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INEQUALITIES FOR SUMS OF INDEPENDENT RANDOM VARIABLES IN LORENTZ SPACES

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ABSTRACT. By using interpolation with a function parameter, we establish a moment inequality for sums of independent random variables in Lorentz spaces $\Lambda^p(\varphi)$. These estimates generalize Rosenthal inequalities in the Lorentz-Zygmund spaces $L^{p,q}(\log L)^\gamma$ as well as Lorentz spaces $L^{p,q}$.

1. Introduction. We begin our work by recalling the classical Khintchine inequalities. Let $\{r_k\}_{k\geq 1}$ be a Rademacher sequence on a probability space $(\Omega, \mathfrak{F}, P)$. Since $\{r_k\}_{k\geq 1}$ is an orthogonal sequence in $L^2(\Omega)$, for any finite sequence $\{\alpha_k\} \subseteq \mathbf{C}$

$$\Big\| \sum_k \alpha_k r_k \Big\|_2 = \bigg(\sum_k |\alpha_k|^2 \bigg)^{1/2}.$$

The classical Khintchine inequalities assert that $\| \sum_k \alpha_k r_k\|_2$ is uniformly equivalent to $\| \sum_k \alpha_k r_k\|_p$ for any $p < \infty$, namely,

$$\Big\| \sum_k \alpha_k r_k \Big\|_p \approx \bigg(\sum_k |\alpha_k|^2 \bigg)^{1/2}.$$

The equivalence $A \approx B$ means that $c_1 A \leq B \leq c_2 A$ for some positive constants c_1 and c_2. Rosenthal **[12]** generalized the Khintchine inequality by replacing $\{r_k\}_{k\geq 1}$ with an arbitrary sequence $\{X_k\}_{k\geq 1}$ of independent symmetric random variables on a probability space $(\Omega, \mathfrak{F}, p)$. More precisely, he proved that, for such a sequence $\{X_k\}_{k\geq 1} \subset L^p(\Omega)$, $p > 2$, we have

$$(1.1) \qquad \Big\| \sum_{k=1}^n X_k \Big\|_p \approx \max \bigg\{ \Big\| \sum_{k=1}^n X_k \Big\|_2, \bigg(\sum_{k=1}^n \|X_k\|_p^p \bigg)^{1/p} \bigg\}$$

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for all $n \geq 1$. Carothers and Dilworth **[3]** proved an analogous result for some of the Lorentz spaces, namely, for $2 < p < \infty$, $0 < q \leq \infty$, and any independent symmetric random variables $X_1, X_2, \dots, X_n$,

(1.2)
$$\Big\| \sum_{k=1}^{n} X_k \Big\|_{L^{p,q}(\Omega)} \approx \max \Big\{ \Big\| \sum_{k=1}^{n} X_k \Big\|_{L^2(\Omega)}, \Big\| \sum_{k=1}^{n} \bigoplus X_k \Big\|_{L^{p,q}(0,\infty)} \Big\},$$

where $\sum_{k=1}^{n} \bigoplus X_k$ denotes the disjoint sum of $X_1, X_2, \dots, X_n$, which is a function on $(0, \infty)$ with $d_X(t) = \sum_{k=1}^{n} d_{X_k}(t)$. For example, we could take $X(t) = \sum_{i=1}^{n} X_i(t-i+1)\chi_{[i-1,i]}$ for $0 \leq t \leq n$. In the setting symmetric function spaces, Johnson and Schechtman **[7]** established a generalization of Rosenthal inequalities. Recently, Hu **[6]** generalize Rosenthal inequalities to $p \geq 0$ instead of $p > 2$ and replaced the quantity 2 by $r \in [1, 2]$ for conditionally independent mean zero random variables.

In this paper, by use of interpolation with a function parameter, a moment inequality is proved for sums of independent random variables in Lorentz spaces $\Lambda^q(\Omega)$. These estimates generalize Rosenthal inequalities in the Lorentz-Zygmund spaces $L^{p,q}(\log L)^{\gamma}$ as well as Lorentz spaces $L^{p,q}$.

2. Lorentz spaces $\Lambda^q_\Omega(\varphi)$.

Let (Ω, Σ, μ) be a σ-finite nonatomic measure space. For a given weight ω, let $L^p_\mu(\omega)$ denote the Lebesgue space defined by the norm $\|f\|_{L^p_\mu(\omega)} = \|f\omega\|_{L^p(\mu)}$ and $L^p_*(\omega)$ when the measure is dt/t on $\mathbb{R}^+ = (0, \infty)$.

Definition 2.1. We say that function $f : (0, \infty) \to (0, \infty)$ belongs to the class $\mathfrak{B}$ if $f(1) = 1$, f is continuous and

$$\overline{f}(t) = \sup_{s>0} \frac{f(ts)}{f(s)} < \infty,$$

for all $0 < t < \infty$.

