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ELEMENTARY APPROACH TO HOMOGENEOUS C^*-ALGEBRAS

PIOTR NIEMIEC

ABSTRACT. An elementary proof of Fell's theorem on models of homogeneous C^*-algebras is presented. A spectral theorem and a functional calculus for finite systems of elements which generate homogeneous C^*-algebras are proposed.

1. Introduction. In 1961, Fell [**9**] introduced models for n-homogeneous C^*-algebras in terms of certain fibre bundles. It is a natural generalization of the commutative Gelfand-Naimark theorem, which gives models for commutative C^*-algebras. However, Fell's proof involves the machinery of (general) operator fields and, as such, is more advanced than Gelfand's theory of commutative Banach algebras. Tomiyama and Takesaki [**28**] gave another proof of Fell's theorem, which involved techniques of von Neumann algebras. In this paper, we propose a new proof of this theorem (starting from the very beginning), which is elementary and resembles the standard proof of the commutative Gelfand-Naimark theorem. We avoid the abstract language of fibre bundles; instead of them we introduce n-spaces, which are counterparts of locally compact Hausdorff spaces in the commutative case. These are locally compact Hausdorff spaces endowed with a (continuous) free action of the group $\mathfrak{U}_n = \mathcal{U}_n/Z(\mathcal{U}_n)$ where $\mathcal{U}_n$ is the unitary group of $n \times n$-matrices and $Z(\mathcal{U}_n)$ is its center.

Our approach to the subject mentioned above enables us to generalize the spectral theorem (for a normal Hilbert space operator) to the context of finite systems generating homogeneous C^*-algebras. It also

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allows building so-called n-functional calculus for such systems. These and related topics are discussed in the present paper.

The paper is organized as follows. Section 2 is devoted to an operator-valued version of the Stone-Weierstrass theorem, which plays an important role in our proof of Fell's theorem on homogeneous C^*-algebras (presented is Section 5). In Section 3, we define and establish basic properties of so-called n-spaces $(X, .)$ (which, in fact, are the same as Fell's fibre bundles) and, corresponding to them, C^*-algebras $C^*(X, .)$. These investigations are continued in the next part where we define spectral n-measures and characterize by means of them *all* representations of $C^*(X, .)$ for any n-space $(X, .)$. In Section 5, we give a new proof of Fell's characterization of homogeneous C^*-algebras. In Section 6, we formulate the spectral theorem for finite systems of elements which generate n-homogeneous C^*-algebras and build the n-functional calculus for them.

Notation and terminology. If a C^*-algebra $\mathcal{A}$ has a unit e, the spectrum of x is denoted by $\sigma(x)$, and it is the set of all $\lambda \in \mathbb{C}$ for which $x - \lambda e$ is noninvertible in $\mathcal{A}$. For two self-adjoint elements a and b of $\mathcal{A}$ we write $a \leqslant b$ provided $b - a$ is nonnegative. If $a \leqslant b$ and $b - a$ is invertible in $\mathcal{A}$, we shall express this by writing $a < b$ or $b > a$. The C^*-algebra of all bounded operators on a (complex) Hilbert space $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$. Representations of unital C^*-algebras need not preserve unities and they are understood as $*$-homomorphisms into $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. A representation of a C^*-algebra is n-dimensional if it acts on an n-dimensional Hilbert space. A *map* is a continuous function.

2. Operator-valued Stone-Weierstrass theorem. The classical Stone-Weierstrass theorem finds many applications in functional analysis and approximation theory. It reached many generalizations as well, see e.g., [**2, 10, 11, 14, 17, 21, 27**] and the references therein (consult also [**5,** Corollary 11.5.3], [**9,** Theorem 1.4] and [**23,** subsection 4.7]). A first significant counterpart of it for general C^*-algebras was established by Glimm [**11**]. Much later, Longo [**17**] and Popa [**21**] proved independently a stronger version of Glimm's result, solving a long-standing problem in theory of C^*-algebras. In comparison to the classical Stone-Weierstrass theorem or, for example, to its generalization by Timofte [**27**], Glimm's and Longo's-Popa's theorems are not settled in function spaces. In this section, we propose another version of the theorem under discussion which takes place in spaces of functions taking values in C^*-algebras. As such, it may be considered as its very natural generalization. Although the results of Glimm and Longo and Popa are stronger and more general than ours, they involve advanced machinery of C^*-algebras and advanced language of this theory, while our approach is very elementary and its proof is similar to Stone's [**24, 25**]. To formulate our result, we need to introduce the following notion.

Definition 2.1. Let X be a set, x and y distinct points of X, and let $\mathcal{A}$ be a unital C^*-algebra. A collection $\mathcal{F}$ of functions from X to $\mathcal{A}$ *spectrally separates* points x and y if there is $f \in \mathcal{F}$ such that $f(x)$ and $f(y)$ are normal elements of $\mathcal{A}$ and their spectra are disjoint. If $\mathcal{F}$ spectrally separates any two distinct points of X, we say that $\mathcal{F}$ *spectrally separates points* of X.

The reader should notice that a collection of complex-valued functions spectrally separates two points if and only if it separates them.

Whenever $\mathcal{A}$ is a unital C^*-algebra and a is a self-adjoint element of $\mathcal{A}$, let us denote by $M(a)$ the real number $\max \sigma(a)$. Further, if X is a locally compact Hausdorff space and $f\colon X \to \mathcal{A}$ is a map, we say that f *vanishes at infinity* if and only if for every $\varepsilon > 0$ there is a compact set $K \subset X$ such that $\|f(x)\| < \varepsilon$ for any $x \in X \setminus K$. The set of all $\mathcal{A}$-valued maps on X vanishing at infinity is denoted by $\mathcal{C}_0(X, \mathcal{A})$. Notice that $\mathcal{C}_0(X, \mathcal{A})$ is a C^*-algebra when it is equipped with pointwise actions and the supremum norm induced by the norm of $\mathcal{A}$. Moreover, $\mathcal{C}_0(X, \mathcal{A})$ is unital if and only if X is compact (recall that we assume here that $\mathcal{A}$ is unital).

A full version of our Stone-Weierstrass type theorem has the following form.

Theorem 2.2. *Let X be a locally compact Hausdorff space, and let $\mathcal{A}$ be a unital C^*-algebra. Let $\mathcal{E}$ be a $*$-subalgebra of $\mathcal{C}_0(X, \mathcal{A})$ such that:*

(AX0) *if X is noncompact, then for each $z \in X$ either $f_0(z)$ is invertible in $\mathcal{A}$ for some $f_0 \in \mathcal{E}$ or $f(z) = 0$ for any $f \in \mathcal{E}$;*

and for any two points x and y of X one of the following two conditions is fulfilled:

(AX1) *either x and y are spectrally separated by $\mathcal{E}$, or*
(AX2) $M(f(x)) = M(f(y))$ *for any self-adjoint $f \in \mathcal{E}$.*

Then the (uniform) closure of $\mathcal{E}$ in $\mathcal{C}_0(X, \mathcal{A})$ coincides with the $*$-algebra $\Delta_2(\mathcal{E})$ of all maps $u \in \mathcal{C}_0(X, \mathcal{A})$ such that for any $x, y \in X$ and each $\varepsilon > 0$ there exists $v \in \mathcal{E}$ with $\|v(z) - u(z)\| < \varepsilon$ for $z \in \{x, y\}$.

As a consequence of the above result we obtain the following result, which is a special case of [**5,** Corollary 11.5.3].

Proposition 2.3. *Let X be a locally compact Hausdorff space, and let $\mathcal{A}$ be a unital C^*-algebra. A $*$-subalgebra $\mathcal{E}$ of $\mathcal{C}_0(X, \mathcal{A})$ is dense in $\mathcal{C}_0(X, \mathcal{A})$ if and only if $\mathcal{E}$ spectrally separates points of X and for every $x \in X$ the set $\mathcal{E}(x) := \{f(x) \colon\ f \in \mathcal{E}\}$ is dense in $\mathcal{A}$.*

It is worth noting that we know no characterization of dense $*$-subalgebras of $\mathcal{C}_0(X, \mathcal{A})$ in case $\mathcal{A}$ does not have a unit.

The proof of Theorem 2.2 is partially based on the original proof of the Stone-Weierstrass theorem given by Stone [**24, 25**]. However, the key tool in our proof is the so-called Loewner-Heinz inequality (for the discussion on this inequality, see [**1,** page 150]), first proved by Loewner [**18**]:

Theorem 2.4. *Let a and b be two self-adjoint nonnegative elements in a C^*-algebra such that $a \leqslant b$. Then, for every $s \in (0, 1)$, $a^s \leqslant b^s$.*

The proof of Theorem 2.2 is preceded by several auxiliary results. For simplicity, the unit of $\mathcal{A}$ will be denoted by 1 and the function from X to $\mathcal{A}$ constantly equal to 1 will be denoted by 1_X. We also preserve the notation of Theorem 2.2. Additionally, $\overline{\mathcal{E}}$ stands for the (uniform) closure of $\mathcal{E}$ in $\mathcal{C}_0(X, \mathcal{A})$.

Lemma 2.5. *Suppose X is compact. Then $1_X \in \overline{\mathcal{E}}$ if and only if for every $x \in X$ there is $f \in \mathcal{E}$ such that $f(x)$ is invertible in $\mathcal{A}$.*

Proof. The necessity is clear (since the set of all invertible elements is open in $\mathcal{A}$).

