sigma(w)g(x)$ as $g(xw + wx) = 0 = g(w)$. Since σ is X-outer, it follows that $g(x) = 0$ for all $x \in R$, as asserted. □

Proof of Theorem 2.1. By [**8,** Theorem 2.4], if σ is not injective, then δ is an X-inner σ-derivation. Thus, we may assume further that σ is an automorphism since σ is an epimorphism of R and, moreover, $\operatorname{char} R = 2$, see [**8,** Corollary 2.3].

In view of Theorem 2.7 we are done if σ is an X-outer automorphism of R. Thus, we may assume that σ is X-inner. There exists a unit $u \in Q_s(R)$ such that $\sigma(x) = uxu^{-1}$ for all $x \in R$. As in the proof of Theorem 2.2, a derivation $d\colon R \to Q_{mr}(R)$ and an additive map $\mu\colon R \to C$ exist such that $u^{-1}\delta(x) = d(x) + \mu(x)$ for $x \in R$, where $\mu(x^2) = 0$ for $x \in R$, that is, $\delta(x) = ud(x) + \mu(x)u$ for all $x \in R$, as asserted. □

3. Proof of Theorem 2.3. In order to prove Theorem 2.3 we need the following result, which is a special case of [**1,** Theorem 1.2].

Theorem 3.1. *Let R be a prime ring, which is not a* GPI-*ring, and let σ be an* X-*outer automorphism of R. Further, suppose that $E_{ij}, F_{\ell s}\colon R^{r-1} \to Q_{mr}(R)$ are $(r-1)$-additive maps, where $1 \leq i$,*

$\ell \le r$ and $1 \le j,\ s \le 2$. Suppose that

$$\sum_{i=1}^{r} E_{i1}^{i}(\overline{x}_r)x_i + \sum_{i=1}^{r} E_{i2}^{i}(\overline{x}_r)x_i^{\sigma} + \sum_{\ell=1}^{r} x_\ell F_{\ell 1}^{\ell}(\overline{x}_r) + \sum_{\ell=1}^{r} x_\ell^{\sigma} F_{\ell 2}^{\ell}(\overline{x}_r) \in V$$

for all $\overline{x}_r \in R^r$, where V is a finite dimensional C-subspace of $Q_{mr}(R)$. Then, there exist unique $(r-2)$-additive maps $p_{ij\ell s}\colon R^{r-2} \to Q_{mr}(R)$ and $(r-1)$-additive maps $\lambda_{ij}\colon R^{r-1} \to C$, where $1 \le i,\ \ell \le r$ and $1 \le j,\ s \le 2$, such that

$$E_{ij}^{i}(\overline{x}_r) = \sum_{\substack{1 \le \ell \le r \\ \ell \ne i}} x_\ell p_{ij\ell 1}^{i\ell}(\overline{x}_r) + \sum_{\substack{1 \le \ell \le r \\ \ell \ne i}} x_\ell^{\sigma} p_{ij\ell 2}^{i\ell}(\overline{x}_r) + \lambda_{ij}^{i}(\overline{x}_r)$$

and

$$F_{\ell s}^{\ell}(\overline{x}_r) = -\sum_{\substack{1 \le i \le r \\ i \ne \ell}} p_{i1\ell s}^{i\ell}(\overline{x}_r)x_i - \sum_{\substack{1 \le i \le r \\ i \ne \ell}} p_{i2\ell s}^{i\ell}(\overline{x}_r)x_i^{\sigma} - \lambda_{\ell s}^{\ell}(\overline{x}_r)$$

for all $\overline{x}_r \in R^r$, where $1 \le i,\ \ell \le r$ and $1 \le j,\ s \le 2$.

From now on, we assume that R is a prime ring, which is not a PI-ring, and let $\sigma\colon R \to R$ be an X-outer automorphism. By $I \lhd R$, we mean that I is an ideal of R.

To begin, we need the following, see [**7,** Proof of Proposition 2], [**12,** Theorem 3.13] or [**10,** Theorem 1.1].

