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APPLICATION OF STRONG DIFFERENTIAL SUPERORDINATION TO A GENERAL EQUATION

R. AGHALARY, P. ARJOMANDINIA AND A. EBADIAN

ABSTRACT. In this paper, we study the notion of strong differential superordination as a dual concept of strong differential subordination, introduced in [**1**]. The notion of strong differential superordination has recently been studied by many authors, see, for example, [**2, 3, 5**]. Let $q(z)$ be an analytic function in $\mathbb{D}$ that satisfies the first order differential equation

$$\theta(q(z)) + F(z)q'(z)\varphi(q(z)) = h(z).$$

Suppose that $p(z)$ is analytic and univalent in the closure of the open unit disk $\overline{\mathbb{D}}$ with $p(0) = q(0)$. We shall find conditions on $h(z), G(z), \theta(z)$ and $\varphi(z)$ such that

$$h(z) \prec\prec \theta(p(z)) + \frac{G(\xi)}{\xi} zp'(z)\varphi(p(z)) \Longrightarrow q(z) \prec p(z).$$

Applications and examples of the main results are also considered.

1. Introduction. Let $\mathcal{H} = \mathcal{H}(\mathbb{D})$ be the class of all analytic functions in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ consisting of functions $f(z)$ of the form

$$f(z) = z + a_2 z^2 + \cdots .$$

For two functions $f, g \in \mathcal{H}$ we say that f is subordinate to g (or g is superordinate to f) and write $f \prec g$ or $f(z) \prec g(z)$ if there exists an analytic function $w(z)$ in $\mathbb{D}$ such that

$$w(0) = 0, \qquad |w(z)| < 1 \quad \text{and} \quad f(z) = g(w(z)),$$

see [**4**]. If g is univalent in $\mathbb{D}$, then

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{D}) \subseteq g(\mathbb{D}).$$

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Suppose that $F(z)$ is analytic and univalent in $\mathbb{D}$ and $F(0) = 0$. The class of F-starlike functions FS^* is defined as follows:

$$FS^* = \left\{ f \in \mathcal{A} : \operatorname{Re}\left(F(z) \frac{f'(z)}{f(z)} \right) > 0 \right\},$$

see [**1**]. If we set $F(z) = z$, then we obtain the usual class of starlike functions.

Let $f(z)$ be analytic in $\mathbb{D}$ and $g(z, \xi)$ analytic in $\mathbb{D} \times \overline{\mathbb{D}}$. We say that $f(z)$ is strongly subordinate to $g(z, \xi)$, or $g(z, \xi)$ is strongly superordinate to $f(z)$, and use $f(z) \prec\prec g(z, \xi)$ if there exists an analytic function $w(z)$ in $\mathbb{D}$ such that

$$w(0) = 0, \qquad |w(z)| < 1 \quad \text{and} \quad f(z) = g(w(z), \xi)$$

for all $\xi \in \overline{\mathbb{D}}$, see [**5**]. If $g(z, \xi)$ is univalent in $\mathbb{D}$ for all $\xi \in \overline{\mathbb{D}}$, then

$$f(z) \prec\prec g(z, \xi) \Longleftrightarrow f(0) = g(0, \xi), \quad \xi \in \overline{\mathbb{D}} \text{ and } f(\mathbb{D}) \subseteq g(\mathbb{D} \times \overline{\mathbb{D}}).$$

A function $L : \mathbb{D} \times [0, \infty) \to \mathbb{C}$ is a subordination (or Loewner) chain if $L(z, t)$ as a function of z is analytic and univalent in $\mathbb{D}$ and is a continuously differentiable function of t on $[0, +\infty)$ for all $z \in \mathbb{D}$, and $L(z, t_1) \prec L(z, t_2)$ when $0 \leq t_1 \leq t_2$.

Throughout this paper, we assume that $F(z)$ and $q(z)$ are analytic in $\mathbb{D}$, $F(0) = 0$ and $p(z)$ is analytic and univalent in $\overline{\mathbb{D}}$ with $p(0) = q(0)$, $G(z)$ is analytic in $\overline{\mathbb{D}}$, $G(0) = 0$ and that θ and φ are analytic in a domain D containing $p(\mathbb{D})$ and $q(\mathbb{D})$, unless expressly stated. We define the analytic function $g(z, \xi)$ in $\mathbb{D} \times \overline{\mathbb{D}}$ by

$$g(z, \xi) = \theta(p(z)) + \frac{G(\xi)}{\xi} z p'(z) \varphi(p(z)). \tag{1.1}$$

In this paper, we aim to find conditions on $h(z)$, $Q(z) = zq'(z)\varphi(q(z))$, $F(z)$ and $G(z)$ such that

$$h(z) \prec\prec g(z, \xi) \Longrightarrow q(z) \prec p(z).$$

In order to prove our main results, we need the next lemmas.