To prove the sufficiency, for each $x \in X$ take $f_x \in \mathcal{E}$ such that $f_x(x)$ is invertible. Put $u_x = f_x^* f_x \in \mathcal{E}$, and let $V_x \subset X$ consist of all $y \in X$ such that $u_x(y) > 0$. It follows from the continuity of u_x that V_x is open. By the compactness of X, $X = \bigcup_{j=1}^p V_{x_j}$ for some finite system $x_1, \dots, x_p$. Put $u = \sum_{j=1}^p u_{x_j} \in \mathcal{E}$ and note that $u(x) > 0$ for each $x \in X$. This implies that u is invertible in $\mathcal{C}_0(X, \mathcal{A})$. Let $f\colon [0, \|u\|] \to \mathbb{R}$ be a map with $f(0) = 0$ and $f\big|_{\sigma(u)} \equiv 1$. There is a sequence of real polynomials $p_1, p_2, \dots$ which converge uniformly to f on $[0, \|u\|]$. Then $p_n(u) \to f(u) = 1_X$ (in the norm topology) and hence $1_X \in \overline{\mathcal{E}}$. □

Lemma 2.6. *Suppose X is compact and $1_X \in \overline{\mathcal{E}}$. Let $x \in X$ and $\delta > 0$ be arbitrary. For any self-adjoint $f \in \Delta_2(\mathcal{E})$, there are self-adjoint $g, h \in \overline{\mathcal{E}}$ such that $g(x) = f(x) = h(x)$ and $g - \delta \cdot 1_X \leqslant f \leqslant h + \delta \cdot 1_X$.*

Proof. It follows from the definition of $\Delta_2(\mathcal{E})$ (and the fact that $*$-homomorphisms between C^*-algebras have closed ranges) that for every $y \in X$ there is an $f_y \in \overline{\mathcal{E}}$ with $f_y(z) = f(z)$ for $z \in \{x, y\}$. Replacing, if needed, f_y by $(f_y + f_y^*)/2$, we may assume that f_y is self-adjoint. Let $U_y \subset X$ consist of all $z \in X$ such that $\|f_y(z) - f(z)\| < \delta$. Take a finite number of points $x_1, \dots, x_p$ for which $X = \bigcup_{j=1}^p U_{x_j}$. For simplicity, put $V_j = U_{x_j}$ and $g_j = f_{x_j}$ $(j = 1, \dots, p)$. Observe that $f - \delta \cdot 1_X \leqslant g_j$ on V_j and $g_j(x) = f(x)$. We define, by induction, functions $h_1, \dots, h_p \in \overline{\mathcal{E}}$: $h_1 = g_1$ and $h_k = (h_{k-1} + g_k + |h_{k-1} - g_k|)/2$ for $k = 2, \dots, p$ where $|u| = \sqrt{u^* u}$ for each $u \in \overline{\mathcal{E}}$. Since $\overline{\mathcal{E}}$ is a C^*-algebra, we clearly have $h_k \in \overline{\mathcal{E}}$. Use induction to show that $h_j(x) = f(x)$ and $g_j \leqslant h_p$ for $j = 1, \dots, p$. Then $h = h_p$ is the function we searched for. Indeed, $h(x) = f(x)$, and for any $y \in X$, there is $j \in \{1, \dots, p\}$ such that $y \in V_j$, which implies that $f(y) - \delta \cdot 1 \leqslant g_j(y) \leqslant h(y)$.

Now if we apply the above argument to the function $-f$, we shall obtain a self-adjoint function $h' \in \overline{\mathcal{E}}$ such that $-f(x) = h'(x)$ and $-f \leqslant h' + \delta \cdot 1_X$. Then put $g := -h'$ to complete the proof. □

Lemma 2.7. *Let $\varepsilon > 0$, $r > 0$ and $k \geqslant 1$ be given. There is a natural number $N = N(\varepsilon, r, k)$ with the following property. If $a_1, \dots, a_k, b$ are* *self-adjoint elements of* $\mathcal{A}$ *such that* $0 \leqslant a_j \leqslant b$, $ba_j = a_jb$ $(j = 1, \dots, k)$ *and* $\|b\| \leqslant r$, *then* $a_s \leqslant (\sum_{j=1}^k a_j^n)^{1/n} \leqslant b + \varepsilon \cdot 1$ *for any* $s \in \{1, \dots, k\}$ *and* $n \geqslant N$.

Proof. Let $N \geqslant 2$ be such that $\sqrt[n]{k} \leqslant 1 + \varepsilon/r$ for each $n \geqslant N$, and let $a_1, \dots, a_k, b$ be as in the statement of the lemma. Then since $a_s^n \leqslant \sum_{j=1}^k a_j^n$, Theorem 2.4 yields $a_s \leqslant (\sum_{j=1}^k a_j^n)^{1/n}$. Further, since b commutes with a_j, we get $a_j^n \leqslant b^n$, and consequently, $\sum_{j=1}^k a_j^n \leqslant kb^n$. So, another application of Theorem 2.4 gives us $(\sum_{j=1}^k a_j^n)^{1/n} \leqslant \sqrt[n]{k}b$. So, it suffices to have $\sqrt[n]{k}b \leqslant b+\varepsilon \cdot 1$ which is fulfilled for $n \geqslant N$ because $\|(\sqrt[n]{k} - 1)b\| \leqslant (\sqrt[n]{k} - 1)r \leqslant \varepsilon$. □

Lemma 2.8. *Suppose* X *is compact and* $1_X \in \overline{\mathcal{E}}$. *If* $f \in \Delta_2(\mathcal{E})$ *commutes with every member of* $\mathcal{E}$, *then* $f \in \overline{\mathcal{E}}$.

Proof. Since $\Delta_2(\mathcal{E})$ is a $*$-algebra, we may assume that f is self-adjoint. Fix $\delta > 0$. By Lemma 2.6, for every $x \in X$, there is an $f_x \in \overline{\mathcal{E}}$ with $f_x(x) = f(x)$ and $f_x \leqslant f + \delta \cdot 1_X$. Let $U_x \subset X$ consist of all $y \in X$ such that $f_x(y) > f(y) - \delta \cdot 1$. We infer from the compactness of X that $X = \bigcup_{j=1}^k U_{x_j}$ for some points $x_1, \dots, x_k \in X$. For simplicity, we put $V_j = U_{x_j}$ and $g_j = f_{x_j}$. We then have

$$g_j(x) \geqslant f(x) - \delta \cdot 1 \quad \text{for any } x \in V_j \tag{2.1}$$

and

$$g_j(x) \leqslant f(x) + \delta \cdot 1 \quad \text{for any } x \in X. \tag{2.2}$$

It follows from the compactness of X that there is a constant $c > 0$ such that $g_j + c \cdot 1_X \geqslant 0$ $(j = 1, \dots, k)$ and $f + (c - \delta) \cdot 1_X \geqslant 0$. Further, there is an $r > 0$ such that $f + (c + \delta) \cdot 1_X \leqslant r \cdot 1_X$. Now let $N = N(\delta, r, k)$ be as in Lemma 2.7. Since f commutes with each member of $\overline{\mathcal{E}}$, we conclude from that lemma and from (2.2) that $g_s(x) + c \cdot 1 \leqslant [\sum_{j=1}^k (g_j(x) + c \cdot 1)^n]^{1/n} \leqslant f(x) + (c + 2\delta) \cdot 1$ for any $x \in X$. Finally, since $1_X \in \overline{\mathcal{E}}$, the function

$$g := \left[\sum_{j=1}^k (g_j + c \cdot 1_X)^n\right]^{1/n} - c \cdot 1_X$$

belongs to $\overline{\mathcal{E}}$. What is more, $g \leqslant f + 2\delta \cdot 1_X$ and $g(x) \geqslant g_j(x) \geqslant f(x) - \delta \cdot 1$ for $x \in V_j$ (cf., (2.1)). This gives $f - \delta \cdot 1_X \leqslant g$ on the whole space X, and therefore $-\delta \cdot 1_X \leqslant g - f \leqslant 2\delta \cdot 1_X$, which is equivalent to $\|g - f\| \leqslant 2\delta$ and finishes the proof. □

Lemma 2.9. *Suppose X is compact, $1_X \in \overline{\mathcal{E}}$ and there exists an equivalence relation $\mathcal{R}$ on X such that two points x and y are spectrally separated by $\mathcal{E}$ whenever $(x, y) \notin \mathcal{R}$. Then every map $g\colon X \to \mathbb{C} \cdot 1 \subset \mathcal{A}$ which is constant on each equivalence class with respect to $\mathcal{R}$ belongs to $\overline{\mathcal{E}}$.*

Proof. By Lemma 2.8, we only need to check that $g \in \Delta_2(\mathcal{E})$. We may assume that $g\colon X \to \mathbb{R} \cdot 1$. Let x and y be arbitrary. Write $g(x) = \alpha \cdot 1$ and $g(y) = \beta \cdot 1$. If $(x, y) \in \mathcal{R}$, then both x and y belong to the same equivalence class, and hence $\alpha = \beta$. Then $g(z) = (\alpha \cdot 1_X)(z)$ for $z \in \{x, y\}$ (and $\alpha \cdot 1_X \in \overline{\mathcal{E}}$). Now assume that $(x, y) \notin \mathcal{R}$. Then, by assumption, there is an $f \in \mathcal{E}$ such that both $f(x)$ and $f(y)$ are normal and $\sigma(f(x)) \cap \sigma(f(y)) = \varnothing$. Let $\varphi\colon \mathbb{C} \to \mathbb{R}$ be a map such that $\varphi\big|_{\sigma(f(x))} \equiv \alpha$ and $\varphi\big|_{\sigma(f(y))} \equiv \beta$. There is a sequence of polynomials $p_1(z, \overline{z}), p_2(z, \overline{z}), \ldots$, which converge uniformly to φ on $K := \sigma(f(x)) \cup \sigma(f(y))$. Then $p_n(f, f^*) \in \mathcal{E}$ and, for $w \in \{x, y\}$, $[p_n(f, f^*)](w) = p_n(f(w), [f(w)]^*)$. Since $f(w)$ is normal and its spectrum is contained in K, we see that

$$\lim_{n\to\infty} [p_n(f, f^*)](w) = \varphi(f(w)).$$

Now notice that $\varphi(f(x)) = \alpha \cdot 1 = g(x)$ and $\varphi(f(y)) = \beta \cdot 1 = g(y)$ finishes the proof. □

We recall that, if X is a compact Hausdorff space and $\mathcal{R}$ is a closed equivalence relation on X, then the quotient topological space $X/\mathcal{R}$ is Hausdorff as well.

