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FIRST ORDER DEFORMATIONS OF PAIRS AND NON-EXISTENCE OF RATIONAL CURVES

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ABSTRACT. Let X_0 be a smooth hypersurface (assumed not to be generic) in projective space $\mathbf{P}^n$, $n \geq 4$, over complex numbers, and C_0 a smooth rational curve on X_0. We are interested in deformations of the pair C_0 and X_0. In this paper, we prove that, if the first order deformations of the pair exist along each deformation of the hypersurface X_0, then $\deg(C_0)$ cannot be in the range

$$\left(m\frac{2\deg(X_0)+1}{\deg(X_0)+1}, \frac{2+m(n-2)}{2n-\deg(X_0)-1}\right),$$

where m is any non negative integer less than

$$\dim(H^0(\mathcal{O}_{\mathbf{P}^n}(1))|_{C_0}) - 1.$$

1. Introduction. Throughout the paper, varieties are over the complex numbers. We are interested in conditions on the first order deformations of a pair of a rational curve and a hypersurface. So let us introduce the first order condition.

Let $H^0(\mathcal{O}_{\mathbf{P}^n}(h))$ denote the vector space of homogeneous polynomials of degree h in $n+1$ variables with $n \geq 4$. Let $f_0 \in H^0(\mathcal{O}_{\mathbf{P}^n}(h))$ be such that

$$X_0 = \operatorname{div}(f_0)$$

is a smooth hypersurface. Let

$$[f_0] \in \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h)))$$

denote the corresponding point of f_0 in the projectivization. Let

$$c_0 : \mathbf{P}^1 \longrightarrow X_0 \subset \mathbf{P}^n \tag{1.1}$$

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be a smooth embedding of $\mathbf{P}^1$, whose image is C_0. Let

$$\mathbb{H}^1(T_{X_0} \longrightarrow N_{C_0/X_0})$$

be the hypercohomology of the complex that is isomorphic to the tangent space of the deformations of the pair $C_0 \subset X_0$ and $H^1(T_{X_0})$, the space that is isomorphic to the tangent space of the moduli space of hypersurfaces at the point X_0. There is a known exact sequence:

$$\mathbb{H}^1(T_{X_0} \longrightarrow N_{C_0/X_0}) \xrightarrow{\phi} H^1(T_{X_0}) \longrightarrow H^1(N_{C_0/X_0}). \tag{1.2}$$

Theorem 1.1. *Let* $\mu(C_0) = \dim(c_0^*(H^0(\mathcal{O}_{\mathbf{P}^n}(1))))$.

(i) *If*

$$\phi \text{ is surjective,} \tag{1.3}$$

then $\deg(C_0)$ *cannot be in the range*

$$\left(m\frac{2h+1}{h+1}, \frac{2+m(n-2)}{2n-h-1}\right)$$

in the case

$$m\frac{2h+1}{h+1} < \frac{2+m(n-2)}{2n-h-1},$$

where m *is any non negative integer less than* $\mu(C_0)-1$. *It follows that, if assumption* (1.3) *holds,* $\deg(C_0)$ *cannot be in the range*

$$\left(1, \frac{n}{2n-h-1}\right), \quad \text{for } m=1,$$

in the case $n-h-1<0$.

In particular,

(ii) *if*

$$H^1(N_{C_0/X_0}) = 0,$$

all results in part (i) *hold*;

(iii) *if* X_0 *is a generic hypersurface and contains a rational curve* C_0, *all results in part* (i) *hold.*

1.1. Remark. The result in this theorem is new but is related to many previously known results by Clemens [**1, 2, 3**], Chiantini, Lopez and Ran [**4**], Ein [**5**], Katz [**6**], Pacienza [**8**], Voisin [**9, 10**] and Xu [**12**], etc.[1] They all studied pairs of rational curves (or higher dimensional subvarieties) and *generic* hypersurfaces of a certain smooth projective variety. The study of the exact sequence (1.2) appeared in many previous papers above. But here we use it as the only assumption for our result.