Lemma 1.1 ([**4**]). *Let $L(z, t) = a_1(t) z + a_2(t) z^2 + \cdots$, with $a_1(t) \neq 0$ for all $t \geq 0$ and $\lim_{t \to +\infty} |a_1(t)| = +\infty$. Suppose that $L(z, t)$ as a function of z is analytic in $\mathbb{D}$ and a continuously differentiable function of t on*

$[0, +\infty)$ *for all* $z \in \mathbb{D}$. *Then,* $L(z,t)$ *is a subordination chain if and only if* $\mathrm{Re}[(z\partial L/\partial z)/(\partial L/\partial t)] > 0$ *for all* $z \in \mathbb{D}$ *and* $t \geq 0$.

Lemma 1.2 ([**5**]). *Let* $h(z)$ *be analytic in* $\mathbb{D}$, $q(z) \in \mathcal{H}$, $q(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$, $n \in \mathbb{N}$ *and* $\psi : \mathbb{C}^2 \times \overline{\mathbb{D}} \times \overline{\mathbb{D}} \to \mathbb{C}$. *Suppose that*

$$\psi(q(z), tzq'(z); \zeta, \xi) \in h(\mathbb{D}),$$

where $z \in \mathbb{D}$, $\zeta \in \partial\mathbb{D}$, $\xi \in \overline{\mathbb{D}}$ *and* $0 < t \leq 1/n \leq 1$. *If* $p(z)$ *is analytic and univalent in* $\overline{\mathbb{D}}$, $p(0) = a$ *and* $\psi(p(z), zp'(z); z, \xi)$ *is analytic in* $\mathbb{D} \times \overline{\mathbb{D}}$ *and univalent in* $\mathbb{D}$ *for all* $\xi \in \overline{\mathbb{D}}$, *then*

$$h(z) \prec\prec \psi(p(z), zp'(z); z, \xi) \Longrightarrow q(z) \prec p(z).$$

2. Main results.

Theorem 2.1. *Let* $h(z)$ *be convex* (*univalent*) *in* $\mathbb{D}$. *Suppose that* $q(z)$ *is an analytic solution of the differential equation*

$$\theta(q(z)) + F(z)q'(z)\varphi(q(z)) = h(z), \quad z \in \mathbb{D}.$$

If $g(z, \xi)$ *is given by equation* (1.1) *and is analytic in* $\mathbb{D} \times \overline{\mathbb{D}}$ *and univalent in* $\mathbb{D}$ *for all* $\xi \in \overline{\mathbb{D}}$,

(i) $\theta(q(z)) \prec h(z)$, *and*
(ii) $\theta(q(z)) + (G(\xi)/\xi)Q(z) \in h(\mathbb{D})$, $(Q(z) = zq'(z)\varphi(q(z)),\ z \in \mathbb{D},\ \xi \in \overline{\mathbb{D}})$,

then

$$h(z) \prec\prec g(z, \xi) \Longrightarrow q(z) \prec p(z).$$

Proof. Define the function $\psi : \mathbb{C}^2 \times \overline{\mathbb{D}} \longrightarrow \mathbb{C}$ by

$$\psi(r, s; \xi) = \theta(r) + \frac{G(\xi)}{\xi} s\varphi(r).$$

Then, we have $h(z) \prec\prec \psi(p(z), zp'(z); \xi)$. It is sufficient to show that

$$\psi(q(z), tzq'(z); \xi) \in h(\mathbb{D}), \quad z \in \mathbb{D},\ \xi \in \overline{\mathbb{D}},\ 0 < t \leq 1. \tag{2.1}$$

We have

$$\begin{aligned}\psi(q(z),tzq'(z);\xi) &= \theta(q(z)) + \frac{G(\xi)}{\xi}tzq'(z)\varphi(q(z))\\ &= (1-t)\theta(q(z)) + t\left(\theta(q(z)) + Q(z)\frac{G(\xi)}{\xi}\right).\end{aligned}$$

From (i), (ii) and the convexity of $h(\mathbb{D})$, we conclude that equation (2.1) is satisfied. Now, the result follows from Lemma 1.2. □