Lemma 2.10. *Suppose X is compact and there is a closed equivalence relation $\mathcal{R}$ on X such that $M(f(x)) = M(f(y))$ for each self-adjoint $f \in \mathcal{E}$ whenever $(x, y) \in \mathcal{R}$. Let $\pi\colon X \to X/\mathcal{R}$ denote the canonical projection, $f \in \overline{\mathcal{E}}$ be self-adjoint, a and b two real numbers, and let $U = \{x \in X\colon\ a \cdot 1 < f(x) < b \cdot 1\}$. Then $\pi^{-1}(\pi(U)) = U$ and $\pi(U)$ is open in $X/\mathcal{R}$.*

Proof. Recall that $\pi(U)$ is open in $X/\mathcal{R}$ if and only if $\pi^{-1}(\pi(U))$ is open in X. Therefore, it suffices to show that $\pi^{-1}(\pi(U)) = U$. Of course, the inclusion '$\supset$' is immediate. And, if $y \in \pi^{-1}(\pi(U))$, then there is an $x \in U$ such that $(x, y) \in \mathcal{R}$. We then have $a \cdot 1 < f(x) < b \cdot 1$, $M(f(x)) = M(f(y))$ and $M(-f(x)) = M(-f(y))$ (the last two relations follow from the fact that $f \in \overline{\mathcal{E}}$). The first of these relations says that $[-M(-f(x)), M(f(x))] \subset (a, b)$, from which we infer that $[-M(-f(y)), M(f(y))] \subset (a, b)$, and consequently $y \in U$. □

The following is a special case of Theorem 2.2.

Lemma 2.11. *Suppose X is compact, $1_X \in \overline{\mathcal{E}}$ and, for any $x, y \in X$, one of conditions* (AX1)–(AX2) *is fulfilled. Then $\Delta_2(\mathcal{E}) = \overline{\mathcal{E}}$.*

Proof. We only need to show that $\Delta_2(\mathcal{E})$ is contained in $\overline{\mathcal{E}}$. Let $f \in \Delta_2(\mathcal{E})$ be self-adjoint, and let $\delta > 0$. We shall construct $w \in \overline{\mathcal{E}}$ such that $\|w - f\| \leqslant 3\delta$. By Lemma 2.6, for each $x \in X$, there are functions $u_x, v_x \in \overline{\mathcal{E}}$ such that $u_x(x) = f(x) = v_x(x)$ and $u_x - \delta \cdot 1_X < f < v_x + \delta \cdot 1_X$. Let $G_x \subset X$ consist of all $y \in X$ such that $v_x(y) - \delta \cdot 1 < f(y) < u_x(y) + \delta \cdot 1$. Since $x \in G_x$ and X is compact, there is a finite system $x_1, \ldots, x_k \in X$ for which $X = \bigcup_{j=1}^{k} G_{x_j}$. For simplicity, we put $W_j = G_{x_j}$, $p_j = u_{x_j}$ and $q_j = v_{x_j}$. Observe that then

$$p_j(x) - \delta \cdot 1 < f(x) < q_j(x) + \delta \cdot 1 \quad \text{for any } x \in X \tag{2.3}$$

and

$$q_j(x) - \delta \cdot 1 < f(x) < p_j(x) + \delta \cdot 1 \quad \text{for any } x \in W_j. \tag{2.4}$$

Let D_j consist of all $x \in X$ such that $-2\delta \cdot 1 < p_j(x) - q_j(x) < 2\delta \cdot 1$. We infer from (2.3) and (2.4) that $W_j \subset D_j$, and thus $X = \bigcup_{j=1}^{k} D_j$. Further, let $\mathcal{R}$ be an equivalence relation on X given by the rule: $(x, y) \in \mathcal{R} \iff M(u(x)) = M(u(y))$ for each self-adjoint $u \in \mathcal{E}$. It follows from the definition of $\mathcal{R}$ that $\mathcal{R}$ is closed in $X \times X$. Denote by $\pi\colon X \to X/\mathcal{R}$ the canonical projection. We deduce from Lemma 2.10 that the sets $\pi(D_1), \ldots, \pi(D_k)$ form an open cover of the space $X/\mathcal{R}$ (which is compact and Hausdorff). Now let $\beta_1, \ldots, \beta_k\colon X/\mathcal{R} \to [0, 1]$ be a partition of unity such that $\beta_j^{-1}((0, 1]) \subset \pi(D_j)$ for $j = 1, \ldots, k$. Put $\alpha_j = (\beta_j \circ \pi) \cdot 1\colon X \to \mathbb{C} \cdot 1 \subset \mathcal{A}$. Lemma 2.9 combined with conditions (AX1)–(AX2) yields that $\alpha_1, \dots, \alpha_k \in \overline{\mathcal{E}}$. Define $w \in \overline{\mathcal{E}}$ by $w = \sum_{j=1}^k \alpha_j p_j$. Since $\sum_{j=1}^k \alpha_j = 1_X$, we conclude from (2.3) that $w \leqslant f + \delta \cdot 1_X$. So, to end the proof, it is enough to check that $f(x) \leqslant w(x) + 3\delta \cdot 1$ for each $x \in X$. This inequality will be satisfied, provided

$$\alpha_j(x)(f(x) - 3\delta \cdot 1) \leqslant \alpha_j(x) p_j(x) \tag{2.5}$$

for any j. We consider two cases. If $x \in D_j$, then $p_j(x) > q_j(x) - 2\delta \cdot 1 > f(x) - 3\delta \cdot 1$ (by (2.3)) and consequently (2.5) holds. Finally, if $x \notin D_j$, then $\pi(x) \notin \pi(D_j)$ (see Lemma 2.10) and therefore $\alpha_j(x) = 0$, which easily gives (2.5). □

Proof of Theorem 2.2. We only need to check that $\Delta_2(\mathcal{E}) \subset \overline{\mathcal{E}}$. We consider two cases.

First assume X is compact. Let $\mathcal{E}' = \mathcal{E} + \mathbb{C} \cdot 1_X$. Observe that $\mathcal{E}'$ is a $*$-algebra and, for any two points x and y, one of the conditions (AX1)–(AX2) is fulfilled with $\mathcal{E}$ replaced by $\mathcal{E}'$. Consequently, it follows from Lemma 2.11 that $\overline{\mathcal{E}'} = \Delta_2(\mathcal{E}')$. But $\overline{\mathcal{E}'} = \overline{\mathcal{E}} + \mathbb{C} \cdot 1_X$. So, for any $g \in \Delta_2(\mathcal{E})$, we clearly have $g \in \Delta_2(\mathcal{E}')$, and hence $g = f + \lambda \cdot 1_X$ for some $f \in \overline{\mathcal{E}}$ and $\lambda \in \mathbb{C}$. If $\lambda = 0$, then $g = f \in \overline{\mathcal{E}}$, and we are done. Otherwise, $1_X = (g - f)/\lambda \in \Delta_2(\mathcal{E})$, which implies that the assumptions of Lemma 2.5 are satisfied. We infer from that lemma that $1_X \in \overline{\mathcal{E}}$ and, therefore, $g \in \overline{\mathcal{E}}$ as well.

Now assume that X is noncompact. Let $\widehat{X} = X \cup \{\infty\}$ be the one-point compactification of X. Every function $f \in \mathcal{C}_0(X, \mathcal{A})$ admits a unique continuous extension $\widehat{f} \colon \widehat{X} \to \mathcal{A}$, given by $\widehat{f}(\infty) = 0$. Denote by $\widehat{\mathcal{E}}$ the $*$-subalgebra of $\mathcal{C}(\widehat{X}, \mathcal{A})$ consisting of all extensions of (all) functions from $\mathcal{E}$. We claim that, for any $x, y \in \widehat{X}$, one of the conditions (AX1)–(AX2) is fulfilled with $\mathcal{E}$ replaced by $\widehat{\mathcal{E}}$. Indeed, if both x and y differ from ∞, this follows from our assumptions about $\mathcal{E}$. And if, for example, $y = \infty \neq x$, condition (AX0) implies that either $M(\widehat{f}(x)) = M(\widehat{f}(y))$ for each $f \in \mathcal{E}$ or $\widehat{u}(x)$ is invertible in $\mathcal{A}$ for some $u \in \mathcal{E}$. But then $f = u^* u \in \mathcal{E}$ is normal and $0 \notin \sigma(\widehat{f}(x))$, while $\sigma(\widehat{f}(y)) = \{0\}$, which shows that x and y are spectrally separated by $\widehat{\mathcal{E}}$. So, it follows from the first part of the proof that the closure of $\widehat{\mathcal{E}}$ in $\mathcal{C}(\widehat{X}, \mathcal{A})$ coincides with $\Delta_2(\widehat{\mathcal{E}})$. But the closure of $\widehat{\mathcal{E}}$ coincides with

$\{\widehat{f}\colon\ f \in \overline{\mathcal{E}}\}$ and $\Delta_2(\widehat{\mathcal{E}}) = \{\widehat{f}\colon\ f \in \Delta_2(\mathcal{E})\}$. We infer from these that $\Delta_2(\mathcal{E}) = \overline{\mathcal{E}}$, and the proof is complete. $\square$

Proof of Proposition 2.3. The necessity of the condition is clear (since, for any two distinct points x and y in X and any elements a and b of $\mathcal{A}$, there is a function $f \in \mathcal{C}_0(X,\mathcal{A})$ such that $f(x) = a$ and $f(y) = b$). To prove the sufficiency, assume $\mathcal{E}$ spectrally separates points of X and, for each $x \in X$, the set $\mathcal{E}(x)$ is dense in $\mathcal{A}$. First notice that then for each $x \in X$ there is an $f \in \mathcal{E}$ such that $f(x)$ is invertible in $\mathcal{A}$. This shows that all assumptions of Theorem 2.2 are satisfied. According to that result, we only need to show that, for any two distinct points x and y of X, the set $L := \{(f(x), f(y))\colon\ f \in \mathcal{E}\}$ is dense in $\mathcal{A} \times \mathcal{A}$. Since x and y are spectrally separated by $\mathcal{E}$, the proof of Lemma 2.9 shows that $(1,0), (0,1) \in \overline{L}$. Further, since both $\mathcal{E}(x)$ and $\mathcal{E}(y)$ are dense in $\mathcal{A}$, we conclude that $\{f(x)\colon\ f \in \overline{\mathcal{E}}\} = \{f(y)\colon\ f \in \overline{\mathcal{E}}\} = \mathcal{A}$ and, therefore, for arbitrary two elements a and b of $\mathcal{A}$, there are $u, v \in \overline{\mathcal{E}}$ for which $u(x) = a$ and $v(y) = b$. Then $(a,b) = (u(x), u(y)) \cdot (1,0) + (v(x), v(y)) \cdot (0,1) \in \overline{L}$ (we use here the coordinatewise multiplication), and we are done. $\square$

