ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 46, Number 1, 2016

THE SYMBOLIC GENERIC INITIAL SYSTEM OF ALMOST LINEAR POINT CONFIGURATIONS IN $\mathbb{P}^2$

SARAH MAYES

ABSTRACT. Consider an ideal $I \subseteq K[x, y, z]$ corresponding to a point configuration in $\mathbb{P}^2$ where all but one of the points lies on a single line. In this paper, we study the symbolic generic initial system $\{\mathrm{gin}\,(I^{(m)})\}_m$ obtained by taking the reverse lexicographic generic initial ideals of the uniform fat point ideals $I^{(m)}$. We describe the limiting shape of $\{\mathrm{gin}\,(I^{(m)})\}_m$ and, in proving this result, demonstrate that infinitely many of the ideals $I^{(m)}$ are componentwise linear.

1. Introduction. Given a set of distinct points $\{p_1, \ldots, p_r\}$ of $\mathbb{P}^2$, we may consider the fat point subscheme $Z = m(p_1 + \cdots + p_r)$, whose ideal $I_Z \subseteq K[x, y, z]$ consists of functions vanishing to at least order m at each point. If I is the ideal of $\{p_1, \ldots, p_r\}$, I_Z is equal to the mth symbolic power of I, denoted $I^{(m)}$. While uniform fat point ideals are relatively easy to describe, computing even simple invariants such as Hilbert functions or the degree of least degree elements has proved very difficult. Understanding how the configuration of the points $\{p_1, \ldots, p_r\}$ is related to invariants of the ideals $I^{(m)}$ is an active area of research (see, for example, **[2, 3, 7, 11]**).

Our main objective is to describe the limiting behavior of the Hilbert functions of the uniform fat point ideals $\{I^{(m)}\}_m$ as m gets large. We study the case where I is the ideal of a point configuration where all but one of the points lies on a single line. The study of the asymptotic behavior of algebraic objects has been a significant research trend over the past 20 years; it is motivated by the philosophy that the limiting behavior of a collection of objects is often simpler than the individual elements within the collection. For example, within the study of fat

2010 AMS *Mathematics subject classification.* Primary 13P10, Secondary 13C40.

Keywords and phrases. Generic initial ideals, fat points, symbolic powers, asymptotic behavior.

Received by the editors on April 29, 2013, and in revised form on March 27, 2014.

DOI:10.1216/RMJ-2016-46-1-283

points, more can be said about the limit $\lim_{m\to\infty} \alpha(I^{(m)})/m$ than the individual invariants $\alpha(I^{(m)})$, where $\alpha(I^{(m)})$ denotes the degree of the least degree element of $I^{(m)}$ (for example, see [**9**]).

It is well known that the Hilbert function of an ideal and its generic initial ideal are equal. Thus, to describe the limiting behavior of the Hilbert functions of $\{I^{(m)}\}_m$ we will study the reverse lexicographic *symbolic generic initial system* $\{\text{gin}\,(I^{(m)})\}_m$ of I and describe its limiting shape. The *limiting shape* P of $\{\text{gin}\,(I^{(m)})\}_m$ is defined to be the limit

$$\lim_{m\to\infty} \frac{1}{m} P_{\text{gin}(I^{(m)})}$$

where $P_{\text{gin}\,(I^{(m)})}$ denotes the Newton polytope of $\text{gin}(I^{(m)})$. When I is an ideal corresponding to a point configuration in $\mathbb{P}^2$ each reverse lexicographic generic initial ideal $\text{gin}\,(I^{(m)})$ is generated in two variables; thus, $P_{\text{gin}\,(I^{(m)})}$, and P itself, may be thought of as a subset of $\mathbb{R}^2$. There is evidence that this limiting shape captures geometric information about the corresponding arrangement of points (see discussion in [**12,** Section 5]).

The main result of this paper is the following theorem describing the limiting shape of $\{\text{gin}\,(I^{(m)})\}_m$ when I is an ideal of a point configuration where all but one of the points lies on a single line.

Theorem 1.1. *Fix some integer $l > 2$, and let $I \subset K[x, y, z]$ be the ideal corresponding to the arrangement of $l + 1$ points $p_1, \ldots, p_{l+1}$ of $\mathbb{P}^2$ such that $p_1, \ldots, p_l$ lie on a line L and p_{l+1} does not lie on L. Then the limiting shape of the symbolic generic initial system $\{\text{gin}\,(I^{(m)})\}_m$ of I is the shaded polytope pictured in Figure* 1.

In proving this theorem, we will show that when I is the ideal of such an almost linear point configuration, $I^{(m)}$ is *componentwise linear* for infinitely many m (Theorem 3.1). This property means that the minimal free resolution of the ideal has a very simple form. Other classes of ideals that are componentwise linear include stable monomial ideals, Gotzmann ideals and ideals of at most $n+1$ fat points in general position in $\mathbb{P}^n$ ([**5, 10**]).