To explain in detail how we utilize assumption (1.3) at a non-generic hypersurface f_0, we let M_d be the parameter space of smooth embeddings $\mathbf{P}^1 \to \mathbf{P}^n$, whose image has degree d. So M_d is an open set of

$$\mathbf{P}(\oplus_{n+1}\mathcal{O}_{\mathbf{P}^n}(d)).$$

The smooth embedding map c_0 represents a point in M_d which is still denoted by c_0. Let

$$\begin{gathered}\Gamma \subset M_d \times \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h))) \\ \Gamma = \{(c, [f]) : c^*(f) = 0\}.\end{gathered} \tag{1.4}$$

Then the assumption in part (iii) that X_0 is generic is equivalent to the assumption that there is an irreducible component Γ_0 of Γ dominating

$$\mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h))),$$

and $(c_0, [f_0])$ is generic in Γ_0. Then it is well known that this implies the surjectivity of ϕ at $(c_0, [f_0])$, which is our assumption (1.3) at this point (see Lemma 2.2 below). The converse may not be true, i.e., the surjectivity of ϕ (at $(c_0, [f_0])$) may not imply the existence of Γ_0 containing the fixed point $(c_0, [f_0])$.

1.2. Idea of the proof. The main idea of the proof is to use special pencils of hypersurfaces constructed from general plane sections L_i (the pencil of $L_1 \cdots L_h$ and f_0). If the rational curve C_0 can deform to all hypersurfaces in first order, i.e., assumption (1.3) holds, then the collection V of the first order deformations of the rational curve along the directions of all such pencils generates some rank-2 quotient bundle $c_0^*(T_{\mathbf{P}^n})/E'$ (see subsection 3.2). But, if the numerical condition on C_0,

$$m\left(\frac{2h+1}{h+1}\right) \le \deg(C_0) \le \frac{2+m(n-2)}{2n-h-1}, \tag{1.5}$$

is satisfied, V fails to generate $c_0^*(T_{\mathbf{P}^n})/E'$ because all the first order deformations in V lie in a fixed, proper sub-bundle of $c_0^*(T_{\mathbf{P}^n})/E'$. The failure is forced by the numerical bounds (1.5) (through carefully designed first order deformations of the hypersurface constructed from the pencils above). Thus, the numerical bounds (1.5) contradict the first order deformation assumption (1.3).

In Section 2, we give another description of assumption (1.3) on first order deformations of the pair. In Section 3, we give the property on the global generation of the bundle $N_{C_0/\mathbf{P}^n}$. This property, Proposition 3.3, requires a long set-up. In Section 4, we show that the numerical condition (1.5) forces the property on the global generation, Proposition 3.3, to fail. This proves Theorem 1.1. In Section 5, we give three examples for Theorem 1.1.

2. First order deformations of the pair. In this section, we give another description of assumption (1.3). Let

$$S \subset \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h)))$$

be an irreducible subvariety (quasi or projective) that contains $[f_0]$ and is smooth at $[f_0]$ (so S could be a Zariski open set). Let

$$\mathcal{X}_S \subset \mathbf{P}^n \times S, \tag{2.1}$$

and

$$\mathcal{X}_S = \{(x, [f]) : [f] \in S, f(x) = 0\}, \tag{2.2}$$

be the universal hypersurface for $S \subset \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h)))$.

Let

$$\begin{aligned} \bar{c}_0 : \mathbf{P}^1 &\longrightarrow C_0 \times \{[f_0]\} \subset \mathcal{X}_S \\ t &\longrightarrow (c_0(t), [f_0]) \end{aligned}$$

be the smooth embedding determined by the above embedding c_0. The projection

$$P_S : \mathcal{X}_S \longrightarrow S$$

has a differential map

$$T_{(q,[f_0])}\mathcal{X}_S \longrightarrow T_{[f_0]}S, \quad q \in C_0,$$

which can be extended to a bundle map

$$(P_S)_* : \bar{c}_0^*(T_{\mathcal{X}_S}) \longrightarrow T_{[f_0]}S \otimes \mathcal{O}_{\mathbf{P}^1}.$$

At last, we obtain a homomorphism on the vector spaces

$$P_S^s : H^0(\bar{c}_0^*(T_{\mathcal{X}_S})) \longrightarrow T_{[f_0]}S, \tag{2.3}$$

where $T_{[f_0]}S \simeq H^0(T_{[f_0]}S \otimes \mathcal{O}_{\mathbf{P}^1})$ is the space of global sections of the trivial bundle, each of whose fibre is $T_{[f_0]}S$.

Now consider the diagram

$$\begin{array}{ccc} & & \mathbb{H}^1(T_{X_0} \to N_{C_0/X_0}) \\ & & \downarrow\phi \\ T_{[f_0]}\mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h))) & \stackrel{\psi}{\to} & H^1(T_{X_0}), \end{array}$$

where the map ψ is the differential (surjective for $n \geq 4$) at $[f_0]$ from $\mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h))$ to the deformation space of complex structures of the differential manifold X_0.