3. Topological n-spaces. In Fell's characterization of homogeneous C^*-algebras [**9**] (consult also [**3,** Theorem IV.1.7.23] and [**28**]) special fibre bundles appear. To make our lecture as simple and elementary as possible, we avoid this language and, instead of using fibre bundles, we shall introduce so-called *n-spaces* (see Definition 3.1 below). To this end, let M_n be the C^*-algebra of all complex $n \times n$-matrices. Let $\mathcal{U}_n$ be the unitary group of M_n and I its neutral element. Let $\mathbb{T} = \{z \in \mathbb{C}\colon\ |z| = 1\}$. Let $\mathfrak{U}_n$ denote the compact topological group $\mathcal{U}_n/(\mathbb{T} \cdot I)$, and let $\pi_n\colon \mathcal{U}_n \to \mathfrak{U}_n$ be the canonical homomorphism. Members of $\mathfrak{U}_n$ will be denoted by $\mathfrak{u}$ and $\mathfrak{v}$. The (probabilistic) Haar measure on $\mathfrak{U}_n$ will be denoted by $d\mathfrak{u}$. For any $A \in M_n$ and $\mathfrak{u} \in \mathfrak{U}_n$, let $\mathfrak{u}.A$ denote the matrix UAU^{-1} where $U \in \mathcal{U}_n$ is such that $\pi_n(U) = \mathfrak{u}$. It is easily seen that the function

$$\mathfrak{U}_n \times M_n \ni (\mathfrak{u}, A) \longmapsto \mathfrak{u}.A \in M_n$$

is a well-defined continuous action of $\mathfrak{U}_n$ on M_n (which means that $\mathfrak{j}.A = A$ where $\mathfrak{j}$ is the identity of $\mathfrak{U}$, and $\mathfrak{u}.(\mathfrak{v}.A) = (\mathfrak{u}\mathfrak{v}).A$ for any $\mathfrak{u}, \mathfrak{v} \in \mathfrak{U}_n$ and $A \in M_n$). More generally, for any C^*-algebra $\mathcal{A}$, let $M_n(\mathcal{A})$ be the algebra of all $n \times n$-matrices with entries in $\mathcal{A}$. $(M_n(\mathcal{A})$

may naturally be identified with $\mathcal{A} \otimes M_n$.) For any matrix $A \in M_n(\mathcal{A})$ and each $\mathfrak{u} \in \mathfrak{U}_n$, $\mathfrak{u}.A$ is defined as UAU^{-1} where $U \in \mathcal{U}_n$ is such that $\pi_n(U) = \mathfrak{u}$, and UAU^{-1} is computed in a standard manner.

Definition 3.1. A pair $(X, .)$ is said to be an *n-space* if X is a locally compact Hausdorff space and $\mathfrak{U}_n \times X \ni (\mathfrak{u}, x) \mapsto \mathfrak{u}.x \in X$ is a continuous *free* action of $\mathfrak{U}_n$ on X. Recall that the action is free if and only if the equality $\mathfrak{u}.x = x$ (for some $x \in X$) implies that $\mathfrak{u}$ is the identity of $\mathfrak{U}$.

Let $(X, .)$ be an n-space. Let $C^*(X, .)$ be the $*$-algebra of all maps $f \in \mathcal{C}_0(X, M_n)$ such that $f(\mathfrak{u}.x) = \mathfrak{u}.f(x)$ for any $\mathfrak{u} \in \mathfrak{U}_n$ and $x \in X$. $C^*(X, .)$ is a C^*-subalgebra of $\mathcal{C}_0(X, M_n)$.

By a *morphism* between two n-spaces $(X, .)$ and $(Y, *)$, we mean any proper map $\psi \colon X \to Y$ such that $\psi(\mathfrak{u}.x) = \mathfrak{u} * \psi(x)$ for any $\mathfrak{u} \in \mathfrak{U}_n$ and $x \in X$. (A map is proper if the inverse images of compact sets under this map are compact.) A morphism which is a homeomorphism is said to be an *isomorphism*. Two n-spaces are *isomorphic* if there exists an isomorphism between them.

The reader should notice that the (natural) action of $\mathfrak{U}_n$ on M_n is *not* free. However, one may check that the set $\mathfrak{M}_n$ of all irreducible matrices $A \in M_n$ (that is, $A \in \mathfrak{M}_n$ if and only if every matrix $X \in M_n$ which commutes with both A and A^* is of the form λI where $\lambda \in \mathbb{C}$) is open in M_n (and, thus, $\mathfrak{M}_n$ is locally compact) and the action $\mathfrak{U}_n \times \mathfrak{M}_n \ni (\mathfrak{u}, A) \mapsto \mathfrak{u}.A \in \mathfrak{M}_n$ is free, which means that $(\mathfrak{M}_n, .)$ is an n-space.

In this section, we establish basic properties of C^*-algebras of the form $C^*(X, .)$ where $(X, .)$ is an n-space. To this end, recall that, whenever $(\Omega, \mathfrak{M}, \mu)$ is a finite measure space and $f \colon \Omega \ni \omega \mapsto (f_1(\omega), \ldots, f_k(\omega)) \in \mathbb{C}^k$ is an $\mathfrak{M}$-measurable (which means that $f^{-1}(U) \in \mathfrak{M}$ for every open set $U \subset \mathbb{C}^k$) bounded function, then $\int_\Omega f(\omega)\,\mathrm{d}\mu(\omega)$ is (well) defined as

$$\Bigl(\int_\Omega f_1(\omega)\,\mathrm{d}\mu(\omega), \ldots, \int_\Omega f_k(\omega)\,\mathrm{d}\mu(\omega)\Bigr).$$

If $\|\cdot\|$ is any norm on $\mathbb{C}^k$, then

$$\Bigl\| \int_\Omega f(\omega)\,\mathrm{d}\mu(\omega) \Bigr\| \leqslant \int_\Omega \|f(\omega)\|\,\mathrm{d}\mu(\omega).$$

In particular, the above rules apply to matrix-valued measurable functions.

From now on, $n \geqslant 1$ and an n-space $(X, .)$ are fixed. A set $A \subset X$ is said to be *invariant* provided $\mathfrak{u}.a \in A$ for any $\mathfrak{u} \in \mathfrak{U}_n$ and $a \in A$. Observe that, if A is closed or open and A is invariant, then A is locally compact and consequently $(A, .)$ is an n-space (when the action of $\mathfrak{U}_n$ is restricted to A). We begin with:

Lemma 3.2. *For each $f \in \mathcal{C}_0(X, M_n)$, let $f^{\mathfrak{U}}\colon X \to M_n$ be given by:*

$$f^{\mathfrak{U}}(x) = \int_{\mathfrak{U}_n} \mathfrak{u}^{-1}.f(\mathfrak{u}.x)\,\mathrm{d}\mathfrak{u} \quad (x \in X).$$

(a) *For any $f \in \mathcal{C}_0(X, M_n)$, $f^{\mathfrak{U}} \in C^*(X, .)$.*
(b) *If $f \in \mathcal{C}_0(X, M_n)$ and $x \in X$ are such that $f(\mathfrak{u}.x) = \mathfrak{u}.f(x)$ for any $\mathfrak{u} \in \mathfrak{U}_n$, then $f^{\mathfrak{U}}(x) = f(x)$.*
(c) *Let $A \subset X$ be a closed invariant nonempty set. Every map $g \in C^*(A, .)$ extends to a map $\widetilde{g} \in C^*(X, .)$ such that $\sup_{a \in A} \|g(a)\| = \sup_{x \in X} \|\widetilde{g}(x)\|$.*
(d) *For any $x \in X$ and $A \in M_n$, there is an $f \in C^*(X, .)$ with $f(x) = A$.*
(e) *Let x and y be two points of X such that there is no $\mathfrak{u} \in \mathfrak{U}_n$ for which $\mathfrak{u}.x = y$. Then, for any $A, B \in M_n$, there is an $f \in C^*(X, .)$ such that $f(x) = A$ and $f(y) = B$.*
(f) *$C^*(X, .)$ has a unit if and only if X is compact.*

Proof. It is clear that $f^{\mathfrak{U}}$ is continuous for every $f \in \mathcal{C}_0(X, M_n)$. Further, if $K \subset X$ is a compact set such that $\|f(x)\| \leqslant \varepsilon$ for each $x \in X \setminus K$, then $\|f^{\mathfrak{U}}(z)\| \leqslant \varepsilon$ for any $z \in X \setminus \mathfrak{U}_n.K$ where $\mathfrak{U}_n.K = \{\mathfrak{u}.x\colon\ \mathfrak{u} \in \mathfrak{U}_n,\ x \in K\}$. The note that $\mathfrak{U}_n.K$ is compact leads to the conclusion that $f^{\mathfrak{U}} \in \mathcal{C}_0(X, M_n)$. Finally, for any $\mathfrak{v} \in \mathfrak{U}_n$, any representative $V \in \mathcal{U}_n$ of $\mathfrak{v}$ and each $x \in X$, we have:

$$\begin{aligned} f^{\mathfrak{U}}(\mathfrak{v}.x) &= \int_{\mathfrak{U}_n} \mathfrak{u}^{-1}.f(\mathfrak{u}\mathfrak{v}.x)\,\mathrm{d}\mathfrak{u} = \int_{\mathfrak{U}_n} (\mathfrak{u}\mathfrak{v}^{-1})^{-1}.f(\mathfrak{u}.x)\,\mathrm{d}\mathfrak{u} \\ &= \int_{\mathfrak{U}_n} \mathfrak{v}.[\mathfrak{u}^{-1}.f(\mathfrak{u}.x)]\,\mathrm{d}\mathfrak{u} = \int_{\mathfrak{U}_n} V[\mathfrak{u}^{-1}.f(\mathfrak{u}.x)]V^{-1}\,\mathrm{d}\mathfrak{u} \\ &= V \cdot \Big(\int_{\mathfrak{U}_n} \mathfrak{u}^{-1}.f(\mathfrak{u}.x)\,\mathrm{d}\mathfrak{u}\Big) \cdot V^{-1} = \mathfrak{v}.f^{\mathfrak{U}}(x), \end{aligned}$$

which proves (a). Point (b) is a simple consequence of the definition of $f^{\mathfrak{U}}$. Further, if g is as in (c), it follows from Tietze's type theorem that there is a $G \in \mathcal{C}_0(X, M_n)$ which extends g and satisfies $\sup_{a\in A} \|g(a)\| = \sup_{x\in X} \|G(x)\|$ (if X is noncompact, consider the one-point compactification $\widehat{X} = X \cup \{\infty\}$ of X, and note that then the set $\widehat{A} = A \cup \{\infty\}$ is closed in $\widehat{X}$ and g extends continuously to $\widehat{A}$). Then $\widetilde{g} = G^{\mathfrak{U}}$ is a member of $C^*(X,.)$ (by (a)) which we searched for (see (b)).