The almost linear point configuration addressed by Theorem 1.1 may be viewed as one step more complex than the case where all

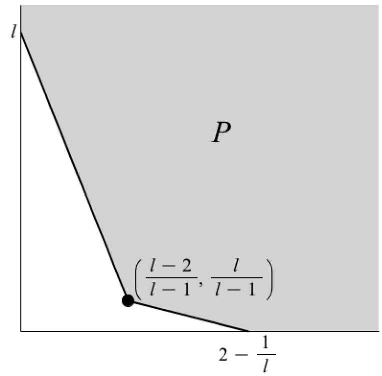

FIGURE 1. The limiting shape of $\{\text{gin}\,(I^{(m)})\}_m$ where I is an ideal corresponding to a point configuration with l points on a line and one point off of that line.

points lie on a smooth conic. Recent work by Denkert and Janssen in [**4**] demonstrates that the relationship between the symbolic and ordinary powers of ideals of such point configurations is moderately more intricate than in the case where all points lie on a smooth conic. Similarly, Theorem 1.1 tells us that the limiting shape of the symbolic generic initial system of almost linear point configurations is slightly more complex than the limiting shape that arises when all points lie on a smooth conic (see [**13**]).

Background information necessary for the proof of the main result is contained in Section 2. In Section 3, we prove results on componentwise linearity for individual fat point ideals. Section 4 uses these results to prove Theorem 1.1.

2. Background. In this section, we will review facts about componentwise linearity, generic initial ideals of fat points and blow-ups of points in $\mathbb{P}^2$. Throughout, $R = K[x, y, z]$ is a polynomial ring over a field of characteristic 0 with the standard grading and the reverse lexicographic order where $x > y > z$.

2.1. Componentwise linearity. Componentwise linear ideals are homogeneous ideals with particularly nice minimal free resolutions.

Definition 2.1 (**[10]**). Let $R = K[x_1, \ldots, x_n]$ and M be a graded R-module. Then M has a *d-linear resolution* if the graded minimal free resolution of M is of the form

$$0 \longrightarrow R(-d-s)^{\beta_s} \longrightarrow \cdots \longrightarrow R(-d-1)^{\beta_1} \longrightarrow R(-d)^{\beta_0} \longrightarrow M \longrightarrow 0.$$

For any homogeneous ideal $I \subset R$, let (I_k) be the ideal generated by all homogeneous polynomials of degree k contained in I. A homogeneous ideal I is said to be *componentwise linear* if (I_k) has a linear resolution for all k.

The following theorem of Aramova, Herzog and Hibi connects componentwise linearity to the study of generic initial ideals and will be our main tool for detecting this property.

Theorem 2.2 (**[1]**). *Let I be a homogeneous ideal of $K[x_1, \ldots, x_n]$. Then I is componentwise linear if and only if I and its reverse lexicographic generic initial ideal* gin (I) *have the same Betti numbers.*

2.2. Generic initial ideals of fat point ideals. When I is the ideal of distinct points of $\mathbb{P}^2$, the reverse lexicographic generic initial ideals gin $(I^{(m)})$ have a very simple form detailed in the following proposition.

Proposition 2.3. *Suppose that $I \subseteq K[x, y, z]$ is the ideal of a set of distinct points of $\mathbb{P}^2$. Then the minimal generators of* gin $(I^{(m)})$ *under the reverse lexicographic order are of the form*

$$\{x^{\alpha(m)}, x^{\alpha(m)-1}y^{\lambda_{\alpha(m)-1}(m)}, \ldots, xy^{\lambda_1(m)}, y^{\lambda_0(m)}\},$$

where $\lambda_0(m) > \lambda_1(m) > \cdots > \lambda_{\alpha(m)-1}(m) \geq 1$ and $\alpha(m) = \alpha(I^{(m)})$ is the degree of the least degree generator of $I^{(m)}$.

This follows from the fact that generic initial ideals are Borel-fixed and the ideals $I^{(m)}$ and gin $(I^{(m)})$ are saturated; see [**14,** Corollary 2.9] for a proof. The following corollary now follows from Theorem 2.2 and Proposition 2.3.

Corollary 2.4. *Let I be the ideal of a set of distinct points in $\mathbb{P}^2$ and m be an integer such that $I^{(m)}$ is componentwise linear. The generators* *of* $\operatorname{gin}(I^{(m)})$ *are completely determined by the degrees of the minimal generators of* $I^{(m)}$.

2.3. Blow-ups of points in $\mathbb{P}^2$. The algorithms that we will use to prove Theorems 1.1 and 3.1 come from **[8]** and are very similar to the procedures outlined in **[12]**. The key to these algorithms is to consider divisors on the blow-ups of each point arrangement.

Suppose that $\pi : X \to \mathbb{P}^2$ is the blow-up of distinct points $p_1, \dots, p_r$ of $\mathbb{P}^2$. Let $E_i = \pi^{-1}(p_i)$ for $i = 1, \dots, r$, and let L be the total transform in X of a line not passing through any of the points $p_1, \dots, p_r$. The classes of these divisors form a basis of $\mathrm{Cl}(X)$; for convenience, we will write e_i in place of $[E_i]$ and e_0 in place of $[L]$. Further, the intersection product in $\mathrm{Cl}(X)$ is defined by $e_i^2 = -1$ for $i = 1, \dots, r$; $e_0^2 = 1$; and $e_i \cdot e_j = 0$ for all $i \neq j$.