Theorem 3.2 ([**7**], Kharchenko)**.** *Let R be a prime* GPI*-ring, and let τ be an automorphism of R. Suppose that $\tau(\beta) = \beta$ for all $\beta \in C$. Then τ is X-inner.*

For $x \in R$, we define $\deg(x)$ to be the minimal algebraic degree over C if x is algebraic over C and $\deg(x) = \infty$ otherwise. For a subset T of R, we define $\deg(T) = \sup\{\deg(t) \mid t \in T\}$. It is known that $\deg(R) = \infty$ if R is not a PI-ring. For any map $f\colon R^{r-1} \to Q_{mr}(R)$ and $t \ne i$, we let

$$f^{i}(\overline{x}_r; \{y\}_t) = f(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_r),$$

where $z_j = x_j$ if $j \ne t$ and $z_t = y$.

Lemma 3.3. *Suppose that* $E_i, F_\ell\colon R^{r-1} \to Q_{mr}(R)$ *are* $(r-1)$*-additive maps satisfying*

$$\sum_{i=1}^{r} x_i E_i^i(\overline{x}_r) + \sum_{\ell=1}^{r} x_\ell^\sigma F_\ell^\ell(\overline{x}_r) \in C \tag{3.1}$$

for all $\overline{x}_r \in R^r$. *Then there exists a nonzero ideal* I *of* R *such that* $E_i^i = 0 = F_\ell^\ell$ *on* I^r *for* $1 \le i,\ \ell \le r$.

Proof. If R is not a GPI-ring, it follows from Theorem 3.1 that $E_i^i = 0 = F_\ell^\ell$ on R^r for $1 \le i,\ \ell \le r$. The lemma is proved.

Suppose now that R is a GPI-ring but not a PI-ring. Let $A := \{1, 2, \ldots, r\}$ and

$$L := \{\ell \in A \mid \text{there exists } 0 \ne J \lhd R \text{ such that } F_\ell^\ell = 0 \text{ on } J^r\}.$$

We proceed with the proof by induction on $r - |L|$. First, suppose that $L = A$. Then, $F_\ell^\ell = 0$ on J^r for $1 \le \ell \le r$, where J is a nonzero ideal of R. Thus,

$$\sum_{i=1}^{r} x_i E_i^i(\overline{x}_r) \in C$$

for all $\overline{x}_r \in J^r$. Note that $Q_{mr}(J) = Q_{mr}(R)$ and $\deg(R) = \infty$. In view of [**2,** Theorem 2.4], $E_i^i = 0$ on J^r for $1 \le i \le r$, as asserted.

Next, suppose that $r - |L| \ge 1$. We may assume without loss of generality that $r \notin L$, that is, $F_r^r \ne 0$ on U^r for any nonzero ideal U of R. Since σ is X-outer and R is a GPI-ring, it follows from Theorem 3.2 that $\sigma(\beta) \ne \beta$ for some $\beta \in C$. Choose a nonzero ideal K satisfying $\beta K \subseteq R$. Then, by (3.1), we have

$$\begin{aligned} \sum_{i=1}^{r-1} x_i \big(E_i^i(\overline{x}_r; \{\beta x_r\}_r) - \beta E_i^i(\overline{x}_r)\big) \\ + \sum_{\ell=1}^{r-1} x_\ell^\sigma \big(F_\ell^\ell(\overline{x}_r; \{\beta x_r\}_r) - \beta F_\ell^\ell(\overline{x}_r)\big) \\ + x_r^\sigma(\sigma(\beta) - \beta) F_r^r(\overline{x}_r) \in C \end{aligned} \tag{3.2}$$

for all $\overline{x}_r \in K^r$. Choose a nonzero ideal K_1 of R contained in K such that $K_1^{\sigma^{-1}} \subseteq K$. Then, by (3.2), we have

$$\sum_{i=1}^{r-1} x_i \widetilde{E}_i^i(\overline{x}_r) + x_r F_r^r(\overline{x}_r) + \sum_{\ell=1}^{r-1} x_\ell^\sigma \widetilde{F}_\ell^\ell(\overline{x}_r) \in C \tag{3.3}$$

for all $\overline{x}_r \in K_1^r$, where

$$\widetilde{E}_i^i(\overline{x}_r) = (\sigma(\beta) - \beta)^{-1}\big(E_i^i(\overline{x}_r; \{\beta x_r^{\sigma^{-1}}\}_r) - \beta E_i^i(\overline{x}_r; \{x_r^{\sigma^{-1}}\}_r)\big)$$

and

$$\widetilde{F}_\ell^\ell(\overline{x}_r) = (\sigma(\beta) - \beta)^{-1}\big(F_\ell^\ell(\overline{x}_r; \{\beta x_r^{\sigma^{-1}}\}_r) - \beta F_\ell^\ell(\overline{x}_r; \{x_r^{\sigma^{-1}}\}_r)\big).$$