Example 2.2. Let A and B be positive real numbers, and let $C < 0$. Suppose that $B > 4A$ and $B + AC \le -1$. Setting $q(z) = 1 - z$, $F(z) = 2Cz/(1-z)^2$, $G(z) = z + z^2$, $\varphi(z) = Az$ and $\theta(z) = 2B/z$, we obtain

$$h(z) = \theta(q(z)) + F(z)q'(z)\varphi(q(z)) = \frac{2B - 2ACz}{1-z}, \quad z \in \mathbb{D}.$$

It is clear that $h(z)$ is convex (univalent) in $\mathbb{D}$ and that $\operatorname{Re}(h(z)) \ge B + AC$. We have

$$\frac{2B}{1-z} = \theta(q(z)) \prec h(z) = \frac{2B - 2ACz}{1-z},$$

and Theorem 2.1 (i) is satisfied. Condition (ii) is

$$\theta(q(z)) + Q(z)\frac{G(\xi)}{\xi} = \frac{2B}{1-z} + A(z^2 + z^2\xi - z - \xi z).$$

By an easy calculation we obtain

$$\begin{aligned}\operatorname{Re}\left(\theta(q(z)) + Q(z)\frac{G(\xi)}{\xi}\right) &= 2B\operatorname{Re}\left(\frac{1}{1-z}\right) + A\operatorname{Re}(z^2 + z^2\xi - z - \xi z)\\ &> B - 4A > 0,\end{aligned}$$

and Theorem 2.1 (ii) is satisfied. Hence, if

$$\frac{2B}{p(z)} + A(1+\xi)zp'(z)p(z)$$

is analytic in $\mathbb{D} \times \overline{\mathbb{D}}$ and univalent in $\mathbb{D}$ for all $\xi \in \overline{\mathbb{D}}$, then

$$\frac{2B - 2ACz}{1-z} \prec\prec \frac{2B}{p(z)} + A(1+\xi)zp'(z)p(z) \Longrightarrow 1 - z \prec p(z).$$

In the case that $h(z)$ is analytic in $\mathbb{D}$, but not convex, we have the next theorem.

Theorem 2.3. *Let $h(z)$ and $q(z)$ be analytic in $\mathbb{D}$ and*

$$\theta(q(z)) + F(z)q'(z)\varphi(q(z)) = h(z), \quad z \in \mathbb{D}.$$

Suppose that $g(z,\xi)$, given by equation (1.1), *is analytic in $\mathbb{D}\times\overline{\mathbb{D}}$ and univalent in $\mathbb{D}$ for all $\xi \in \overline{\mathbb{D}}$. If $\theta'(q(0))q'(0) \neq 0$,*

(i) *$Q(z) = zq'(z)\varphi(q(z))$ is starlike in $\mathbb{D}$;*
(ii) *$\operatorname{Re}[(G(\xi)/\xi)(\varphi(q(z))/\theta'(q(z)))] > 0$, $z \in \mathbb{D}$, $\xi \in \overline{\mathbb{D}}$; and*
(iii) *$\theta(q(z)) + (G(\xi)/\xi)Q(z) \in h(\mathbb{D})$,*

then

$$h(z) \prec\prec g(z,\xi) \Longrightarrow q(z) \prec p(z).$$

Proof. The function $L : \mathbb{D}\times[0,+\infty)\times\overline{\mathbb{D}} \longrightarrow \mathbb{C}$ given by

$$L(z,t;\xi) = \theta(q(z)) + t\frac{G(\xi)}{\xi}Q(z)$$

is analytic in $\mathbb{D}$ for all $t \geq 0$ and $\xi \in \overline{\mathbb{D}}$ and is a continuously differentiable function of t on $[0,+\infty)$ for all $z \in \mathbb{D}$ and $\xi \in \overline{\mathbb{D}}$. We have that

$$\begin{aligned} a_1(t) = \left.\frac{\partial L}{\partial z}\right|_{z=0} &= \theta'(q(0))q'(0) + t\frac{G(\xi)}{\xi}Q'(0) \\ &= \theta'(q(0))q'(0)\left(1 + t\frac{G(\xi)}{\xi}\frac{\varphi(q(0))}{\theta'(q(0))}\right). \end{aligned}$$