We turn to (d) and (e). Let $K = \mathfrak{U}_n.\{x\}$ and $f_0 \colon K \to M_n$ be given by $f_0(\mathfrak{u}.x) = \mathfrak{u}.A$ ($\mathfrak{u} \in \mathfrak{U}_n$). Since the action of $\mathfrak{U}_n$ on X is free, f_0 is a well-defined map. Since K is compact, (c) yields the existence of $f \in C^*(X,.)$ which extends f_0. To prove (e), we argue similarly: put $L = \mathfrak{U}_n.\{x, y\}$, and let $g_0 \colon L \to M_n$ be given by $g_0(\mathfrak{u}.x) = \mathfrak{u}.A$ and $g_0(\mathfrak{u}.y) = \mathfrak{u}.B$ ($\mathfrak{u} \in \mathfrak{U}_n$). We infer from the assumption of (e) that g_0 is a well defined map. Consequently, since L is compact, there exists, by (c), a map $g \in C^*(X,.)$ which extends g_0. This finishes the proof of (e), while point (f) immediately follows from (d). □

Proposition 3.3.

(a) *For every closed two-sided ideal $\mathfrak{I}$ in $C^*(X,.)$, there exists a (unique) closed invariant set $A \subset X$ such that $\mathfrak{I}$ coincides with the ideal $\mathfrak{I}_A$ of all functions $f \in C^*(X,.)$ which vanish on A. Moreover, $C^*(X,.)/\mathfrak{I}$ is "naturally" isomorphic to $C^*(A,.)$.*

(b) *Let $k \leqslant n$, and let $\pi \colon C^*(X,.) \to M_k$ be a nonzero representation. Then $k = n$, and there is a unique point $x \in X$ such that $\pi(f) = f(x)$ for $f \in C^*(X,.)$.*

(c) *Let $(Y, *)$ be an n-space. For every $*$-homomorphism $\Phi \colon (X,.) \to (Y, *)$, there is a unique pair (U, φ) where U is an open invariant subset of Y, $\varphi \colon (U, *) \to (X,.)$ is a morphism of n-spaces and*

$$[\Phi(f)](y) = \begin{cases} f(\varphi(y)) & \text{if } y \in U \\ 0 & \text{if } y \notin U. \end{cases} \tag{3.1}$$

In particular, $C^*(X,.)$ and $C^*(Y, *)$ are isomorphic if and only if so are $(X,.)$ and $(Y, *)$.

Proof. The uniqueness of the set A in (a) follows from point (e) of

Lemma 3.2. To show its existence, let A consist of all $x \in X$ such that $f(x) = 0$ for any $f \in \mathcal{I}$. It is clear that A is closed and invariant and that $\mathcal{I} \subset \mathcal{I}_A$. To prove the converse inclusion we shall involve Theorem 2.2 for $\mathcal{E} = \mathcal{I}$. First of all, it follows from Lemma 3.2 (d) that, for each $x \in X$, the set $\mathcal{I}(x) := \{f(x)\colon\ f \in \mathcal{I}\}$ is a two-sided ideal in M_n. Since $\{0\}$ is the only proper ideal of M_n, we conclude that $\mathcal{I}(x) = \{0\}$ for $x \in A$ and $\mathcal{I}(x) = M_n$ for $x \in X \setminus A$. This shows that condition (AX0) of Theorem 2.2 is satisfied. Further, if x and y are arbitrary points of X, then either:

- $x, y \in A$; in that case, (AX2) is fulfilled; or
- $x \in A$ and $y \notin A$ (or conversely); in that case, there is $f \in \mathcal{I}$ such that $f(y) = I$, and $f(x) = 0$ (since $x \in A$)—this implies that x and y are spectrally separated by $\mathcal{I}$; or
- $x, y \notin A$ and $y = \mathfrak{u}.x$ for some $\mathfrak{u} \in \mathfrak{U}_n$; in that case, (AX2) is fulfilled since, for any self-adjoint $f \in \mathcal{I}$, $f(y) = \mathfrak{u}.f(x)$ and consequently $\sigma(f(x)) = \sigma(f(y))$; or
- $x, y \notin A$ and $y \notin \mathfrak{U}_n.\{x\}$; in that case, there are $f_1 \in \mathcal{I}$ and $f_2 \in C^*(X,.)$ such that $f_1(x) = I = f_2(x)$ and $f_2(y) = 0$ (cf., Lemma 3.2 (e)), then $f = f_1 f_2 \in \mathcal{I}$ is such that $f(x) = I$ and $f(y) = 0$, and hence x and y are spectrally separated by $\mathcal{I}$.

Now, according to Theorem 2.2, it suffices to check that $\mathcal{I}_A \subset \Delta_2(\mathcal{I})$ (since $\mathcal{I}$ is closed). To this end, we fix $f \in \mathcal{I}_A$ and two arbitrary points x and y of X. We consider similar cases as above:

(1°) If $x, y \in A$, we have nothing to do because then $f(x) = f(y) = 0$.
(2°) If $x \in A$ and $y \notin A$ (or conversely), then there is $g \in \mathcal{I}$ such that $g(y) = f(y)$. But also $g(x) = 0 = f(x)$, and we are done.
(3°) If $x, y \notin A$ and $y = \mathfrak{u}.\{x\}$ for some $\mathfrak{u} \in \mathfrak{U}_n$, then there is a $g \in \mathcal{I}$ with $g(x) = f(x)$. Then also $g(y) = g(\mathfrak{u}.x) = \mathfrak{u}.g(x) = \mathfrak{u}.f(x) = f(y)$, and we are done.
(4°) Finally, if $x, y \notin A$ and $y \notin \mathfrak{U}_n.\{x\}$, there are functions $g_1, g_2 \in \mathcal{I}$ and $h_1, h_2 \in C^*(X,.)$ such that $g_1(x) = f(x)$, $g_2(y) = f(y)$, $h_1(x) = I = h_2(y)$ and $h_1(y) = 0 = h_2(x)$. Then $g = g_1h_1 + g_2h_2 \in \mathcal{I}$ satisfies $g(z) = f(z)$ for $z \in \{x, y\}$.

The arguments (1°)–(4°) show that $f \in \Delta_2(\mathcal{I})$, and thus $\mathcal{I} = \mathcal{I}_A$. It follows from Lemma 3.2 (c) that the $*$-homomorphism $C^*(X,.) \ni f \mapsto f\big|_A \in C^*(A,.)$ is surjective. What is more, its kernel coincides with $\mathcal{I}_A = \mathcal{I}$ and therefore $C^*(X,.)/\mathcal{I}$ and $C^*(A,.)$ are isomorphic.

We now turn to (b). We infer from (a) that there is a closed invariant set $A \subset X$ such that $\ker(\pi) = \mathfrak{I}_A$. Since π is nonzero, A is nonempty. Further, $k^2 \geqslant \dim \pi(C^*(X,.)) = \dim(C^*(X,.)/\ker(\pi)) = \dim C^*(A,.) \geqslant n^2$ (by Lemma 3.2 (d) and by (a)), and thus $k = n$, $\dim C^*(A,.) = n^2$ and π is surjective. Fix $a \in A$, and observe that $A = \mathfrak{U}_n.\{a\}$ because otherwise $\dim C^*(A,.) > n^2$ (thanks to Lemma 3.2 (e)). Now define $\Phi\colon M_n \to M_n$ by the rule $\Phi(X) = f(a)$ where $\pi(f) = X$. It may easily be checked (using the fact that $\ker(\pi) = \mathfrak{I}_{\mathfrak{U}_n.\{a\}}$) that Φ is a well defined one-to-one $*$-homomorphism of M_n. We conclude that there is a $\mathfrak{u} \in \mathfrak{U}_n$ for which $\Phi(X) = \mathfrak{u}.X$ (in the algebra of matrices this is quite an elementary fact; however, this follows also from [**23,** Corollary 2.9.32]). Put $x = \mathfrak{u}^{-1}.a$ and note that then $f(a) = \Phi(\pi(f)) = \mathfrak{u}.\pi(f)$, and consequently $\pi(f) = \mathfrak{u}^{-1}.f(a) = f(x)$, for each $f \in C^*(X,.)$. The uniqueness of x follows from Lemma 3.2 (d), (e).