Let $Z = m(p_1 + \cdots + p_r)$ be a uniform fat point subscheme with sheaf of ideals $\mathcal{I}_Z$; set

$$F_d = dE_0 - m(E_1 + E_2 + \cdots + E_r)$$

and $\mathcal{F}_d = \mathcal{O}_X(F_d)$. Much of our interest in the blow-ups comes from the fact that the Hilbert function of $I^{(m)}$ is related to the divisors F_d (see **[12]**):

$$h^0(X, \mathcal{F}_d) = H_{I^{(m)}}(d).$$

For convenience, we will sometimes write $h^0(X, F) = h^0(X, \mathcal{O}_X(F))$. Recall that, if $[F]$ is not the class of an effective divisor, then $h^0(X, F) = 0$. On the other hand, if F is effective, then we will see that we can compute $h^0(X, F)$ by finding $h^0(X, H)$ for some *numerically effective* divisor H.

Definition 2.5. A divisor H is *numerically effective* if $[F] \cdot [H] \geq 0$ for every effective divisor F. The cone of classes of numerically effective divisors in $\mathrm{Cl}(X)$ is denoted $\mathrm{NEF}(X)$.

Lemma 2.6. *Suppose that X is the blow-up of $\mathbb{P}^2$ at distinct points $p_1, \dots, p_r$. Let $F \in \mathrm{NEF}(X)$. Then F is effective, and*

$$h^0(X, F) = ([F]^2 - [F] \cdot [K_X])/2 + 1,$$

where $K_X = -3E_0 + E_1 + \cdots + E_r$.

Proof. This is a consequence of Riemann-Roch and the fact that $h^1(X, F) = 0$ for any numerically effective divisor F on X. See [**8,** Lemmas III.1.1 (b) and 2.2] for a discussion. □

Knowing how to compute $h^0(X, H)$ for a numerically effective divisor H will allow us to compute $h^0(X, F)$ for *any* divisor F. In particular, given a divisor F, there exists a divisor H such that $h^0(X, F) = h^0(X, H)$ and either:

(a) H is numerically effective so

$$h^0(X, F) = h^0(X, H) = ([H]^2 - [H] \cdot [K_X])/2 + 1$$

by Lemma 2.6; or

(b) there is a numerically effective divisor G such that $[G] \cdot [H] < 0$ so $[H]$ is not the class of an effective divisor and $h^0(X, F) = h^0(X, H) = 0$.

The set of classes of effective, reduced and irreducible curves of negative self-intersection in X is denoted by

$$\begin{aligned} \mathrm{NEG}\,(X) := \{[C] \in \mathrm{Cl}\,(X) : [C]^2 < 0, \\ C \text{ is effective, reduced and irreducible}\}. \end{aligned}$$

NEG (X) is significant because it allows us to reduce a divisor in the sense of the following lemma. The proof involves looking at a long exact sequence of cohomology; see the discussion in [**6,** Section 3] for details.

Lemma 2.7. *Suppose that* $[C] \in \mathrm{NEG}\,(X)$ *is such that* $[F] \cdot [C] < 0$. *Then* $h^0(X, F) = h^0(X, F - C)$.

We have the following enumeration of the elements of NEG (X) from [**8,** Lemma 3.1.1 (c)]. For convenience, we set

$$A := E_0 - E_1 - \cdots - E_l, \qquad B_i := E_0 - E_i - E_{l+1},$$

where $i = 1, \ldots, l$.

Lemma 2.8 ([**8**])**.** *Let X be the blow-up of points $p_1, \dots, p_{l+1} \in \mathbb{P}^2$, where $p_1, \dots, p_l$ lie on a line and p_{l+1} lies off of that line. Then*

$$\mathrm{NEG}\,(X) = \{[A]; [B_i] \text{ for } i = 1, \dots, l;\ e_i \text{ for } i = 1, \dots, l+1\}.$$

The method for finding an H satisfying (a) or (b) above is as follows.

Procedure 2.9 (Remark 2.4 of [**7**])**.** Let X be the blow-up of points $p_1, \dots p_r \in \mathbb{P}^2$. Given a divisor F, we can find a divisor H with $h^0(X, F) = h^0(X, H)$ satisfying either condition (a) or (b) above as follows.

(i) Reduce to the case where $[F] \cdot e_i \geq 0$ for all $i = 1, \dots, r$: if $[F] \cdot e_i < 0$ for some i, $h^0(X, F) = h^0(X, F - ([F] \cdot e_i)E_i)$, so we can replace F with $F - ([F] \cdot e_i)E_i$.

(ii) Since L is numerically effective, if $[F] \cdot e_0 < 0$ then $[F]$ is not the class of an effective divisor, and we can take $H = F$ (case (b)).

(iii) If $[F] \cdot [C] \geq 0$ for every $[C] \in \mathrm{NEG}\,(X)$, then, by Lemma 2.8, F is numerically effective, so we may take $H = F$ (case (a)).