Lemma 2.1.

$$\psi(T_{[f_0]}S) \subset \text{image}\,(\phi)$$

if and only if P_S^s *is surjective.*

Proof. Let M_d be the parameter space of smooth embedding $\mathbf{P}^1 \to \mathbf{P}^n$, whose image has degree d. So M_d is an open set of

$$\mathbf{P}(\oplus_{n+1}\mathcal{O}_{\mathbf{P}^n}(d)).$$

The map c_0 represents a point in M_d which is still denoted by c_0. Let $\mathcal{X}_n$ be the universal hypersurface for $S = \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h)))$ (defined in formula (2.2)). Let

$$\begin{aligned} &\Gamma \subset M_d \times \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h))) \\ &\Gamma = \{(c, [f]) : c^*(f) = 0\} \end{aligned} \tag{2.4}$$

be the incidence scheme containing the point $(c_0, [f_0])$. Let $T_{(c_0,[f_0])}\Gamma$ be the Zariski tangent space of Γ. Let e be the evaluation map

$$e : \Gamma \times \mathbf{P}^1 \longrightarrow \mathcal{X}_n \tag{2.5}$$

$$(c, [f], t) \longrightarrow (c(t), [f]). \tag{2.6}$$

Its differential map induces a bundle map

$$e_* : T_{(c_0,[f_0])}\Gamma \otimes \mathcal{O}_{\mathbf{P}^1} \to c_0^*(T_{\mathcal{X}_n}).$$

It further induces a homomorphism on the cohomologies:

$$e^s : T_{(c_0,[f_0])}\Gamma \longrightarrow H^0(c_0^*(T_{\mathcal{X}_n})),$$

where $T_{(c_0,[f_0])}\Gamma = H^0(T_{(c_0,[f_0])}\Gamma \otimes \mathcal{O}_{\mathbf{P}^1})$. Also, there is a surjective map η:

$$T_{(C_0,[f_0])}\Gamma \quad \twoheadrightarrow \quad \mathbb{H}^1(T_{X_0} \to N_{C_0/X_0}),$$

such that the following diagram commutes
(2.7)

$$\begin{array}{ccccc} T_{(C_0,[f_0])}\Gamma & = & T_{(C_0,[f_0])}\Gamma & \stackrel{\eta}{\twoheadrightarrow} & \mathbb{H}^1(T_{X_0} \to N_{C_0/X_0}) \\ \downarrow e^s & & \downarrow & & \downarrow \phi \\ H^0(\bar{c}_0^*(T_{\mathcal{X}_n})) & \stackrel{P_n^s}{\to} & T_{[f_0]}\mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h))) & \stackrel{\psi}{\twoheadrightarrow} & H^1(T_{X_0}), \end{array}$$

where P_n^s is the corresponding map in formula (2.3) for

$$S = \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h))).$$

Because

$$T_{c_0}M_d \longrightarrow H^0(c_0^*(T_{\mathbf{P}^n}))$$

is surjective (it is an isomorphism), e^s has to be surjective. Then, the lemma is true for $S = \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h)))$.

Now we consider the subvariety $S \subset \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h)))$ in the lemma. If $\psi(T_{[f_0]}S) \subset \text{image}\,(\phi)$, for any $\alpha \in T_{[f_0]}S$, we apply the diagram to find a section $\sigma \in H^0(\bar{c}_0^*(T_{\mathcal{X}_n}))$ such that $P_n^s(\sigma) = \alpha$. Because $P_n^s(\sigma) = \alpha \in T_{[f_0]}S$, σ must be in the subspace $H^0(\bar{c}_0^*(T_{\mathcal{X}_S}))$ of $H^0(\bar{c}_0^*(T_{\mathcal{X}_n}))$. Thus, P_S^s is surjective. Conversely, we suppose P_S^s is surjective. For any $\alpha \in T_{[f_0]}S$, using the commutative diagram, we obtain

$$\psi(\alpha) \in \phi \circ \eta \circ (e^s)^{-1} \circ (P_S^s)^{-1}(\alpha).$$

This completes the proof. □

Lemma 2.2. *If there is an irreducible component* Γ_0 *of the incidence scheme*

$$\{(c, [f]) \in M_d \times \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h))) : c^*(f) = 0\},$$

such that Γ_0 *dominates* $\mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h)))$ *and* $(c_0, [f_0]) \in \Gamma_0$ *is generic, then* ϕ *is surjective.*

Proof. In this proof, we consider the entire space of hypersurfaces, i.e.,

$$S = \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h))).$$

As before, $\mathcal{X}_n$ denotes the universal hypersurface corresponding to $\mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h)))$.