We turn to (c). Let $\Phi\colon C^*(X,.) \to C^*(Y,*)$ be a $*$-homomorphism of C^*-algebras. Put

$$U = Y \setminus \{y \in Y\colon\ [\Phi(f)](y) = 0 \quad \text{for each } f \in C^*(X,.)\}.$$

It is clear that U is invariant and open in Y. For any $y \in U$, the function $C^*(X,.) \ni f \mapsto [\Phi(f)](y) \in M_n$ is a nonzero representation and therefore, thanks to (b), there is a unique point $\varphi(y) \in X$ such that $[\Phi(f)](y) = f(\varphi(x))$ for each $f \in C^*(X,.)$. In this way, we have obtained a function $\varphi\colon U \to X$ for which (3.1) holds. By the uniqueness in (b), we see that $\varphi(\mathfrak{u}.y) = \mathfrak{u}.\varphi(y)$ for any $\mathfrak{u} \in \mathfrak{U}_n$ and $y \in U$. So, to prove that φ is a morphism of n-spaces, it remains to check that φ is a proper map. First we shall show that φ is continuous. Suppose, to the contrary, that there is a set $D \subset U$ and a point $b \in U \cap \overline{D}$ ($\overline{D}$ is the closure of D in Y) such that $a := \varphi(b) \notin \overline{\varphi(D)}$ (the closure taken in X). Let V be an open neighborhood of a whose closure is compact and disjoint from $F := \overline{\varphi(D)}$. Let $\langle \cdot, - \rangle$ be the standard inner product on M_n, that is, $\langle X, Y \rangle = \operatorname{tr}(Y^* X)$ ('tr' is the trace) and let $\|X\|_2 := \sqrt{\operatorname{tr}(X^* X)}$. Take an irreducible matrix $Q \in M_n$ with $\|Q\|_2 = 1$. For simplicity, put $\mathcal{B} = \{X \in M_n\colon\ \|X\|_2 \leqslant 1\}$. Our aim is to construct $f \in C^*(X,.)$ such that $f(a) = Q$ and $f^{-1}(\{Q\}) \subset V$. Observe that there is a compact convex nonempty set $\mathcal{K}$ such that

$$Q \notin \mathcal{K} \subset \mathcal{B} \quad \text{and} \quad \{\mathfrak{u}.a\colon\ \mathfrak{u} \in \mathfrak{U}_n,\ \mathfrak{u}.Q \notin \mathcal{K}\} \subset V. \tag{3.2}$$

(Indeed, it suffices to define $\mathcal{K}$ as the convex hull of the set $\{X \in \mathcal{B}\colon\ \|X - Q\|_2 \geqslant r\}$ where $r > 0$ is such that $\mathfrak{u}.a \in V$ whenever $\mathfrak{u} \in \mathfrak{U}_n$ satisfies $\|\mathfrak{u}.Q - Q\|_2 < r$. Such an r exists because Q is irreducible, and hence the maps $\mathfrak{U}_n \ni \mathfrak{u} \mapsto \mathfrak{u}.b \in X$ and $\mathfrak{U}_n \ni \mathfrak{u} \mapsto \mathfrak{u}.Q \in M_n$ are embeddings.) Let $W = \mathfrak{U}_n.\{a\}$, and let $g_0\colon W \to M_n$ be given by $g_0(\mathfrak{u}.a) = \mathfrak{u}.Q$. Since $g_0(W \setminus V) \subset \mathcal{K}$ (by (3.2)) and the set $\mathcal{K}$ (being compact, convex and nonempty) is a retract of M_n, there is a map $g_1 \in \mathcal{C}_0(X \setminus V, M_n)$ such that $g_1(X \setminus V) \subset \mathcal{K}$ and $g_1(x) = g_0(x)$ for $x \in W \setminus V$. Finally, there is a $g \in \mathcal{C}_0(X, M_n)$ which extends both g_0 and g_1, and $g(X) \subset \mathcal{B}$. Now put $f = g^{\mathfrak{U}} \in C^*(X, .)$, and notice that $f(a) = Q$ (by Lemma 3.2 (b)). We claim that

$$f^{-1}(\{Q\}) \subset V. \tag{3.3}$$

Let us prove the above relation. Let $x \in X \setminus V$. Then $g(x) = g_1(x) \in \mathcal{K}$, and hence $g(x) \neq Q$ (see (3.2)). The set $\mathfrak{G} := \{\mathfrak{u} \in \mathfrak{U}_n\colon\ \mathfrak{u}^{-1}.g(\mathfrak{u}.x) \neq Q\}$ is open in $\mathfrak{U}_n$ and nonempty, which implies that its Haar measure is positive. Further, $|\left\langle \mathfrak{u}^{-1}.g(\mathfrak{u}.x), Q\right\rangle| \leqslant 1$ for any $\mathfrak{u} \in \mathfrak{U}_n$ and $\left\langle \mathfrak{u}^{-1}.g(\mathfrak{u}.x), Q\right\rangle \neq 1$ for $\mathfrak{u} \in \mathfrak{G}$ (since $g(X) \subset \mathcal{B}$). We infer from these remarks that $\int_{\mathfrak{U}_n} \left\langle \mathfrak{u}^{-1}.g(\mathfrak{u}.x), Q\right\rangle \, d\mathfrak{u} \neq 1$. Equivalently, $\langle f(x), Q\rangle \neq 1$, which implies that $f(x) \neq Q$ and finishes the proof of (3.3). For $m \geqslant 1$, let

$$C_m = \{y \in Y\colon\ \|[\Phi(f)](y) - Q\|_2 \leqslant 2^{-m}\}$$

and

$$F_m = \{x \in X\colon\ \|f(x) - Q\|_2 \leqslant 2^{-m}\}.$$

Since $f \in \mathcal{C}_0(X, M_n)$ and $\Phi(f) \in \mathcal{C}_0(Y, M_n)$, F_m is compact and C_m is a compact neighborhood of b. Consequently, $C_m \cap D \neq \varnothing$. We infer from (3.1) that $\varphi(C_m \cap D) \subset F_m \cap F$. Now the compactness argument gives $F \cap \bigcap_{m=1}^{\infty} F_m \neq \varnothing$. Let c belong to this intersection. Then $f(c) = Q$ and $c \notin V$, which contradicts (3.3) and finishes the proof of the continuity of φ.

To see that φ is proper, take a compact set $K \subset X$ and note that $L = \mathfrak{U}_n.K$ is compact as well. Let $G \subset X$ be an open neighborhood of L with compact closure. Take a map $\beta \in \mathcal{C}_0(X, M_n)$ such that $\beta(x) = I$ for $x \in L$ and β vanishes off G. Let $f = \beta^{\mathfrak{U}} \in C^*(X, .)$ and observe that $f(x) = I$ for $x \in L$. Since $\Phi(f) \in \mathcal{C}_0(Y, M_n)$, the set $Z := \{y \in Y\colon\ [\Phi(f)](y) = I\}$ is a compact subset of Y. But (3.1) implies that $Z \subset U$ and $\varphi^{-1}(K) \subset Z$. This finishes the proof of the fact that φ is a morphism. The uniqueness of the pair (U, φ) follows from Lemma 3.2 and is left to the reader.

Now if Φ is a $*$-isomorphism of C^*-algebras, then $U = Y$ (by Lemma 3.2 (d)) and thus $\Phi(f) = f \circ \varphi$. Similarly, Φ^{-1} is of the form $\Phi^{-1}(g) = g \circ \psi$ for some morphism $\psi\colon (X, .) \to (Y, *)$. Then $f = f \circ (\varphi \circ \psi)$ for each $f \in C^*(X, .)$, and the uniqueness in (c) gives $(\varphi \circ \psi)(x) = x$ for each $x \in X$. Similarly, $(\psi \circ \varphi)(y) = y$ for any $y \in Y$, and consequently φ is an isomorphism of n-spaces. The proof is complete. □

4. Representations of $C^*(X, .)$. In this section, we will characterize all representations of $C^*(X, .)$ for an arbitrary n-space $(X, .)$. But first we shall give a 'canonical' description of all continuous linear functionals on $C^*(X, .)$. We underline here that we are not interested in the formula for the norm of a functional. The results of the section will be applied in the next two parts where we formulate our version of Fell's characterization of homogeneous C^*-algebras (Section 5) and a counterpart of the spectral theorem for finite systems of operators which generate n-homogeneous C^*-algebras (Section 6).

Definition 4.1. Let $(X, .)$ be an n-space. Let $\mathfrak{B}(X)$ denote the σ-algebra of all Borel subsets of X; that is, $\mathfrak{B}(X)$ is the smallest σ-algebra of subsets of X which contains all open sets. For any $\mathfrak{u} \in \mathfrak{U}_n$ and $A \in \mathfrak{B}(X)$, the set $\mathfrak{u}.A := \{\mathfrak{u}.a\colon\ a \in A\}$ is Borel as well. We shall denote by $\chi_A\colon X \to \{0, 1\}$ the characteristic function of A. Further, $\mathfrak{B}C^*(X, .)$ stands for the C^*-algebra of all bounded Borel (i.e., $\mathfrak{B}(X)$-measurable) functions $f\colon X \to M_n$ such that $f(\mathfrak{u}.x) = \mathfrak{u}.f(x)$ for any $\mathfrak{u} \in \mathfrak{U}_n$ and $x \in X$.

An *n-measure* on $(X, .)$ is an $n \times n$-matrix $\mu = [\mu_{jk}]$ where $\mu_{jk}\colon \mathfrak{B}(X) \to \mathbb{C}$ is a regular (complex-valued) measure and $\mu(\mathfrak{u}.A) = \mathfrak{u}.\mu(A)$ for any $\mathfrak{u} \in \mathfrak{U}_n$ and $A \in \mathfrak{B}(X)$ (here, of course, $\mu(A) = [\mu_{jk}(A)] \in M_n$). The set of all n-measures on $(X, .)$ is denoted by $\mathcal{M}(X, .)$.

For any bounded Borel function $f\colon X \to M_n$ and an $n \times n$-matrix $\mu = [\mu_{jk}]$ of complex-valued regular Borel measures we define the integral $\int f \,\mathrm{d}\mu$ as the complex number

$$\sum_{j,k} \int_X f_{jk} \,\mathrm{d}\mu_{kj},$$

where $f(x) = [f_{jk}(x)]$ for $x \in X$. We emphasize that in the formula for $\int f \,\mathrm{d}\mu$, f_{jk} meets μ_{kj} (not μ_{jk} (!)).