(iv) If $[F] \cdot [C] < 0$ for some $[C] \in \mathrm{NEG}\,(X)$ then $h^0(X, F) = h^0(X, F - C)$ by Lemma 2.7. Replace F with $F - C$ and repeat from Step 2.

There are only a finite number of elements in $\mathrm{NEG}\,(X)$ to check (by Lemma 2.8) so it is possible to complete step (iii). Further, $[F] \cdot e_0 > [F - C] \cdot e_0$ when $[C] \in \mathrm{NEG}(X)$, so the condition in step (ii) will be satisfied after at most $[F] \cdot e_0 + 1$ repetitions. Thus, the process will terminate.

Denote the number of minimal generators of $I^{(m)}$ of degree d by $v_d(I^{(m)})$. Then

$$\begin{aligned} v_{d+1}(I^{(m)}) &= \dim\left(\mathrm{coker}\left((I^{(m)})_d \otimes R_1 \longrightarrow (I^{(m)})_{d+1}\right)\right) \\ &= \dim\left(\mathrm{coker}\left(H^0(X, \mathcal{F}_d) \otimes H^0(X, e_0) \longrightarrow H^0(X, \mathcal{F}_{d+1})\right)\right) \\ &:= s(\mathcal{F}_d, e_0). \end{aligned}$$

If $[F_d]$ is not the class of an effective divisor, then $h^0(X, \mathcal{F}_d) = 0$ and

$$s(\mathcal{F}_d, e_0) = h^0(X, \mathcal{F}_{d+1}). \tag{2.1}$$

In the case that $[F_d]$ is the class of an effective divisor, let H_d be the numerically effective divisor produced by Procedure 2.9. Then

$$s(\mathcal{F}_d, e_0) = s(\mathcal{H}_d, e_0) + h^0(X, \mathcal{F}_{d+1}) - h^0(X, \mathcal{H} + e_0)$$

by [**8,** Lemma 2.10]. Further, since H_d is numerically effective by definition, $s(\mathcal{H}_d, e_0) = 0$ when the points $p_1, \ldots, p_r$ lie on a conic by [**8,** Theorem 3.1.2]. Thus, in the cases we are interested in,

$$s(\mathcal{F}_d, e_0) = h^0(X, \mathcal{F}_{d+1}) - h^0(X, \mathcal{H} + e_0). \tag{2.2}$$

Therefore, to find the number of generators $s(\mathcal{F}_d, e_0)$ of each degree $d+1$, we will proceed as follows

(a) Follow Procedure 2.9 to determine if F_d is effective or non-effective. If F_d is effective, the procedure will yield a numerically effective divisor H_d.
(b) Compute $v_{d+1}(I^{(m)}) = s(\mathcal{F}_d, e_0)$ for each d using expressions (2.1) or (2.2) together with the formula from Lemma 2.6.

3. Generators of $I^{(m)}$ and componentwise linearity. Throughout this section, $R = K[x, y, z]$ is a polynomial ring over a field of characteristic zero with the reverse lexicographic order. Also, I is the ideal of a point configuration $\{p_1, \ldots, p_{l+1}\}$ in $\mathbb{P}^2$ where $p_1, \ldots, p_l$ lie on a single line, p_{l+1} is off of that line, and $l > 2$. The purpose of this section is to enumerate the generators of the fat point ideals $I^{(m)}$ and, in doing so, prove the following theorem.

Theorem 3.1. *Let $I \subseteq K[x, y, z]$ be an ideal of $l + 1 > 3$ points in $\mathbb{P}^2$ where l points lie on a single line and the other point lies off of the line. Then an infinite number of the uniform fat point ideals $I^{(m)}$ are componentwise linear. In particular, $I^{(m)}$ is componentwise linear when $l(l-1)$ divides m.*

The following proposition gives a specific criterion for an ideal of fat points to be componentwise linear.

Proposition 3.2. *Let J be a homogeneous ideal of $K[x_1, \ldots, x_n]$ such that the reverse lexicographic generic initial ideal* gin (J) *is generated in two variables. If α is the degree of the smallest degree generator of*

J and

$$\alpha = \{\textit{number of minimal generators of } J\} - 1,$$

then J is componentwise linear.

Proof. By Theorem 2.2, J is componentwise linear if and only if J and $\operatorname{gin}(J)$ have the same Betti numbers. Since the Betti numbers of J are obtained from those of $\operatorname{gin}(J)$ by making a series of consecutive cancelations (see **[15,** Section I.22]), J is componentwise linear if and only if no consecutive cancellations occur. However, since the minimal free resolution of $\operatorname{gin}(J)$ is of the form

$$0 \longrightarrow \bigoplus_j R(-j)^{\beta_{1,j}} \longrightarrow \bigoplus_j R(-j)^{\beta_{0,j}} \longrightarrow \operatorname{gin}(J) \longrightarrow 0,$$

any consecutive cancelation must involve canceling a $\beta_{0,j}$; these Betti numbers correspond to minimal generators of $\operatorname{gin}(J)$. Therefore, showing that J is componentwise linear in this case is equivalent to showing that the minimal generators of J and $\operatorname{gin}(J)$ are of the same degrees or, equivalently by consecutive cancelation, that J and $\operatorname{gin}(J)$ have the same number of generators.