Since $t \geq 0$, from (ii), we deduce that $a_1(t) \neq 0$ and $\lim_{t\to+\infty}|a_1(t)| = +\infty$ for all $\xi \in \overline{\mathbb{D}}$. A simple calculation along with (i) and (ii) yields

$$\begin{aligned} \operatorname{Re}\left(\frac{z\partial L/\partial z}{\partial L/\partial t}\right) &= \operatorname{Re}\left(\frac{z(\theta'(q(z))q'(z) + t(G(\xi)/\xi)Q'(z))}{(G(\xi)/\xi)Q(z)}\right) \\ &= \operatorname{Re}\left(\frac{\xi\theta'(q(z))}{G(\xi)\varphi(q(z))}\right) + t\operatorname{Re}\left(\frac{zQ'(z)}{Q(z)}\right) > 0, \end{aligned}$$

for all $\xi \in \overline{\mathbb{D}}$. Hence, by Lemma 1.1, $L(z,t;\xi)$ is a subordination chain for all $\xi \in \overline{\mathbb{D}}$. Therefore, we have

$$L(z,t;\xi) \prec L(z,1;\xi), \quad z \in \mathbb{D},\ 0 < t \leq 1,\ \xi \in \overline{\mathbb{D}}.$$

Using (iii), the last relation gives

$$\theta(q(z)) + t\frac{G(\xi)}{\xi}Q(z) \in h(\mathbb{D}), \quad z \in \mathbb{D},\ 0 < t \le 1,\ \xi \in \overline{\mathbb{D}}.$$

Now, consider the function $\psi : \mathbb{C}^2 \times \overline{\mathbb{D}} \to \mathbb{C}$ defined by

$$\psi(r, s; \xi) = \theta(r) + \frac{G(\xi)}{\xi}s\varphi(r).$$

Then, we have

$$\psi(q(z), tzq'(z); \xi) \in h(\mathbb{D}), \quad 0 < t \le 1,\ z \in \mathbb{D},\ \xi \in \overline{\mathbb{D}}.$$

Since all conditions of Lemma 1.2 are satisfied, we obtain $q(z) \prec p(z)$. This completes the proof. □

Example 2.4. In this example, we investigate the conditions of Theorem 2.3. Let $0 < C < A,\ B > 1$ and

$$\frac{B+1}{B-1} < M < \frac{(B-1)(A-C)}{B+1} - (C+1).$$

Suppose that $q(z) = C/(B-z)$, $F(z) = Az$, $G(z) = Mz + z^2$, $\theta(z) = z$ and $\varphi(z) = 1/z$. From this, we obtain

$$h(z) = \theta(q(z)) + F(z)q'(z)\varphi(q(z)) = \frac{C + Az}{B - z}, \quad z \in \mathbb{D}.$$

We also have

$$Q(z) = zq'(z)\varphi(q(z)) = \frac{z}{B-z} \in S^* \quad \text{(or starlike)},$$

which satisfies (i). It is easy to see that

$$\begin{aligned}\operatorname{Re}\left(\frac{G(\xi)}{\xi}\frac{\varphi(q(z))}{\theta'(q(z))}\right) &= \frac{1}{C}\operatorname{Re}\left(MB - Mz + B\xi - z\xi\right)\\ &> \frac{1}{C}(M(B-1) - (B+1)) > 0;\end{aligned}$$

thus, condition (ii) is true. In order to satisfy (iii), it is sufficient to show that

$$\left|\frac{C + (M+\xi)z}{B-z} - \frac{A + BC}{B^2 - 1}\right| < \frac{C + AB}{B^2 - 1}, \quad z \in \mathbb{D},\ \xi \in \overline{\mathbb{D}}.$$

Note that

$$\theta(q(z)) + \frac{G(\xi)}{\xi}Q(z) = \frac{C + (M+\xi)z}{B - z}.$$

We have

$$\begin{aligned}\left|\frac{C + (M+\xi)z}{B - z} - \frac{A + BC}{B^2 - 1}\right| &\leq \frac{C + M + 1}{B - 1} + \frac{A + BC}{B^2 - 1}\\ &\leq \frac{(B-1)(A-C) + A + BC}{B^2 - 1}\\ &= \frac{C + AB}{B^2 - 1}.\end{aligned}$$

Therefore, (iii) is also satisfied. Hence, if

$$p(z) + (M+\xi)\frac{zp'(z)}{p(z)}$$

is analytic in $\mathbb{D} \times \overline{\mathbb{D}}$ and univalent in $\mathbb{D}$ for all $\xi \in \overline{\mathbb{D}}$, then

$$\frac{C + Az}{B - z} \prec\prec p(z) + (M+\xi)\frac{zp'(z)}{p(z)} \Longrightarrow \frac{C}{B - z} \prec p(z).$$

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Urmia University, Department of Mathematics, Urmia, Iran
Email address: **raghalary@yahoo.com**

Urmia University, Department of Mathematics, Urmia, Iran
Email address: **p.arjomandinia@gmail.com**

Department of mathematics, Payame Noor university, Tehran, Iran
Email address: **ebadian.ali@gmail.com**