For such a function f, the Boyd upper and lower indices $\alpha_{\overline{f}}$ and $\beta_{\overline{f}}$ (**[10]**) of $\overline{f}$, which is submultiplicative and Lebesgue-measurable, are then defined by

$$\alpha_{\overline{f}} = \lim_{t\to+\infty} \frac{\log \overline{f}(t)}{\log t}, \qquad \beta_{\overline{f}} = \lim_{t\to 0} \frac{\log \overline{f}(t)}{\log t}$$

with

$$-\infty < \beta_{\overline{f}} \le \alpha_{\overline{f}} < +\infty.$$

For example, if $\theta, \gamma \in \mathbf{R}$, then $f(t) = t^\theta(1+|\log t|)^\gamma \in \mathfrak{B}$, $\overline{f}(t) = t^\theta(1+|\log t|)^{|\gamma|}$ and $\alpha_{\overline{f}} = \beta_{\overline{f}} = \theta$.

Let $\varphi \in \mathfrak{B}$ and $0 < q \le \infty$; the Lorentz space $\Lambda^q(\varphi)$ is the set of (classical of) μ-measurable functions from Ω in $\mathbf{C}$ such that

$$\|f\|_{\Lambda^q_\Omega(\varphi)} := \|f^*\|_{L^q_*(\varphi)} = \left(\int_0^\infty (\varphi(t)f^*(t))^q \frac{dt}{t}\right)^{1/q} < \infty$$
$$(0 < q < \infty)$$
$$\|f\|_{\Lambda^\infty_\Omega(\varphi)} := \|f^*\|_{L^\infty_*(\varphi)} = \sup_{t>0} \varphi(t)f^*(t) < \infty,$$

where f^* denotes the decreasing rearrangement of $|f|$, i.e.,

$$f^*(t) = \inf\{s > 0 : d_f(s) = \mu(\{|f| > s\}) \le t\}.$$

It is known that $\Lambda^q_\Omega(\varphi)$ is a rearrangement quasi–Banach space.

Remark 2.2. It is also well known that the inclusion relations between Lorentz spaces are determined by their fundamental functions, since $\Lambda^q_\Omega(\varphi_1) \subset \Lambda^q_\Omega(\varphi_2)$ if and only if $\omega_2(t) \le C\omega_1(t)$ for all $t > 0$, and both spaces agree if and only if $\omega_1 \approx \omega_2$, where

$$\omega_i(t) = \left(\int_0^t \varphi(s)^q \frac{ds}{s}\right)^{1/q}$$

is the fundamental function for $\Lambda^q_\Omega(\varphi_i)$, $i = 1, 2$, [**4**].

Example 2.3. For $\varphi(t) = t^{1/p}(1+|\log t|)^\gamma$ with $0 < p < \infty$ and $-\infty < \gamma < +\infty$, $\Lambda^q_\Omega(\varphi)$ is the Lorentz-Zygmund space $L^{p,q}(\log L)^\gamma$. This is the classical Lorentz space $L^{p,q}$ if $\gamma = 0$.

We let $(\mathcal{A}_1, \mathcal{A}_2)$ denote a compatible couple of quasi-Banach spaces pair (i.e., $\mathcal{A}_1$ and $\mathcal{A}_2$ are quasi-Banach spaces, which are continuously embedded in some Hausdorff topological vector space) and K is the classical interpolation functional of Peetre.

$$K(t,a) = K(t,a,\mathcal{A}_1,\mathcal{A}_2) = \inf\{\|a_1\|_{\mathcal{A}_1} + t\|a_2\|_{\mathcal{A}_2} : a = a_1 + a_2\}, \quad t > 0.$$

We can define, for each p, $0 < p \leq \infty$ and each Lebesgue-measurable function $f : (0,\infty) \to (0,\infty)$, the space

$$(\mathcal{A}_1,\mathcal{A}_2)_{f,p;K} = \Big\{a : a \in \mathcal{A}_1 + \mathcal{A}_2, \|a\|_{f,p;K} = \|K(t,a;\mathcal{A}_1,\mathcal{A}_2)/f(t)\|_{L^q_*(0,\infty)} < \infty\Big\}.$$