Let c_0 be as above, and let

$$\bar{c}_0 : \mathbf{P}^1 \longrightarrow X_0 \times \{[f_0]\} \subset \mathcal{X}_n$$

be the morphism that lifts the image C_0 to $\mathcal{X}_n$. The projection

$$P : \mathcal{X}_n \longrightarrow S$$

induces a map on the sections of bundles over $\mathbf{P}^1$,

$$P^s : H^0(\bar{c}_0^*(T_{\mathcal{X}_n})) \longrightarrow T_{[f_0]}S, \tag{2.8}$$

where $T_{[f_0]}S \simeq H^0(T_{[f_0]}S \otimes \mathcal{O}_{\mathbf{P}^1})$ is the space of global sections of the trivial bundle, each of whose fibre is $T_{[f_0]}S$. Observe the commutative diagram

$$\begin{array}{ccc} T_{(c_0,[f_0])}\Gamma & \stackrel{(e_\Gamma)_*}{\longrightarrow} & H^0(\bar{c}_0^*(T_{\mathcal{X}_n})) \\ \downarrow (\pi_\Gamma)_* & & \downarrow P_n^s \\ T_{[f_0]}S & = & T_{[f_0]}S \end{array}$$

(see (2.7) for P_n^s) where $(e_\Gamma)_*$ is induced from the differential of the evaluation e_Γ:

$$\begin{array}{rcl} e_\Gamma : \Gamma \times \mathbf{P}^1 & \longrightarrow & \mathcal{X}_n \\ (c, [f], t) & \longrightarrow & c(t) \times \{[f]\}. \end{array}$$

Since f_0 is generic and π_Γ is dominant (by the assumption of the lemma), then $(c_0, [f_0]) \in \Gamma$ is a generic point in Γ_0. Then the dominance of π_Γ implies the surjectivity of $(\pi_\Gamma)_*$. Thus, P_n^s is surjective. By Lemma 2.1, we have proved Lemma 2.2. □

3. The quotient bundle $c_0^*(T_{\mathbf{P}^n})/E'$. In this section, we prove the quotient bundle $c_0^*(T_{\mathbf{P}^n})/E'$ is generated by a special type V of global sections. Also, these special sections come from a particular type of first order deformation of hypersurfaces. Let us introduce a long setting that defines this special type of section.

3.1. A parameter space of hypersurfaces. Let $S \simeq C^N$ be an affine open set of $\mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h)))$. Let $f_0 \in S$. Specifically, we can let

$$S = \left\{ f_0 + \sum_{i=1}^{N} a_i f_i \right\},$$

where $\{f_i\}$, $i = 0, \ldots, N$, is a basis for the linear space $H^0(\mathcal{O}_{\mathbf{P}^n}(h))$, and $(a_1, \ldots, a_N)$ are coordinates of $S = \mathbf{C}^N$.

Let

$$\mathcal{X} \subset \mathbf{P}^n \times \mathbf{S} \tag{3.1}$$

and

$$\mathcal{X} = \{(x, [f]) : f(x) = 0\}. \tag{3.2}$$

Let

$$F(a, x) = f_0(x) + \sum_{i=1}^{N} a_i f_i(x), \quad a \in S,\ x \in \mathbf{P}^n, \tag{3.3}$$

be the corresponding universal polynomial that defines $\mathcal{X}_n$.

Definition 3.1. There is an identification of the tangent space at the origin $[f_0] \in S$ such that, at any point $[f]$ in general,

$$T_{[f_0]}S = S = T_{[f]}S. \tag{3.4}$$

More specifically, if $[f] \in S$ is a polynomial $\neq f_0$, $\overrightarrow{f}$ is defined to be the corresponding vector field in $T_{[f_0]}S$ via formula (3.4). So $\overrightarrow{f}$ represents the direction of the line connecting $[f_0]$ and $[f]$.

3.2. A quotient vector bundle via decomposition. Now we introduce a rational curve. Let $\overline{c}_0$ be the smooth embedding map from

$$\begin{array}{cccc} \overline{c}_0 : \mathbf{P}^1 & \stackrel{\overline{c}_0}{\longrightarrow} & \mathbf{P}^n \times \{[f_0]\} \subset \mathbf{P}^n \times S \\ t & \longrightarrow & (c_0(t), [f_0]) \end{array}$$

induced from c_0 in formula (1.1), and $\overline{C}_0 = C_0 \times \{[f_0]\}$. From now on, we assume that assumption (1.3) holds. By Lemma 2.1, P_n^s is surjective. Thus, we can choose sections