Since α is the degree of the least degree generator of J, it is also the degree of the least degree generator of $\operatorname{gin}(J)$. By Borel-fixedness,

$$\operatorname{gin}(J) = (x_1^{\alpha}, x_1^{\alpha-1}x_2^{\lambda_{\alpha-1}}, \ldots, x_1 x_2^{\lambda_1}, x_2^{\lambda_0})$$

for some invariants $\{\lambda_i\}_i$ and $\operatorname{gin}(J)$ has $\alpha+1$ generators. Since J also has $\alpha+1$ generators, it must be componentwise linear. $\square$

To prove Theorem 3.1, it remains to show that the conditions of Proposition 3.2 are satisfied when $J = I^{(m)}$ and $l(l-1)$ divides m. That is, we need to show that the degree of the smallest degree generator of $I^{(m)}$ is one less than the number of minimal generators of $I^{(m)}$. To prove that this holds we will compute the number of generators of each degree using the procedure outlined at the end of subsection 2.3.

3.1. Finding H_d. In this section, we follow Procedure 2.9 to determine if each divisor $F_d = dE_0 - m(E_1 + \cdots + E_{l+1})$ is numerically effective and, if it is, to compute the associated numerically effective divisor H_d satisfying equation (2.2). These results will be used in the next section to compute the number of generators of $I^{(m)}$ of each degree. As d varies, the sign of the product $[F_d] \cdot [C]$, $C \in \mathrm{NEG}(X)$, changes, influencing the divisors that are obtained throughout Procedure 2.9. The four cases below correspond to the different directions that Procedure 2.9 may take. In most cases, obtaining H_d requires several iterations of the procedure and lengthy computations; we only include the results here.

Recall that $p_1, \dots, p_l$ are fixed points of $\mathbb{P}^2$ lying on a single line, p_{l+1} is an additional point lying off of that line (where $l > 2$), and I is the ideal of $\{p_1, \dots, p_{l+1}\}$. Let X be the blow-up of the points $p_1, \dots, p_{l+1}$. Throughout this section, we will assume that $m = \rho l(l-1)$ for some $\rho \in \mathbb{N}$ and write

$$a_0 E_0 - a_1 E_1 - \cdots - a_r E_{l+1} := (a_0; a_1, \dots, a_{l+1}).$$

As before we will write elements of $\mathrm{NEG}(X)$ as

$$A := E_0 - E_1 - \cdots - E_l, \qquad B_i := E_0 - E_i - E_{l+1},$$

where $i = 1, \dots, l$.

3.1.1. Case $d \geq lm$. When $d \geq lm$, $[F_d] \cdot [C] \geq 0$ for all $[C] \in \mathrm{NEG}(X)$. Therefore, $\mathcal{F}_d$ is already numerically effective, so

$$\mathcal{H}_d = \mathcal{F}_d.$$

3.1.2. Case $2m \leq d < lm$. When $2m \leq d < lm$, $[F_d] \cdot [A] < 0$ and $[F_d] \cdot [B_i] \geq 0$. Thus, only copies of A are subtracted in Procedure 2.9, and we obtain

$$\begin{aligned} H_d &= F_d - \left\lceil \frac{lm-d}{l-1} \right\rceil A \\ &= \left(d - \left\lceil \frac{lm-d}{l-1} \right\rceil; m - \left\lceil \frac{lm-d}{l-1} \right\rceil, \dots, m - \left\lceil \frac{lm-d}{l-1} \right\rceil, m \right). \end{aligned}$$

3.1.3. Case $2m - m/l \leq d < 2m$. When $d < 2m$, both $[F_d] \cdot [B_i] < 0$ and $[F_d] \cdot [A] < 0$, so copies of both A and B_i are subtracted in Procedure 2.9. Writing $d = 2m - \gamma$ where $\gamma \leq m/l$, we obtain

$$H_d = F_d - \left\lceil \frac{lm-d}{l-1} \right\rceil A - \gamma(B_1 + \cdots + B_l)$$

$$= \left(d - \left\lceil \frac{lm-d}{l-1} \right\rceil - \gamma l; m - \left\lceil \frac{lm-d}{l-1} \right\rceil - \gamma, \ldots, \right.$$
$$\left. m - \left\lceil \frac{lm-d}{l-1} \right\rceil - \gamma, m - \gamma l \right).$$

Notice that $d - \lceil (lm-d)/(l-1) \rceil - \gamma l > 0$ exactly when $2m - m/l \leq d$.

3.1.4. Case $d < 2m - m/l$. When $d < 2m - m/l$, Procedure 2.9 will eventually yield a divisor class equal to that of H_d from the previous case. However, since $d < 2m - m/l$, this intermediate divisor is of the form $a_0E_0 - a_1E_1 - \cdots - a_{l+1}E_{l+1}$ with $a_0 < 0$ and therefore is not effective. It follows that F_d is not effective so $h^0(X, \mathcal{F}_d) = 0$.