The space $(\mathcal{A}_1,\mathcal{A}_2)_{f,p;K}$ is quasi-normed by $\|\cdot\|_{f,p;K}$. To generalize to $(\mathcal{A}_1,\mathcal{A}_2)_{f,p;K}$ the very well known properties of this space when $f(t) = t^\theta$ (i.e., $(\mathcal{A}_1,\mathcal{A}_2)_{\theta,p;K}$), one takes the function f in the class $\mathfrak{B}$. In **[10]**, Merucci showed that interpolation with a function parameter is perfectly suited for identifying interpolation spaces between two quasi-normed Lorentz spaces $\Lambda^q_\Omega(\varphi)$. We refer the reader to **[5, 10, 11, 13]** for the theory and bibliography concerning these spaces. Recall also that intersection of two Lorentz spaces $\Lambda^q_\Omega(\varphi_1)$ and $\Lambda^q_\Omega(\varphi_2)$ is a quasi–Banach space under the quasi-norm $\max\{\|\cdot\|_{\Lambda^q_\Omega(\varphi_1)}, \|\cdot\|_{\Lambda^q_\Omega(\varphi_2)}\}$.

3. Main results. In the sequel, we assume that $(\Omega,\mathfrak{F},P)$ is probability space and establish an extension of Rosenthal inequalities in Lorentz spaces $\Lambda^q_\Omega(\varphi)$. To prove the main result, we need the following lemma.

Lemma 3.1. *Let* $0 < r < p < \infty$, $f \in \mathfrak{B}$, *and* $0 < q \leq \infty$. *Then*

$$(L^r(0,\infty), L^r(0,\infty) \cap L^p(0,\infty))_{f,q;K} = L^r(0,\infty) \cap (L^r(0,\infty), L^p(0,\infty))_{f,q;K}.$$

Proof. By use of Holmsted's formula about interpolation with a function parameter **[10, 11]** the proof of this lemma is similar to **[3,** Lemma 2.1]. □

Theorem 3.2. *Given* $1 \le r \le 2 < p < \infty$ *and* $0 < q \le \infty$, *let* $f \in \mathfrak{B}$ *with* $0 < \beta_{\overline{f}} \le \alpha_{\overline{f}} < 1$. *Then*

$$\Big\|\sum_{k=1}^n X_k\Big\|_{\Lambda_\Omega^q(\varphi)} \approx \max\left\{\Big\|\sum_{k=1}^n X_k\Big\|_{L^r(\Omega)}, \Big\|\sum_{k=1}^n \bigoplus X_k\Big\|_{\Lambda_{(0,\infty)}^q(\varphi)}\right\},$$

for all independent symmetric random variables $X_1, X_2, \ldots, X_n$ *in* $\Lambda_\Omega^q(\varphi)$, *where*

$$\varphi(t) = \frac{t^{1/r}}{f\left(t^{1/r-1/p}\right)}.$$

Proof. It follows from [**10,** Theorem 3] that $\varphi \in \mathfrak{B}$. It is convenient to take Ω be $[0,1]^{\mathbf{N}}$ with the product measure and denote a typical element of Ω by the sequence $t = (t_1, t_2, \ldots)$. Define a linear operator $T : L_0(0,\infty) \to L_0(\Omega \times [0,1])$ by

$$T(g) = \sum_{k=1}^\infty g_k(t_k) r_k(s),$$

where $g_k(t_k) = g(t_k + k - 1)$ and $r_k(s)$ is the kth Rademacher function. Then, by Hu's inequality [**6**], T is a bounded operator from $L^r(0,\infty) \cap L^p(0,\infty)$ into $L^p(\Omega \times [0,1])$ for $p > 2$. So, by Lemma 3.1 and the interpolation theorem with a function parameter ([**10,** Theorem 3] and [**5**]), T is bounded from $L^r(0,\infty) \cap \Lambda_{(0,\infty)}^q(\varphi)$ into $\Lambda_\Omega^q(\varphi)$, where

$$\varphi(t) = \frac{t^{1/r}}{f\left(t^{1/r-1/p}\right)}.$$

Therefore, there exists a positive constant C such that

$$\Big\|\sum_{k=1}^n X_k\Big\|_{\Lambda_\Omega^q(\varphi)} \le C \max\left\{\Big\|\sum_{k=1}^n X_k\Big\|_{L^r(\Omega)}, \Big\|\sum_{k=1}^n \bigoplus X_k\Big\|_{\Lambda_{(0,\infty)}^q(\varphi)}\right\}. \tag{3.1}$$

It follows from Remark 2.2 that

$$\Big\|\sum_{k=1}^n X_k\Big\|_{L^2(\Omega)} \le C_1 \Big\|\sum_{k=1}^n X_k\Big\|_{\Lambda_{(\Omega)}^q(\varphi)} \tag{3.2}$$

for a positive constant C_1.