$$\beta_i, \quad i = 1, \dots, N,$$

in $H^0(\overline{c}_0^*(T\mathcal{X}))$ such that $P_n^s(\beta_i) = \partial/\partial a_i$. The inverse β_i is clearly not unique, but we choose one for each i. Let G be the subbundle generated by these N sections $\beta_i \in H^0(\overline{c}_0^*(T\mathcal{X}))$. G is not unique. We may choose G so it is invariant under the PGL $(n+1)$ action (see **[9]**), i.e., G contains all the vectors tangent to the orbit of PGL $(n+1)$ action on $\mathbf{P}^n \times S$. Because of $\partial/\partial a_i$, these sections β_i form a direct pointwise sum in the bundle $\overline{c}_0^*(T\mathcal{X})$. Hence, G is a trivial bundle of rank N. This bundle G gives us the decompositions

$$\overline{c}_0^*(T_{\mathbf{P}^n \times S}) \simeq (\oplus_N \mathcal{O}) \oplus \overline{c}_0^*(T_{(\mathbf{P}^n \times S)/S}), \tag{3.5}$$

$$\overline{c}_0^*(T_{\mathcal{X}}) \simeq (\oplus_N \mathcal{O}) \oplus \overline{c}_0^*(T_{\mathcal{X}/S}), \tag{3.6}$$

where $\mathcal{O}$ is the trivial line bundle on $\mathbf{P}^1$, and $\oplus_N \mathcal{O} \simeq G$. Thus, we have the isomorphisms I_1 and I_2:

$$\begin{aligned} I_1 : \frac{\overline{c}_0^*(T_{\mathcal{X}})}{G} &\simeq \overline{c}_0^*(T_{\mathcal{X}/S}), \\ I_2 : \frac{\overline{c}_0^*(T_{\mathbf{P}^n \times S})}{G} &\simeq \overline{c}_0^*(T_{(\mathbf{P}^n \times S)/S}). \end{aligned} \tag{3.7}$$

Then, we have the isomorphism I_3:

$$\frac{\overline{c}_0^*(T_{\mathcal{X}})/G}{I_1^*(T_{\bar{C}_0})} \simeq N_{C_0/X_0} \simeq \mathcal{O}_{\mathbf{P}^1}(k_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(k_{n-2}). \tag{3.8}$$

Let k_{n-2} be the smallest among all k_i, $i = 1, \dots, n-2$. Let $E \subset \overline{c}_0^*(T_{\mathcal{X}})$ be the inverse image of

$$\mathcal{O}_{\mathbf{P}^1}(k_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(k_{n-3}) \tag{3.9}$$

under the bundle morphism

$$\overline{c}_0^*(T_{\mathcal{X}}) \longrightarrow N_{C_0/X_0} \longrightarrow \mathcal{O}_{\mathbf{P}^1}(k_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(k_{n-2}). \tag{3.10}$$

Also, let E' be the inverse of

$$\mathcal{O}_{\mathbf{P}^1}(k_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(k_{n-3}) \tag{3.11}$$

under the bundle morphism

$$\overline{c}_0^*(T_{X_0}) \longrightarrow N_{C_0/X_0} \longrightarrow \mathcal{O}_{\mathbf{P}^1}(k_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(k_{n-2}). \tag{3.12}$$

Then we have

$$\frac{\bar{c}_0^*(T_{\mathbf{P}^n \times S})}{E} \simeq \mathcal{O}_{\mathbf{P}^1}(d_1) \oplus \mathcal{O}_{\mathbf{P}^1}(d_2) \simeq c_0^*(T_{\mathbf{P}^n})E', \tag{3.13}$$

where

$$d_1 + d_2 = (n+1)\,d - 2 - \sum_{i=1}^{n-3} k_i, d_1 \le d_2.$$

Let D be the inverse image of the summand $\mathcal{O}_{\mathbf{P}^1}(d_2)$ under the map

$$\bar{c}_0^*(T_{\mathbf{P}^n \times S}) \quad \longrightarrow \quad \mathcal{O}_{\mathbf{P}^1}(d_1) \oplus \mathcal{O}_{\mathbf{P}^1}(d_2). \tag{3.14}$$

(D is most likely unique depending on whether inequalities for d_2, k_{n-2} are strict. But the uniqueness of D will not affect the proof. So we fix D.)