Set

$$L_1 := \{\ell \mid 1 \leq \ell \leq r-1 \text{ there exists } 0 \neq J \lhd R \text{ such that } \widetilde{F}_\ell^\ell = 0 \text{ on } J^r\}.$$

Let $\ell \in \{1, \dots, r-1\}$ be such that $\ell \in L$. Then, there exists a nonzero ideal N of R such that $F_\ell^\ell = 0$ on N^r. Clearly, there exists a nonzero ideal M of R contained in N such that $\widetilde{F}_\ell^\ell = 0$ on M^r, that is, $\ell \in L_1$. Since $r \notin L$, we have $|L| \leq |L_1|$, and so, $r - |L| \geq r - |L_1| > r - 1 - |L_1|$. By the inductive hypothesis, it follows from (3.2) that $F_r^r = 0$ on W^r, where W is a nonzero ideal of R. This is a contradiction. $\square$

Proof of Theorem 2.3. We divide the proof into two cases.

Case 1. R is not a GPI-ring. We let $E_{i2} = 0$ for $1 \leq i \leq r$, where $E_{i2}\colon R^{r-1} \to Q_{mr}(R)$ and rewrite (2.2) as

$$\sum_{i=1}^{r} E_{i1}^i(\overline{x}_r) x_i + \sum_{i=1}^{r} E_{i2}^i(\overline{x}_r) x_i^\sigma + \sum_{\ell=1}^{r} x_\ell F_{\ell 1}^\ell(\overline{x}_r) + \sum_{\ell=1}^{r} x_\ell^\sigma F_{\ell 2}^\ell(\overline{x}_r) \in C \tag{3.4}$$

for all $\overline{x}_r \in R^r$. By Theorem 3.1, there exist unique additive maps $p_{ij\ell s}\colon R^{r-2} \to Q_{mr}(R)$ and $\lambda_{is}\colon R^{r-1} \to C$, $1 \leq i,\ \ell \leq r$ and $s = 1, 2$, such that

$$E_{ij}^i(\overline{x}_r) = \sum_{\substack{1 \leq \ell \leq r \\ \ell \neq i}} x_\ell p_{ij\ell 1}^{i\ell}(\overline{x}_r) + \sum_{\substack{1 \leq \ell \leq r \\ \ell \neq i}} x_\ell^\sigma p_{ij\ell 2}^{i\ell}(\overline{x}_r) + \lambda_{ij}^i(\overline{x}_r) \tag{3.5}$$

and

$$F^{\ell}_{\ell s}(\overline{x}_r) = -\sum_{\substack{1\leq i\leq r\\ i\neq \ell}} p^{i\ell}_{i1\ell s}(\overline{x}_r)x_i - \sum_{\substack{1\leq i\leq r\\ i\neq \ell}} p^{i\ell}_{i2\ell s}(\overline{x}_r)x_i^{\sigma} - \lambda^{\ell}_{\ell s}(\overline{x}_r) \tag{3.6}$$

for all $\overline{x}_r \in R^r$, where $1 \leq i,\, \ell \leq r$ and $1 \leq j,\, s \leq 2$. Since $E_{i2} = 0$ for $1 \leq i \leq r$, it follows from (3.5) with $j = 2$ that

$$\sum_{\substack{1\leq \ell\leq r\\ \ell\neq i}} x_\ell p^{i\ell}_{i2\ell 1}(\overline{x}_r) + \sum_{\substack{1\leq \ell\leq r\\ \ell\neq i}} x_\ell^{\sigma} p^{i\ell}_{i2\ell 2}(\overline{x}_r) + \lambda^{i}_{i2}(\overline{x}_r) = 0.$$

But, R is not a PI-ring. By Lemma 3.3, there exists a nonzero ideal I of R such that

$$p^{i\ell}_{i2\ell 1} = 0 = p^{i\ell}_{i2\ell 2} \quad \text{and} \quad \lambda^{i}_{i2} = 0 \text{ on } I^r,$$

where $1 \leq i,\, \ell \leq r$ with $i \neq \ell$. Hence, (3.6) is reduced to

$$F^{\ell}_{\ell s}(\overline{x}_r) = -\sum_{\substack{1\leq i\leq r\\ i\neq \ell}} p^{i\ell}_{i1\ell s}(\overline{x}_r)x_i - \lambda^{\ell}_{\ell s}(\overline{x}_r)$$

for all $\overline{x}_r \in I^r$, where $1 \leq \ell \leq r$ and $1 \leq s \leq 2$, as asserted.