3.2. Determining $s(\mathcal{F}_d, e_0)$. Fix I, X, $l > 2$ and $m = \rho l(l-1)$ as in subsection 3.1. In this section, we will compute the number of generators of $I^{(m)}$ of each degree $d+1$, i.e. , $s(\mathcal{F}_d, e_0)$. The following facts are used to evaluate these expressions.

- The divisors H_d and H_{d+1} computed in the previous section.
- When $[F_{d+1}]$ is the class of an effective divisor,

$$h^0(X, \mathcal{F}_{d+1}) = h^0(X, \mathcal{H}_{d+1}) = ([H_{d+1}]^2 - [K_X] \cdot [H_{d+1}])/2 + 1$$

by Lemma 2.6 and Procedure 2.9.

3.2.1. Case $d \geq lm$. We will find the number of generators of $I^{(m)}$ and $\operatorname{gin}(I^{(m)})$ of degree $d+1$, assuming that $d \geq lm$. In this case, $H_d + E_0 = F_d + E_0 = F_{d+1}$, so

$$s(\mathcal{F}_d, e_0) = h^0(X, \mathcal{F}_{d+1}) - h^0(X, \mathcal{F}_{d+1}) = 0.$$

3.2.2. Case $2m \leq d \leq lm - 2$. To obtain the number of generators of $I^{(m)}$ of degree $d+1$ for $2m \leq d \leq lm-1$, we begin by writing $d = jm + w(l-1) + p$ where $j = 2, \ldots, l-1$, $w = 0, \ldots, \rho l - 1$ and $p = 0, \ldots, l-2$. Then $\lceil (lm-d)/(l-1) \rceil = (l-j)l\rho - w$ and, referring to the expression for H_d from subsection 3.1.2, we see that $H_d + E_0 = H_{d+1}$ exactly when $p \neq l-2$. Thus, when $p \neq l-2$,

$$s(\mathcal{F}_d, e_0) = h^0(X, \mathcal{H}_{d+1}) - h^0(X, \mathcal{H}_d + e_0) = 0,$$

and, when $p = l-2$,

$$s(\mathcal{F}_d, e_0) = h^0(X, \mathcal{F}_{d+1}) - h^0(X, \mathcal{H}_d + e_0) = 1.$$

3.2.3. Case $2m-m/l \leq d < 2m$. To find the number of generators of $I^{(m)}$ of degree $d+1$ for $2m-m/l \leq d < 2m$, we begin by writing $d = 2m-(p+w(l-1))$, where: $p = 1,\dots,l-2$ when $w=0$; $p = 0,\dots,l-2$ when $w = 1,\dots,\rho-1$; and $p=0$ when $w=\rho$. The expressions for H_d from subsection 3.1.3 yield

$$s(\mathcal{F}_d, e_0) = \begin{cases} l & \text{if } p \neq 1 \\ l+1 & \text{if } p = 1. \end{cases}$$

3.2.4. Case $d = 2m-m/l-1$. When $d = 2m-m/l-1$, $[F_d]$ is not in the class of an effective divisor but $[H_{d+1}]$ is. Therefore, the formula for $s(\mathcal{F}_d, e_0)$ yields

$$s(\mathcal{F}_d, e_0) = h^0(X, \mathcal{F}_{d+1}) = h^0(X, \mathcal{H}_{d+1}) = 1.$$

3.2.5. Case $d < 2m-m/l-1$. When $d < 2m-m/l-1$, neither $[F_d]$ nor $[F_{d+1}]$ is in the class of an effective divisor, so

$$s(\mathcal{F}_d, e_0) = h^0(X, \mathcal{F}_{d+1}) = 0.$$

3.3. Generators of $I^{(m)}$. Let I, X, $l>2$ and $m = \rho l(l+1)$ be as in subsections 3.1 and 3.2. In this section we write down the number of generators of each degree, using the results of the previous section and the fact that $v_{d+1}(I^{(m)}) = s(\mathcal{F}_d, e_0)$.

3.3.1. Case $2m < d \leq lm$. We saw in subsection 3.2.2 that, when $2m \leq d \leq lm-1$, we only obtain nonzero values for $s(\mathcal{F}_d, e_0)$ when we can write $d = jm + y(l-1)$ for $j, y \in \mathbb{N}$. Therefore, when $2m < d \leq lm$, there is one generator of $I^{(m)}$ of degree

$$jm + w(l-1) + (l-2) + 1 = jm + w(l-1) + l - 1$$

for each $j = 2,\dots,l-1$, $w = 0,\dots,\rho l-1$.

3.3.2. Case $2m-m/l+1 \leq d \leq 2m$. As in the previous case, the number of generators of degree d when $2m-m/l+1 \leq d \leq 2m$ depends on the form of d.

- There are l generators of degrees

$$2m-(p+w(l-1))+1$$

when $w = 0$ and $p = 2, \dots, l-2$ or $w = 1, \dots, \rho - 1$ and $p = 0, 2, 3, \dots, l-2$.