3.3. A property on the global generation of the quotient bundle. Let $\theta = \{p_1, \ldots, p_m\}$ be an element in the symmetric product $Sy^m(\mathbf{P}^1)$,

$$0 \le m < \mu(C_0) - 1,$$

where Sy^m is the symmetric product. Let $H^0_\theta(\mathcal{O}_{\mathbf{P}^n}(1))$ be the sublinear system of $H^0(\mathcal{O}_{\mathbf{P}^n}(1))$ with base points p_i, $i = 1, \ldots, m$, i.e.,

$$H^0_\theta(\mathcal{O}_{\mathbf{P}^n}(1)) = H^0(\mathcal{O}_{\mathbf{P}^n}(1) \otimes \mathcal{I}_{p_1} \otimes \cdots \otimes \mathcal{I}_{p_m}).$$

Let

$$V = \bigcup_{\theta \in Sy^m(\mathbf{P}^1)} \mathrm{Sym}^h(H^0_\theta(\mathcal{O}_{\mathbf{P}^n}(1))),$$

where Sym^h is the hth symmetric algebra of a vector space.

Lemma 3.2.

$$\mathrm{span}\,(V) = H^0(\mathcal{O}_{\mathbf{P}^n}(h)).$$

Proof. Consider the Veronese map v_h

$$\begin{array}{ccc} H^0(\mathcal{O}_{\mathbf{P}^n}(1)) & \xrightarrow{v_h} & H^0(\mathcal{O}_{\mathbf{P}^n}(h)) \\ f & \longrightarrow & f^h. \end{array} \tag{3.15}$$

Since the Veronese variety is non-degenerated, the linear span of its image is the entire space, i.e.,

$$\mathrm{span}\,(v_h(H^0(\mathcal{O}_{\mathbf{P}^n}(1)))) = H^0(\mathcal{O}_{\mathbf{P}^n}(h)).$$

By the definition,

$$\mathrm{span}\,(v_h(H^0(\mathcal{O}_{\mathbf{P}^n}(1)))) \subset \mathrm{span}\,(V).$$

Thus,

$$\mathrm{span}\,(V) = H^0(\mathcal{O}_{\mathbf{P}^n}(h)).$$

This completes the proof. □

Proposition 3.3. *If assumption* (1.3) *holds, V, identified with the subspace*

$$\{0\} \oplus V \subset \overline{c}_0^*(T_{\mathbf{P}^n \times S}),$$

via formula (3.4), *generates the bundle*

$$\frac{\overline{c}_0^*(T_{\mathbf{P}^n \times S})}{E} \simeq \frac{c_0^*(T_{\mathbf{P}^n})}{E'},$$

after modulo *E.*

To prove this lemma, we need to prove the following.

Lemma 3.4. *If assumption* (1.3) *holds, the global sections of subbundle*

$$\overline{c}_0^*(T_{(\mathbf{P}^n \times S)/\mathbf{P}^n}) \quad (\text{modulo } E)$$

generate the bundle $\overline{c}_0^*(T_{\mathbf{P}^n \times S})/E \simeq c_0^*(T_{\mathbf{P}^n})/E'$.

Proof. There is a PGL $(n+1)$ action on $\mathbf{P}^n$. This action induces a PGL $(n+1)$ action on

$$\mathbf{P}^N = \mathbf{P}(\mathcal{O}_{\mathbf{P}^n}(h)).$$

Thus, there is a PGL $(n+1)$ action on $\mathbf{P}^n \times \mathbf{P}^N$,

$$g(f, x) = (g(f), g(x)) = (f(g^{-1}(x)), g(x)).$$

If $z = (x, [f_0]) \in \overline{C}_0$, infinitesimally we have a linear map α_z,

$$\begin{array}{ccc} T_{Id}(\mathrm{PGL}\,(n+1)) & \xrightarrow{\alpha_z} & T_z(\mathbf{P}^N \times \mathbf{P}^n) \\ g & \longrightarrow & (g(f_0), g(x)). \end{array} \tag{3.16}$$

Because G is invariant under $\mathrm{PGL}\,(n+1)$ action, then

$$\mathrm{image}\,(\alpha_z) \in G|_z.$$

Let

$$W = \{g(f_0) : g \in T_{\mathrm{Id}}(\mathrm{PGL}\,(n+1))\} \subset T_{f_0}\mathbf{P}^N.$$

It is clear that $\alpha_z(T_{\mathrm{Id}}(\mathrm{PGL}\,(n+1)))$ is projected onto $T_x\mathbf{P}^n$. Hence,

$$\{0\} \oplus W \subset H^0(\overline{c}_0^*(T_{\mathbf{P}^n \times S/\mathbf{P}^n})) \quad (\text{modulo } E)$$

generates the bundle

$$\frac{\overline{c}_0^*(T_{\mathbf{P}^n \times S})}{E} = \overline{c}_0^*\left(\frac{T_{\mathbf{P}^n \times S/S}}{\overline{C}_0}\right) \simeq \frac{c_0^*(T_{\mathbf{P}^n})}{E'}.$$