Case 2. R is a GPI-ring. Let $A := \{1, 2, \ldots, r\}$ and

$$L := \{\ell \in A \mid \text{ there exists } 0 \neq J \lhd R \text{ such that } F^{\ell}_{\ell 2} = 0 \text{ on } J^r\}.$$

We proceed with the proof by induction on $r - |L|$. Suppose first that $L = A$. Then, $F^{\ell}_{\ell 2}(\overline{x}_r) = 0$ for all $\overline{x}_r \in U^r$ for $1 \leq \ell \leq r$, where U is a nonzero ideal of R. Thus, (2.2) is reduced to

$$\sum_{i=1}^{r} E^{i}_{i1}(\overline{x}_r)x_i + \sum_{\ell=1}^{r} x_\ell F^{\ell}_{\ell 1}(\overline{x}_r) \in C$$

for all $\overline{x}_r \in U^r$. Note that $Q_{mr}(U) = Q_{mr}(R)$. Since R is not a PI-ring, $\deg(R) = \infty$. In view of [**2,** Corollary 2.11], there exist additive maps $p_{i1\ell 1}\colon R^{r-2} \to Q_{mr}(R)$ and $\lambda_{i1}\colon R^{r-1} \to C$, $1 \leq i,\, \ell \leq r$, such that

$$E^{i}_{i1}(\overline{x}_r) = \sum_{\substack{1\leq \ell\leq r\\ \ell\neq i}} x_\ell p^{i\ell}_{i1\ell 1}(\overline{x}_r) + \lambda^{i}_{i1}(\overline{x}_r)$$

and

$$F^{\ell}_{\ell 1}(\overline{x}_r) = -\sum_{\substack{1\leq i\leq r\\ i\neq \ell}} p^{i\ell}_{i1\ell 1}(\overline{x}_r)x_i - \lambda^{\ell}_{\ell 1}(\overline{x}_r)$$

for all $\overline{x}_r \in R^r$, where $1 \leq i,\, \ell \leq r$, as asserted.

Suppose next that $r - |L| \geq 1$. We may assume without loss of generality that $r \notin L$, that is, $F^r_{r2} \neq 0$ on U^r for any nonzero ideal U of R. Since σ is X-outer and R is a GPI-ring, it follows from Theorem 3.2 that $\sigma(\beta) \neq \beta$ for some $\beta \in C$. Choose a nonzero ideal J of R such that $\beta J \subseteq R$. By (2.2), we have

$$\begin{aligned}
&\sum_{i=1}^{r-1} x_i\big(E^i_i(\overline{x}_r; \{\beta x_r\}_r) - \beta E^i_i(\overline{x}_r)\big)\\
&\quad + \sum_{\ell=1}^{r-1} x_\ell\big(F^{\ell}_{\ell 1}(\overline{x}_r; \{\beta x_r\}_r) - \beta F^{\ell}_{\ell 1}(\overline{x}_r)\big)\\
&\quad + \sum_{\ell=1}^{r-1} x^{\sigma}_\ell\big(F^{\ell}_{\ell 2}(\overline{x}_r; \{\beta x_r\}_r) - \beta F^{\ell}_{\ell 2}(\overline{x}_r)\big)\\
&\quad + x^{\sigma}_r(\sigma(\beta) - \beta)F^r_{r2}(\overline{x}_r) \in C
\end{aligned}$$

for all $\overline{x}_r \in J^r$. Then,

$$\sum_{i=1}^{r-1} x_i \widetilde{E}^i_i(\overline{x}_r) + \sum_{\ell=1}^{r-1} x_\ell \widetilde{F}^{\ell}_{\ell 1}(\overline{x}_r) + \sum_{\ell=1}^{r-1} x^{\sigma}_\ell \widetilde{F}^{\ell}_{\ell 2}(\overline{x}_r) + x^{\sigma}_r F^r_{r2}(\overline{x}_r) \in C \tag{3.7}$$