- There are l generators of degree

$$2m - (\rho(l-1)) + 1 = 2m - \frac{m}{l} + 1.$$

- There are $l+1$ generators of degrees

$$2m - (1 + w(l-1)) + 1 = 2m - w(l-1),$$

where $w = 0, 1, \dots, \rho - 1$.

3.3.3. Case $d = 2m - m/l$. In subsection 3.2.4 we observed that $s(\mathcal{F}_{2m-m/l-1}, e_0) = 1$. Therefore, there is exactly one generator of degree $2m - m/l$.

3.4. Componentwise linearity.

Proof of Theorem 3.1. Let I be the ideal corresponding to a point configuration where l points lie on a line and one point lies off of the line. Fix $m = \rho(l)(l-1)$ for $\rho \in \mathbb{N}$. By Proposition 3.2, it is sufficient to show that the degree of the smallest degree generator of $I^{(m)}$ is one less than the number of elements in a minimal generating set of $I^{(m)}$. By our work in subsection 4, the smallest degree generator of $I^{(m)}$ is of degree $2m - m/l$. The number of minimal generators is equal to

$$\begin{aligned}
&[\text{no. gens. of degree } > 2m] + [\text{no. gens. of degree } d,\\
&2m - \frac{m}{l} + 1 \le d \le 2m] + \left[\text{no. gens. of degree } \le 2m - \frac{m}{l}\right]\\
&= [(l-2)(\rho l)] + [l(l-2)(\rho - 1) + l(l-3) + l + (l+1)\rho] + 1\\
&= [\rho l^2 - 2\rho l] + [\rho l^2 - \rho l + \rho] + 1\\
&= 2m - \rho l + \rho + 1 = 2m - \frac{m}{l} + 1.
\end{aligned}$$ ☐

4. Computation of the limiting shape. In this section we will prove Theorem 1.1 using the fact from Theorem 3.1 that infinitely many of the $I^{(m)}$ are componentwise linear. As before, we assume that l, the number of points on the line, is greater than 2.

Proof of Theorem 1.1. Let m be divisible by l and $l1$ so that $m = \rho l(l1)$ for some $\rho \in \mathbb{N}$. Then $I^{(m)}$ is componentwise linear by Theorem 3.1, so the degrees of the minimal generators of $I^{(m)}$ are equal to the degrees of the minimal generators of $\operatorname{gin}(I^{(m)})$. Proposition 2.3 implies that

$$\operatorname{gin}(I^{(m)}) = (\{x^{\alpha(m)}, x^{\alpha(m)1}y^{\lambda_{\alpha(m)-1}(m)}, \dots, xy^{\lambda_1(m)}, y^{\lambda_0(m)}\}),$$

where $\alpha(m) = 2mm/l$ and $\lambda_0(m) = lm$ by our work in subsection .

We may think of the sequence of invariants $\{\lambda_i(m)\}_i$ of $\operatorname{gin}(I^{(m)})$ as having two phases: the first phase corresponds to $\lambda_i(m)$ where $lm \geq \lambda_i(m) + i > 2m$ and the second corresponds to $\lambda_i(m)$ where $\lambda_i(m) + i \leq 2m$. The $\lambda_i(m)$ within each of these two phases are regularly spaced, that is, there are patterns in the gaps $\lambda_i(m)\lambda_{i+1}(m)$ forced by the patterns within the sequence of degrees $\{\lambda_i(m) + i\}$.

For example, consider the case where $\lambda_i(m) + i \leq 2m$. If there are l generators of a certain degree μ, then the fact that $\lambda_i(m) + i = \mu$ for l consecutive i forces $\lambda_i(m)\lambda_{i+1}(m) = 1$ for l consecutive i. On the other hand, when the degree of the jth generator is μ and the degree of the $(j+1)$st generator is $\mu 1$, $(\lambda_j(m) + j)(\lambda_{j+1}(m) + j + 1) = 1$ so $\lambda_j(m)\lambda_{j+1}(m) = 2$. Putting these observations together with the results of subsection , the gap sequence $\{\lambda_i(m)\lambda_{i+1}(m)\}$ for $\lambda_i(m)+i \leq 2m$ exhibits the following pattern:

$$\underbrace{1,1,\dots,1}_{l\text{ repeats}},2,\underbrace{1,1,\dots,1}_{l-1\text{ repeats}},2,\quad\underbrace{\underbrace{1,1,\dots,1}_{l-1\text{ repeats}}}_{\text{repeats } l-2\text{ times}},\dots,2.$$

This pattern repeats a total of ρ times.

If we plot the points $(i, \lambda_i(m))$ where $\lambda_i(m) + i \leq 2m$ and connect consecutive points, the resulting line segments will have slopes 1 or 2 as in the pattern above. Taking the convex hull of this set of points removes much of this detail; with some sketching, it is not difficult to see that the boundary of the convex hull of this set of points is a single line segment.