The proof is complete. □

Proof of Proposition 3.3. By Lemma 3.2, V linearly spans $H^0(\mathcal{O}_{\mathbf{P}^n}(h))$. Using the identification in (3.4), $\{0\} \oplus V$ spans

$$H^0(\overline{c}_0^*(T_{(\mathbf{P}^n \times S)/\mathbf{P}^n})).$$

Then Lemma 3.4 says the global sections of

$$\overline{c}_0^*(T_{(\mathbf{P}^n \times S)/\mathbf{P}^n})) \quad (\text{modulo } E)$$

generate $\overline{c}_0^*(T_{\mathbf{P}^n \times S})/E$, then so does V. This completes the proof. □

4. The proof of Theorem 1.1.

Proof.

(i) We show the proof by a contradiction. So, suppose otherwise. Such a rational curve c_0 exists on the smooth X_0. Then

$$m\left(\frac{2h+1}{h+1}\right) < d < \frac{2+m(n-2)}{2n-h-1}, \quad 0 \le m < \mu(C_0) - 1, \tag{4.1}$$

where $d = \deg(C_0)$. By Lemma 2.1, the map P^s is surjective. Then we can apply Proposition 3.3, which says that the set of sections in the form

$$L_1 \cdots L_h, \qquad L_i \in H^0_\theta(\mathcal{O}_{\mathbf{P}^n}(1))$$

generates the bundle $\overline{c}_0^*(T_{\mathbf{P}^n \times S})/E$. Since D is a proper subbundle of $\overline{c}_0^*(T_{\mathbf{P}^n \times S})$, Proposition 3.3 implies that there exist generic sections

$L_0, \ldots, L_h \in H^0_\theta(\mathcal{O}_{\mathbf{P}^n}(1))$, such that the $c_0^*(L_i), c_0^*(L_j)$ for $i \neq j$ do not have common zeros except that $p_1, \ldots, p_m \in \theta$ with multiplicity 1,

$$\bar{c}_0^*(\overrightarrow{L_0 \cdots L_{h-1}})$$

is a non-zero section that does not lie in D, where $\overrightarrow{L_0 \cdots L_{h-1}}$ represents the direction of the line connecting f_0 and $L_0 \cdots L_{h-1}$ in S (see Definition 3.1). We should note that, in particular, the section $\bar{c}_0^*(\overrightarrow{L_0 \cdots L_{h-1}})$ does not lie in E.

Next, we show that the numerical condition (4.1) forces

$$\bar{c}_0^*(\overrightarrow{L_0 \cdots L_{h-1}})$$

to lie in D.

Using the chosen sections $L_0, \ldots, L_h$, we construct vector fields

$$u_i = L_h \frac{\partial}{\partial a_h} - L_i \frac{\partial}{\partial a_i}, \quad i = 0, \ldots, h-1, \tag{4.2}$$

on $S \times \mathbf{C}^{n+1}$, where

$$\frac{\partial}{\partial a_i} = \overrightarrow{L_0 \cdots \widehat{L}_i \cdots L_h},$$

as in Definition 3.1, are the tangent fields on S. The u_i annihilate universal polynomial F that defines the universal hypersurface $\mathcal{X}$. Hence, u_i are sections of the bundle

$$T_{\mathcal{X}} \otimes \mathcal{O}_{\mathbf{P}^n}(1).$$

Let $v_i = \bar{c}_0^*(u_i)$ be the pull-back of u_i to $\mathbf{P}^1$. The non-zero nature of sections v_i is the key to the proof of Theorem 1.1. Notice that v_i must lie in $\bar{c}_0^*(T_{\mathcal{X}}(1))$. Using formula (3.8), v_i is reduced to sections of

$$\frac{\bar{c}_0^*(T_{\mathcal{X}})/G}{I_1^*(T_{\overline{C}_0})} \otimes \mathcal{O}_{\mathbf{P}^1}(d) \simeq ((E/G) \otimes \mathcal{O}_{\mathbf{P}^1}(d)) \oplus \mathcal{O}_{\mathbf{P}^1}(k_{n-2} + d).$$