for all $\overline{x}_r \in J^r$, where

$$\begin{aligned}
\widetilde{E}^i_{i1}(\overline{x}_r) &= (\sigma(\beta) - \beta)^{-1}\big(E^i_i(\overline{x}_r; \{\beta x_r\}_r) - \beta E^i_i(\overline{x}_r)\big),\\
\widetilde{F}^{\ell}_{\ell 1}(\overline{x}_r) &= (\sigma(\beta) - \beta)^{-1}\big(F^{\ell}_{\ell 1}(\overline{x}_r; \{\beta x_r\}_r) - \beta F^{\ell}_{\ell 1}(\overline{x}_r)\big),
\end{aligned}$$

and

$$\widetilde{F}^{\ell}_{\ell 2}(\overline{x}_r) = (\sigma(\beta) - \beta)^{-1}\big(F^{\ell}_{\ell 2}(\overline{x}_r; \{\beta x_r\}_r) - \beta F^{\ell}_{\ell 2}(\overline{x}_r)\big).$$

Choose a nonzero ideal J_1 of R contained in J such that $J_1^{\sigma^{-1}} \subseteq J$. It follows from (3.7) that

$$\sum_{i=1}^{r-1} x_i \widetilde{E}_i^i(\overline{x}_r; \{x_r^{\sigma^{-1}}\}_r) + \sum_{\ell=1}^{r-1} x_\ell \widetilde{F}_{\ell 1}^\ell(\overline{x}_r; \{x_r^{\sigma^{-1}}\}_r) + x_r F_{r2}^r(\overline{x}_r) + \sum_{\ell=1}^{r-1} x_\ell^\sigma \widetilde{F}_{\ell 2}^\ell(\overline{x}_r; \{x_r^{\sigma^{-1}}\}_r) \in C \tag{3.8}$$

for all $\overline{x}_r \in J_1^r$. Set $H_{\ell 2}^\ell(\overline{x}_r) := \widetilde{F}_{\ell 2}^\ell(\overline{x}_r; \{x_r^{\sigma^{-1}}\}_r)$ for $\overline{x}_r \in J_1^r$, $1 \leq \ell \leq r-1$ and

$$L_1 := \{\ell \mid 1 \leq \ell \leq r-1, \text{ there exists } 0 \neq J \triangleleft R \text{ such that } H_{\ell 2}^\ell = 0 \text{ on } J^r\}.$$

Let $\ell \in \{1, \ldots, r-1\}$ be such that $\ell \in L$. Then, there exists a nonzero ideal N of R such that $F_\ell^\ell = 0$ on N^r. Clearly, there exists a nonzero ideal M of R contained in N such that $H_{\ell 2}^\ell = 0$ on M^r, that is, $\ell \in L_1$. Since $r \notin L$, we have $|L| \leq |L_1|$, and so, $r - |L| \geq r - |L_1| > r - 1 - |L_1|$.

By the inductive hypothesis, the F_{r2}^r in (3.8) can be solved, that is, there exists a nonzero ideal J_2 of R contained in J_1, $(r-2)$-additive maps $p_{i1\ell 2} \colon J_2^{r-2} \to Q_{mr}(R)$ such that

$$F_{r2}^r(\overline{x}_r) = -\sum_{\substack{1 \leq i \leq r \\ i \neq r}} p_{i1r2}^{ir}(\overline{x}_r) x_i \tag{3.9}$$

for all $\overline{x}_r \in J_2^r$. It follows from (2.2) together with (3.9) that

$$\sum_{i=1}^{r-1} \big(E_{i1}^i(\overline{x}_r) - x_r^\sigma p_{i1r2}^{ir}(\overline{x}_r)\big) x_i + E_{r1}^r(\overline{x}_r) x_r + \sum_{\ell=1}^{r} x_\ell F_{\ell 1}^\ell(\overline{x}_r) + \sum_{\ell=1}^{r-1} x_\ell^\sigma F_{\ell 2}^\ell(\overline{x}_r) \in C \tag{3.10}$$

for all $\overline{x}_r \in J_2^r$. By induction, these $F_{\ell 2}^\ell$ in (3.10) can be solved as follows:

$$F_{\ell 2}^\ell(\overline{x}_r) = -\sum_{\substack{1 \leq i \leq r \\ i \neq \ell}} p_{i1\ell 2}^{i\ell}(\overline{x}_r) x_i \tag{3.11}$$