With similar considerations for the case where $\lambda_i + i > 2m$, one sees that the Newton polytope of $\operatorname{gin}(I^{(m)})$ is defined by the points $(\alpha(m), 0) = (2mm/l, 0)$, $(J, \lambda_J(m))$ where $\lambda_J + J = 2m$, and

$(0, \lambda_0(m)) = (0, lm)$ (see Figure 2). Also note that, when $\lambda_i + i > 2m$,

$$(l1) = (\lambda_{i1} + i1)(\lambda_i + i) = \lambda_{i1}\lambda_i 1,$$

so $\lambda_{i1}\lambda_i = l$, and the slope of the line L_1 in Figure 2 is equal to l. There are a total of $(l2)(\rho l1 + 1) = \rho l^2 2\rho l = mm/(l1)$ generators of degree greater than $2m$, so $J = mm/(l1)$, and

$$\lambda_J(m) = lml\left(m\frac{m}{l1}\right) = l\left(\frac{m}{l1}\right).$$

Note that

$$\begin{aligned} J + \lambda_J(m) &= \left[m\frac{m}{l1}\right] + l\left(\frac{m}{l1}\right) \\ &= m + \frac{m}{l1}(l1) = 2m, \end{aligned}$$

as required.

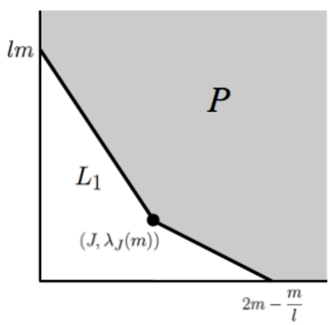

FIGURE 2. The Newton polytope of gin $(I^{(m)})$ where I is an ideal corresponding to a point configuration with l points on a line and one point off of that line.

Thus, the limiting shape of $\{\text{gin}\,(I^{(m)})\}_m$ is defined by the points:

$$\left(\lim_{m\to\infty} \frac{2m - m/l}{m}, 0\right) = \left(2 - \frac{1}{l}, 0\right)$$

$$\left(0, \lim_{m\to\infty} \frac{lm}{m}\right) = (0, l)$$

$$\left(\lim_{m\to\infty}\frac{m-(m/l-1)}{m}, \lim_{m\to\infty}\frac{l(m/l-1)}{m}\right) = \left(1-\frac{1}{l-1}, \frac{l}{l-1}\right),$$

as claimed. $\square$

One can easily check that the area under the limiting shape is equal to $(l+1)/2$. This is consistent with the general fact that the area under the limiting shape of $\{\text{gin}\,(I^{(m)})\}_m$ when I is the ideal of r points is equal to $r/2$ (see [**14,** Proposition 2.14]).

Acknowledgments. I thank Karen Smith and Brian Harbourne for introducing this problem to me and for many useful discussions. I also thank the referee for a careful reading and suggestions that greatly improved this paper.

REFERENCES

1. A. Aramova, J. Herzog and T. Hibi, *Ideals with stable Betti numbers*, Adv. Math. **152** (2000), 72-77.

2. S. Cooper, B. Harbourne and Z. Teitler, *Combinatorial bounds on Hilbert functions of fat points in projective space*, J. Pure Appl. Alg. **215** (2011), 2165-2179.

3. S. Cooper and S. Hartke, *The alpha problem and line count configurations*, `arXiv:1312.4147` [math.AC].

4. A. Denkert and M. Janssen, *Containment problem for points on a reducible conic in* $\mathbb{P}^2$, `arXiv:1207.7153v2`.

5. C.A. Francisco, *Resolutions of small sets of fat points*, J. Pure Appl. Alg. **203** (2005), 220-236.

6. A. Geramita, *Lectures on the non-singular cubic surface in* $\mathbb{P}^3$, Queen's Papers Pure Appl Math. **83**, Queen's University, Kingston, 1989.

7. E. Guardo and B. Harbourne, *Resolutions of ideals of any six fat points in* $\mathbb{P}^2$, J Alg. **318** (2007), 619-640.

8. ______, *Free resolutions of fat point ideals on* $\mathbb{P}^2$, J. Pure Appl. Alg. **125** (1998), 213-234.

9. ______, *Problems and progress: A survey on fat points in* $\mathbb{P}^2$, Queen's Papers Pure Appl. Math. **123**, Queen's University, Kingston, 2002, 85-132.

10. J. Herzog and T. Hibi, *Componentwise linear ideals*, Nagoya Math J. **153** (1999), 141-153.

11. H. Markwig, *Free resolutions of fat points on a conic in* $\mathbb{P}^2$, `http://xcrcg.uni-math.gwdg.de/wiki/images/a/ab/Fatpoints.pdf`.

12. S. Mayes, *The asymptotics of symbolic generic initial systems of six points in* $\mathbb{P}^2$, `arXiv:1202:1317 [math.AC]`, J. Commutative Algebra, to appear.

13. S. Mayes, *The symbolic generic initial system of points on an irreducible conic*, `arXiv:1304.7542`.

14. ______, *The asymptotic behaviour of symbolic generic initial systems of generic points*, J. Pure Appl. Alg. **218** (2014), 381-390.

15. I. Peeva, *Graded syzygies*, Alg. Appl. **14**, Springer, London, 2011.

Quest University Canada, 3200 University Blvd, Squamish BC V8B 0N8
Email address: **sarah.mayes-tang@questu.ca**