Next, we consider the bound for $k_{n-2} + d$. By the result of **[11]**[2], $2n - h - 1 \geq 1$. So, the condition $d < 2 + m(n-2)/(2n-h-1)$ implies that

$$(n+1)d - hd - 2 < -(n-2)(d-m). \tag{4.3}$$

The left hand side of formula (4.3) is

$$c_1(N_{C_0/X_0}) = k_1 + \cdots + k_{n-2}.$$

Since k_{n-2} is the smallest among $k_1, \ldots, k_{n-3}, k_{n-2}$,

$$(n-2)k_{n-2} \leq k_1 + \cdots + k_{n-2} = (n+1)\,d - hd - 2 < -(n-2)(d-m).$$

Hence, $k_{n-2} < -(d-m)$. Then,

$$k_{n-2} + d < -(d-m) + d = m. \tag{4.4}$$

Since v_i has m zeros at $p_1, \ldots, p_m$, by the bound of $k_{n-2} + d$, v_i must be in E along $\mathbf{P}^1$. Then formula (4.2) says that $\overline{c}_0^*(\partial/\partial a_h)$ must lie in E at the zeros of $c_0^*(L_i)$, $i = 0, \ldots, h-1$, other than $p_1, \ldots, p_m$, because $c_0^*(L_h)$ does not vanish at any of the zeros of $c_0^*(L_i)$ along $\mathbf{P}^1$ except at $p_1, \ldots, p_m$. Thus, $\overline{c}_0^*(\partial/\partial a_h)$ does not lie in E (by the choice of L_i), but lies in E at least at $h(d-m)$ points. At last, we see this is impossible because

$$\overline{c}_0^*(T_{\mathbf{P}^n \times S})/E \simeq \mathcal{O}_{\mathbf{P}^1}(d_1) \oplus \mathcal{O}_{\mathbf{P}^1}(d_2),$$

where $d_1 + d_2 = hd + k_{n-2}$ and $d_1 \leq d_2$. Then, d_1 has the following bound (because of the inequality $m(2h + 1/h + 1) < d$ and formula (4.4)),

$$h(d-m) > \frac{hd - (d-m)}{2} \geq \frac{hd + k_{n-2}}{2} \geq d_1.$$

Because of this bound, $\overline{c}_0^*(\partial/\partial a_h)$ modulo E must lie entirely in the summand $\mathcal{O}_{\mathbf{P}^1}(d_2)$. Recall that D is the sub-bundle in $\overline{c}_0^*(T_{\mathbf{P}^n \times S})$ such that $D/E = \mathcal{O}_{\mathbf{P}^1}(d_2)$. Thus, $\overline{c}_0^*(\partial/\partial a_h)$ must lie in D. This contradicts our choice of

$$\frac{\partial}{\partial a_h} = \overrightarrow{L_0 \cdots L_{h-1}},$$

which says that $\overline{c}_0^*(\overrightarrow{L_0 \cdots L_{h-1}})$ does not lie in D. This completes the proof for the first part.

(ii) This part follows from the first part because of the exact sequence (1.2).

(iii) If X_0 is a generic hypersurface containing a smooth rational curve C_0, then using Lemma 2.2, we obtain that assumption (1.3) holds. Thus, all results in part (i) hold. □

5. Examples. We give three examples that follow from Theorem 1.1. The first two, to our knowledge, cannot be derived from previously known results, but the last one can.

Example 5.1. There are no quadratic curves in a general hypersurface of degree 14 in $\mathbf{P}^9$. This is a direct consequence of Theorem 1.1 for $n = 9$, $h = 14$ and $d = 2$.

Example 5.2. There are no irreducible, rational quartic curves in a general hypersurface of degree 54 in $\mathbf{P}^{30}$. This is a direct consequence of Theorem 1.1 for $n = 30$, $h = 54$ and $d = 4$.

Example 5.3. The condition $H^1(N_{C_0/X_0}) = 0$ is actually much stronger than assumption (1.3). Therefore, the application of part (2) of Theorem 1.1 may be limited. Let us observe one in the following. Consider an irreducible quadratic curve C_0 in a smooth sextic threefold X_0 (which is non generic). Then its normal bundle N_{C_0/X_0} cannot have the most balanced splitting

$$\mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1),$$

because, otherwise, $H^1(N_{C_0/X_0}) = 0$, and by Theorem 1.1, the degree of C_0 cannot be 2.

This result can be obtained by using other methods. For instance, consider the combination of results of [**7, 9, 10**] in the following. If $H^1(N_{C_0/X_0}) = 0$, Kodaira's theorem says that C_0 can globally deform to all sextics. Then, a generic sextic threefold contains a rational curve. On the other hand, since, in this case, $\deg(X_0) \geq 2n-2$, Voisin's result says that a generic sextic does not contain a rational curve. This is a contradiction.

ENDNOTES

1. The references listed here are not complete. We apologize for this incompleteness due to the quantity of papers in this area.

2. We cannot use Voisin's result in [**9**] because X_0 in Theorem 1.1 is not generic.

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