for all $\overline{x}_r \in I^r$ and $1 \leq \ell \leq r-1$, where I is a nonzero ideal of R

contained in J_2. By (2.2), (3.9) and (3.11), we have

$$\sum_{i=1}^{r}\big(E^i_{i1}(\overline{x}_r)-\sum_{\substack{1\le\ell\le r\\ \ell\ne i}}x^\sigma_\ell p^{i\ell}_{i1\ell2}(\overline{x}_r)\big)x_i+\sum_{\ell=1}^{r}x_\ell F^\ell_{\ell1}(\overline{x}_r)\in C \tag{3.12}$$

for all $\overline{x}_r\in I^r$. Note that $Q_{mr}(I)=Q_{mr}(R)$. We now apply [**2,** Corollary 2.11] to solve (3.10). Then $(r-2)$-additive maps $p_{i1\ell1}\colon I^{r-2}\to Q_{mr}(R)$ and additive maps $\lambda_{i1}\colon I^{r-1}\to C$ exist such that

$$E^i_{i1}(\overline{x}_r)-\sum_{\substack{1\le\ell\le r\\ \ell\ne i}}x^\sigma_\ell p^{i\ell}_{i1\ell2}(\overline{x}_r)=\sum_{\substack{1\le\ell\le r\\ \ell\ne i}}x_\ell p^{i\ell}_{i1\ell1}(\overline{x}_r)+\lambda^i_{i1}(\overline{x}_r) \tag{3.13}$$

and

$$F^\ell_{\ell1}(\overline{x}_r)=-\sum_{\substack{1\le i\le r\\ i\ne\ell}}p^{i\ell}_{i1\ell1}(\overline{x}_r)x_i-\lambda^\ell_{\ell1}(\overline{x}_r) \tag{3.14}$$

for all $\overline{x}_r\in I^r$, where $1\le i,\ \ell\le r$. The theorem is now proved by (3.9), (3.11), (3.13) and (3.14). $\square$

REFERENCES

1. K.I. Beidar, M. Brešar and M.A. Chebotar, *Generalized functional identities with (anti-)automorphisms and derivations on prime rings*, I, J. Algebra **215** (1999), 644–665.

2. K.I. Beidar and W.S. Martindale, III, *On functional identities in prime rings with involution*, J. Algebra **203** (1998), 491–532.

3. K.I. Beidar, W.S. Martindale, III, and A.A. Mikhalev, *Rings with generalized identities*, in *Monographs and textbooks in pure and applied mathematics*, **196**, Marcel Dekker, Inc., New York, 1996.

4. M. Brešar, M.A. Chebotar and W.S. Martindale, III, *Functional identities*, in *Frontiers in mathematics*, Birkhauser Verlag, Basel, 2007.

5. V. De Filippis, A. Mamouni and L. Oukhtite, *Generalized Jordan semiderivations in prime rings*, Canad. Math. Bull. **58** (2015), 263–270.

6. I.N. Herstein, *Jordan derivations of prime rings*, Proc. Amer. Math. Soc. **8** (1957), 1104–1110.

7. V.K. Kharchenko, *Generalized identities with automorphisms*, Alg. Logik. **14** (1975), 215–237.

8. T.-K. Lee, *Functional identities and Jordan σ-derivations*, Linear Multilin. Alg. **64** (2016), 221–234.

9. T.-K. Lee and J.-H. Lin, *Jordan derivations of prime rings with characteristic two*, Lin. Alg. Appl. **462** (2014), 1–15.

10. T.-K. Lee and K.-S. Liu, *The Skolem-Noether theorem for semiprime rings satisfying a strict identity*, Comm. Alg. **35** (2007), 1949–1955.

11. W.S. Martindale, III, *Prime rings satisfying a generalized polynomial identity*, J. Alg. **12** (1969), 576–584.

12. S. Montgomery, *Fixed rings of finite automorphism groups of associative rings*, Lect. Notes Math. **818**, Springer, Berlin, 1980.

13. L. Rowen, *Some results on the center of a ring with polynomial identity*, Bull. Amer. Math. Soc. **79** (1973), 219–223.

Department of Mathematics, National Taiwan University, Taipei 106, Taiwan
Email address: **tklee@math.ntu.edu.tw**