Since $1 \leq r \leq 2 < p$ and $\alpha_{\overline{f}} < 1$, it follows from [**10,** Propositions 2, 3] that

$$\alpha_{\overline{f}\left(t^{1/r-1/p}\right)} = \left(\frac{1}{r} - \frac{1}{p}\right)\alpha_{\overline{f}},$$

and so $\alpha_{\overline{\varphi}} < 1$. On the other hand, $\overline{\alpha}_{\Lambda^q_\Omega(\varphi)} = \alpha_{\overline{\varphi}} < 1$, where $\overline{\alpha}_{\Lambda^q_\Omega(\varphi)}$ are Boyd indices of $\Lambda^q_\Omega(\varphi)$, [**13**]. Now, by [**8,** Theorem 5.8], $\Lambda^q_\Omega(\varphi)$ has the Kalton property (that is, for

$$\varphi(t) = \frac{t^{1/r}}{f\left(t^{1/r-1/p}\right)},$$

$\Lambda^q_\Omega(\varphi)$ satisfies $\|X\| \leq C\|Y\|$ whenever $X^{**} \leq Y^{**}$ (recall that $X^{**}(t) = t^{-1}\int_0^t X^*(s)\,ds$)).

By the definition of the disjoint sum, it is easy to check that

$$\left(\sum_{k=1}^{n} \bigoplus X_k\right)^{**} \leq \left(\left(\sum_{k=1}^{n} |X_k|^2\right)^{1/2}\right)^{**}.$$

Now, by the Kalton property, we have

$$\left\|\sum_{k=1}^{n} \bigoplus X_k\right\|_{L^q((0,\infty))} \leq C_2\left\|\left(\sum_{k=1}^{n} |X_k|^2\right)^{1/2}\right\|_{\Lambda^q_\Omega(\varphi)} \tag{3.3}$$

for some positive constant C_2. Since $\sum_{k=1}^n X_k$ has the same distribution as $\sum_{k=1}^n X_k(t)r_k(t)$, by the Maurey-Khintchine inequality [**9,** Theorem 1.d.6] and inequality (3.3), we obtain

$$\left\|\sum_{k=1}^{n} \bigoplus X_k\right\|_{\Lambda^q_{(0,\infty)}(\varphi)} \leq C_3\left\|\sum_{k=1}^{n} X_k\right\|_{\Lambda^q_\Omega(\varphi)}, \tag{3.4}$$

for some positive constant C_3. Therefore, by inequalities (3.2) and (3.4), we get

$$C^{'} \max\left\{\left\|\sum_{k=1}^{n} X_k\right\|_{L^2(\Omega)}, \left\|\sum_{k=1}^{n} \bigoplus X_k\right\|_{\Lambda^q_{(0,\infty)}(\varphi)}\right\} \leq \left\|\sum_{k=1}^{n} X_k\right\|_{\Lambda^q_\Omega(\varphi)},$$

where $C^{'} = 1/\max\{C_1, C_3\}$. So, the desired inequality now follows easily since $1 \le r \le 2$, i.e.,

(3.5)

$$C' \max\left\{\left\|\sum_{k=1}^{n} X_k\right\|_{L^r(\Omega)}, \left\|\sum_{k=1}^{n} \bigoplus X_k\right\|_{\Lambda^q_{(0,\infty)}(\varphi)}\right\} \le \left\|\sum_{k=1}^{n} X_k\right\|_{\Lambda^q_\Omega(\varphi)}.$$

Thus, inequalities (3.1) and (3.5) imply that

$$\left\|\sum_{k=1}^{n} X_k\right\|_{\Lambda^q_\Omega(\varphi)} \approx \max\left\{\left\|\sum_{k=1}^{n} X_k\right\|_{L^r(\Omega)}, \left\|\sum_{k=1}^{n} \bigoplus X_k\right\|_{\Lambda^q_{(0,\infty)}(\varphi)}\right\}. \quad \square$$

Corollary 3.3. *Given $1 \le r \le 2 < p < \infty$ and $0 < q \le \infty$, we then have*

$$\left\|\sum_{k=1}^{n} X_k\right\|_{L^{p,q}(\log L)^\gamma} \approx \max\left\{\left\|\sum_{k=1}^{n} X_k\right\|_{L^r(\Omega)}, \left\|\sum_{k=1}^{n} \bigoplus X_k\right\|_{L^{p,q}(\log L)^\gamma}\right\},$$

for all independent symmetric random variables $X_1, X_2, \dots, X_n$ in $L^{p,q}(\log L)^\gamma$.

Proof. It is sufficient to consider

$$f(t) = t^\theta \left(1 + \frac{pr}{p-r} |\log t|\right)^{-|\gamma|}$$

in Theorem 3.2. $\square$

Remark 3.4. In the previous corollary, if $\gamma = 0$ and $r = 2$ $(p = q)$, then this corollary implies Rosenthal inequalities (1.2) in Lorentz spaces $L^{p,q}$ (spaces L^p).

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