The first purpose of this section is to prove the following

Theorem 4.2. *For every continuous linear functional $\varphi\colon C^*(X,.) \to \mathbb{C}$ there exists a unique $\mu \in \mathcal{M}(X,.)$ such that $\varphi(f) = \int f \,\mathrm{d}\mu$ for any $f \in C^*(X,.)$.*

The above result is a simple consequence of the next one.

Proposition 4.3. *Let $\mu = [\mu_{jk}]$ be an $n \times n$-matrix of complex-valued regular Borel measures on X. Then $\mu \in \mathcal{M}(X,.)$ if and only if, for every map $f \in \mathcal{C}_0(X, M_n)$,*

$$\int f \,\mathrm{d}\mu = \int f^{\mathfrak{U}} \,\mathrm{d}\mu. \tag{4.1}$$

Proof. For any $n \times n$-matrix A we shall write A_{jk} to denote the suitable entry of A. We adapt the same rule for functions $f \in \mathcal{C}_0(X, M_n)$ and matrix-valued measures. Further, for two arbitrarily fixed indices (j,k) and (p,q), the function $\mathfrak{U}_n \ni \mathfrak{u} \mapsto \mathfrak{u}_{jk}\overline{\mathfrak{u}}_{pq} \in \mathbb{C}$ is well defined and continuous (although '$\mathfrak{u}_{jk}$' is not well defined). Observe that for any $A \in M_n$, $\mathfrak{u} \in \mathfrak{U}_n$ and an index (p,q) one has:

$$(\mathfrak{u}.A)_{p,q} = \sum_{j,k} \mathfrak{u}_{pj}\overline{\mathfrak{u}}_{qk} \cdot A_{jk}$$

and

$$(\mathfrak{u}^{-1}.A)_{p,q} = \sum_{j,k} \mathfrak{u}_{kq}\overline{\mathfrak{u}}_{jp} \cdot A_{jk}.$$

Further, for $\mathfrak{u} \in \mathfrak{U}_n$ and a complex-valued regular Borel measure ν on X, let $\nu^{\mathfrak{u}}$ be the (complex-valued regular Borel) measure on X given by $\nu^{\mathfrak{u}}(A) = \nu(\mathfrak{u}.A)$ ($A \in \mathfrak{B}(X)$). It follows from the transport measure theorem that, for any $g \in \mathcal{C}_0(X, \mathbb{C})$,

$$\int_X g(\mathfrak{u}.x)\,\mathrm{d}\nu^{\mathfrak{u}}(x) = \int_X g(x)\,\mathrm{d}\nu(x).$$

We adapt the above notation also for $n \times n$-matrix μ of measures: $\mu^{\mathfrak{u}}(A) = \mu(\mathfrak{u}.A)$. Notice that $(\mu^{\mathfrak{u}})_{jk} = (\mu_{jk})^{\mathfrak{u}}$.

Now assume that $\mu \in \mathcal{M}(X, .)$. This means that, for any $\mathfrak{u} \in \mathfrak{U}_n$, $\mathfrak{u}.\mu = \mu^{\mathfrak{u}}$. For $f \in \mathcal{C}_0(X, M_n)$ and $x \in X$, we have

$$(f^{\mathfrak{U}})_{pq}(x) = \sum_{j,k} \int_{\mathfrak{U}_n} \mathfrak{u}_{kq}\overline{\mathfrak{u}}_{jp} \cdot f_{jk}(\mathfrak{u}.x)\,\mathrm{d}\mathfrak{u},$$

and therefore, by Fubini's theorem,

$$\begin{aligned}
\int f^{\mathfrak{U}}\,\mathrm{d}\mu &= \sum_{p,q} \int_X (f^{\mathfrak{U}})_{p,q}\,\mathrm{d}\mu_{qp} \\
&= \sum_{p,q} \sum_{j,k} \int_X \int_{\mathfrak{U}_n} \mathfrak{u}_{kq}\overline{\mathfrak{u}}_{jp} \cdot f_{jk}(\mathfrak{u}.x)\,\mathrm{d}\mathfrak{u}\,\mathrm{d}\mu_{qp}(x) \\
&= \sum_{j,k} \int_{\mathfrak{U}_n} \int_X f_{jk}(\mathfrak{u}.x)\,\mathrm{d}\Big(\sum_{p,q} \mathfrak{u}_{kq}\overline{\mathfrak{u}}_{jp} \cdot \mu_{qp}\Big)(x)\,\mathrm{d}\mathfrak{u} \\
&= \sum_{j,k} \int_{\mathfrak{U}_n} \int_X f_{jk}(\mathfrak{u}.x)\,\mathrm{d}(\mathfrak{u}.\mu)_{kj}(x)\,\mathrm{d}\mathfrak{u} \\
&= \sum_{j,k} \int_{\mathfrak{U}_n} \int_X f_{jk}(\mathfrak{u}.x)\,\mathrm{d}(\mu_{kj})^{\mathfrak{u}}(x)\,\mathrm{d}\mathfrak{u} \\
&= \sum_{j,k} \int_{\mathfrak{U}_n} \int_X f_{jk}(x)\,\mathrm{d}\mu_{kj}(x)\,\mathrm{d}\mathfrak{u} \\
&= \sum_{j,k} \int_X f_{jk}(x)\,\mathrm{d}\mu_{kj}(x) \\
&= \int f\,\mathrm{d}\mu,
\end{aligned}$$

which gives (4.1). Conversely, assume (4.1) if fulfilled for any $f \in \mathcal{C}_0(X, M_n)$ and fix a compact $\mathcal{G}_\delta$ subset K of X and an index (p, q). Let $g \in \mathcal{C}_0(X, \mathbb{C})$ be arbitrary, and let $f \in \mathcal{C}_0(X, M_n)$ be such that $f_{pq} = g$ and $f_{jk} = 0$ for $(j, k) \neq (p, q)$. Applying (4.1) for such an f, we obtain

$$\int_X g \, \mathrm{d}\mu_{qp} = \sum_{j,k} \int_X \int_{\mathfrak{U}_n} \mathfrak{u}_{qk} \overline{\mathfrak{u}}_{pj} \cdot g(\mathfrak{u}.x) \, \mathrm{d}\mathfrak{u} \, \mathrm{d}\mu_{kj}(x). \tag{4.2}$$

Further, since K is compact and $\mathcal{G}_\delta$, there is a sequence $(g_k)_{k=1}^\infty \subset \mathcal{C}_0(X, \mathbb{C})$ such that $g_k(X) \subset [0,1]$ and $\lim_{k\to\infty} g_k(x) = \chi_K(x)$ for any $x \in X$. Substituting $g = g_k$ in (4.2) and letting $k \to \infty$, we obtain (by Lebesgue's dominated convergence theorem as well as Fubini's):

$$\begin{aligned}
\mu_{q,p}(K) &= \sum_{j,k} \int_{\mathfrak{U}_n} \int_X \mathfrak{u}_{qk} \overline{\mathfrak{u}}_{pj} \cdot \chi_K(\mathfrak{u}.x) \, \mathrm{d}\mu_{kj}(x) \, \mathrm{d}\mathfrak{u} \\
&= \int_{\mathfrak{U}_n} \Big(\sum_{j,k} \mathfrak{u}_{qk} \overline{\mathfrak{u}}_{pj} \cdot \mu_{kj}(\mathfrak{u}^{-1}.K) \Big) \, \mathrm{d}\mathfrak{u} \\
&= \int_{\mathfrak{U}_n} (\mathfrak{u}.\mu)_{q,p}(\mathfrak{u}^{-1}.K) \, \mathrm{d}\mathfrak{u}.
\end{aligned}$$

We infer from the arbitrariness of (p, q) in the above formula that

$$\mu(K) = \int_{\mathfrak{U}_n} \mathfrak{u}.\mu(\mathfrak{u}^{-1}.K) \, \mathrm{d}\mathfrak{u}.$$

Now, if $\mathfrak{v} \in \mathfrak{U}_n$, the set $\mathfrak{v}.K$ is also compact and $\mathcal{G}_\delta$, and therefore

$$\begin{aligned}
\mu(\mathfrak{v}.K) &= \int_{\mathfrak{U}_n} \mathfrak{u}.\mu(\mathfrak{u}^{-1}\mathfrak{v}.K) \, \mathrm{d}\mathfrak{u} \\
&= \int_{\mathfrak{U}_n} \mathfrak{v}.[\mathfrak{u}.\mu(\mathfrak{u}^{-1}.K)] \, \mathrm{d}\mathfrak{u} \\
&= \mathfrak{v}.\Big(\int_{\mathfrak{U}_n} \mathfrak{u}.\mu(\mathfrak{u}^{-1}.K) \, \mathrm{d}\mathfrak{u} \Big) \\
&= \mathfrak{v}.\mu(K).
\end{aligned}$$

Finally, since μ is regular, the relation $\mu(\mathfrak{v}.A) = \mathfrak{v}.\mu(A)$ holds for any $A \in \mathfrak{B}(X)$, and we are done. □

Proof of Theorem 4.2. Note that the function $P\colon \mathcal{C}_0(X, M_n) \ni f \mapsto f^{\mathfrak{U}} \in C^*(X, .)$ is a continuous linear projection (that is, $P(f) = f$ for $f \in C^*(X, .)$). So, if $\varphi\colon C^*(X, .) \to \mathbb{C}$ is a continuous linear functional, so is $\psi := \varphi \circ P\colon \mathcal{C}_0(X, M_n) \to \mathbb{C}$. Since $\mathcal{C}_0(X, M_n)$ is isomorphic, as a Banach space, to $[\mathcal{C}_0(X, \mathbb{C})]^{n^2}$, the Riesz-type representation theorem yields that there is a unique $n \times n$-matrix μ of complex-valued regular Borel measures such that $\psi(f) = \int f \,\mathrm{d}\mu$. Observe that $\psi(f^{\mathfrak{U}}) = \psi(f)$ for any $f \in \mathcal{C}_0(X, M_n)$, and hence $\mu \in \mathcal{M}(X, .)$, thanks to Proposition 4.3. The uniqueness of μ follows from the above construction, Proposition 4.3 and the uniqueness in the Riesz-type representation theorem. □

Now we turn to representations of $C^*(X, .)$. To this end, we introduce

Definition 4.4. An operator-valued *n-measure* on the n-space $(X, .)$ is any function of the form $E\colon \mathfrak{B}(X) \ni A \mapsto [E_{jk}(A)] \in M_n(\mathcal{B}(\mathcal{H}))$ (where $(\mathcal{H}, \langle \cdot, - \rangle)$ is a Hilbert space) such that:

(M1) for any $h, w \in \mathcal{H}$ and $j, k \in \{1, \dots, n\}$, the function

$$E_{jk}^{(h,w)}\colon \mathfrak{B}(X) \ni A \mapsto \langle E_{jk}(A)h, w\rangle \in \mathbb{C}$$

is a (complex-valued) measure,

(M2) for any $\mathfrak{u} \in \mathfrak{U}_n$ and $A \in \mathfrak{B}(X)$, $E(\mathfrak{u}.A) = \mathfrak{u}.E(A)$.

In other words, an operator-valued n-measure is an $n \times n$-matrix of operator-valued measures which satisfies axiom (M2). The operator-valued n-measure E is *regular* if and only if $E_{jk}^{(h,w)}$ is regular for any h, w and j, k.

Recall that if $\mu\colon \mathfrak{B}(X) \to \mathcal{B}(\mathcal{H})$ is an operator-valued measure and $f\colon X \to \mathbb{C}$ is a bounded Borel function, $\int_X f \,\mathrm{d}\mu$ is a bounded linear operator on $\mathcal{H}$, defined by an implicit formula:

$$\left\langle \left(\int_X f \,\mathrm{d}\mu \right) h, w \right\rangle = \int_X f \,\mathrm{d}\mu^{(h,w)}, \quad (h, w \in \mathcal{H}),$$

where $\mu^{(h,w)}(A) = \langle \mu(A)h, w\rangle$ $(A \in \mathfrak{B}(X))$. Now assume that $E = [E_{jk}]\colon \mathfrak{B}(X) \to M_n(\mathcal{B}(\mathcal{H}))$ is an n-measure and $f = [f_{jk}]\colon X \to M_n$ is a bounded Borel function. We define $\int f \,\mathrm{d}E$ as a bounded linear operator on $\mathcal{H}$ given by

$$\int f \,\mathrm{d}E = \sum_{j,k} \int_X f_{jk} \,\mathrm{d}E_{kj}.$$

We are now ready to introduce

Definition 4.5. A *spectral* n-measure is any operator-valued regular n-measure $E\colon \mathfrak{B}(X) \to M_n(\mathcal{B}(\mathcal{H}))$ such that

$$\left(\int f \,\mathrm{d}E\right)^* = \int f^* \,\mathrm{d}E, \tag{4.3}$$

$$\int f \cdot g \,\mathrm{d}E = \int f \,\mathrm{d}E \cdot \int g \,\mathrm{d}E \tag{4.4}$$

for any $f, g \in \mathfrak{B}C^*(X, .)$. (The product $f \cdot g$ is computed pointwise as the product of matrices.) In other words, a spectral n-measure is an operator-valued regular n-measure $E\colon \mathfrak{B}(X) \to M_n(\mathcal{B}(\mathcal{H}))$ such that the operator

$$\mathfrak{B}C^*(X, .) \ni f \longmapsto \int f \,\mathrm{d}E \in \mathcal{B}(\mathcal{H})$$

is a representation of a C^*-algebra $\mathfrak{B}C^*(X, .)$.

The main result of this section is the following.

Theorem 4.6. *Let $(X, .)$ be an n-space and $\pi\colon C^*(X, .) \to \mathcal{B}(\mathcal{H})$ a representation. There is a unique spectral n-measure $E\colon \mathfrak{B}(X) \to M_n(\mathcal{B}(\mathcal{H}))$ such that*

$$\pi(f) = \int f \,\mathrm{d}E \quad (f \in C^*(X, .)). \tag{4.5}$$

In particular, every representation of $C^*(X, .)$ admits an extension to a representation of $\mathfrak{B}C^*(X, .)$.

In the proof of the above result we shall involve the following:

Lemma 4.7. *Let $\mu\colon \mathfrak{B}(X) \to \mathbb{R}_+$ be a regular measure. For any $f \in \mathfrak{B}C^*(X, .)$ and $\varepsilon > 0$, there exists $g \in C^*(X, .)$ such that $\sup_{x \in X} \|g(x)\| \leqslant \sup_{x \in X} \|f(x)\|$ and $\int_X \|f(x) - g(x)\| \,\mathrm{d}\mu(x) < \varepsilon$.*

Proof. Let $f = [f_{jk}] \in \mathfrak{B}C^*(X, .)$, and let $M > 0$ be such that

$$\sup_{x \in X} \|f(x)\| \leqslant M.$$

It follows from the regularity of μ that, for each (j,k), there is a compact set L_{jk} such that
$$\mu(X \setminus L_{jk}) \leqslant \frac{\varepsilon}{2Mn^2}$$
and $f_{jk}\big|_{L_{jk}}$ is continuous. Put
$$L = \bigcap_{j,k} L_{jk}$$
and $K = \mathfrak{U}_n.L$. Then K is compact and invariant, and
$$\mu(X \setminus K) \leqslant \frac{\varepsilon}{2M}.$$
What is more, $f\big|_K$ is continuous (this follows from the facts that $f\big|_L$ is continuous and $f(\mathfrak{u}.x) = \mathfrak{u}.f(x)$). Now Lemma 3.2 (c) yields the existence of $g \in C^*(X,.)$ such that $\sup_{x\in X} \|g(x)\| \leqslant \sup_{x\in X} \|f(x)\|$ and $g\big|_K = f\big|_K$. Then:
$$\begin{aligned}\int_X \|f(x) - g(x)\| \,\mathrm{d}\mu(x) &= \int_{X\setminus K} \|f(x) - g(x)\| \,\mathrm{d}\mu \\ &\leqslant 2M \cdot \mu(X \setminus K) \\ &= \varepsilon,\end{aligned}$$
and we are done. □

Proposition 4.8. *Let $E = [E_{jk}]\colon \mathfrak{B}(X) \to M_n(\mathcal{B}(\mathcal{H}))$ be a regular n-measure.*

(a) *E satisfies* (4.3) *for any $f \in \mathfrak{B}C^*(X,.)$ if and only if* (4.3) *is fulfilled for any $f \in C^*(X,.)$, if and only if $(E_{jk}(A))^* = E_{kj}(A)$ for each $A \in \mathfrak{B}(X)$;*
(b) *E is spectral if and only if* (4.3) *and* (4.4) *are satisfied for any $f, g \in C^*(X,.)$.*

Proof. For any complex-valued regular Borel measure ν on X we shall denote by $|\nu|$ the variation of ν. Recall that $|\nu|$ is a nonnegative finite regular Borel measure on X. Further, for any $h, w \in \mathcal{H}$ and $j,k \in \{1,\dots,n\}$, let $E_{jk}^{(h,w)}$ be as in Definition 4.4. Finally, $\langle \cdot, - \rangle$ stands for the scalar product of $\mathcal{H}$.

We begin with (a). Fix $h, w \in \mathcal{H}$ and $j, k \in \{1, \ldots, n\}$. First assume that (4.3) is fulfilled for any $f \in C^*(X, .)$. Let $E^{(h,w)} := [E_{jk}^{(h,w)}]$, and note that $E^{(h,w)} \in \mathcal{M}(X, .)$ since $E_{pq}(\mathfrak{u}.A) = \sum_{j,k} \mathfrak{u}_{pj}\overline{\mathfrak{u}}_{qk} \cdot E_{jk}(A)$. Thus, $E_{pq}^{(h,w)}(\mathfrak{u}.A) = \sum_{j,k} \mathfrak{u}_{pj}\overline{\mathfrak{u}}_{qk} \cdot E_{jk}^{(h,w)}(A) = (\mathfrak{u}.E^{(h,w)}(A))_{pq}$. Observe that $(E^{(h,w)})^* \in \mathcal{M}(X, .)$ as well where $(E^{(h,w)})^*(A) = (E^{(h,w)}(A))^*$ (because $(\mathfrak{u}.P)^* = \mathfrak{u}.P^*$ for any $P \in M_n$). Further, for each $f \in C^*(X, .)$, we have

$$\overline{\int f^* \,\mathrm{d}E^{(h,w)}} = \sum_{j,k} \overline{\int_X (f^*)_{jk} \,\mathrm{d}E_{kj}^{(h,w)}} = \sum_{j,k} \int_X f_{kj} \,\mathrm{d}\overline{E_{kj}^{(h,w)}}$$
$$= \sum_{j,k} \int_X f_{kj} \,\mathrm{d}(E^{(h,w)})_{jk}^* = \int f \,\mathrm{d}(E^{(h,w)})^*$$

and, on the other hand,

$$\overline{\int f^* \,\mathrm{d}E^{(h,w)}} = \overline{\left\langle \left(\int f^* \,\mathrm{d}E \right) h, w \right\rangle} = \overline{\left\langle \left(\int f \,\mathrm{d}E \right)^* h, w \right\rangle}$$
$$= \left\langle \left(\int f \,\mathrm{d}E \right) w, h \right\rangle = \int f \,\mathrm{d}E^{(w,h)}.$$

The uniqueness in Theorem 4.2 implies that $(E^{(h,w)})^* = E^{(w,h)}$, which means that, for each $A \in \mathfrak{B}(X)$, $\langle (E_{jk}(A))w, h\rangle = \overline{\langle (E_{kj}(A))h, w\rangle} = \langle (E_{kj}(A))^* w, h\rangle$. We conclude that $(E_{jk}(A))^* = E_{kj}(A)$. Finally, if the last relation holds for any $j, k \in \{1, \ldots, n\}$, then for every $f \in \mathfrak{B}C^*(X, .)$ we get:

$$\left(\int f \,\mathrm{d}E \right)^* = \sum_{j,k} \left(\int_X f_{jk} \,\mathrm{d}E_{kj} \right)^* = \sum_{j,k} \int_X \overline{f}_{jk} \,\mathrm{d}(E_{kj})^*$$
$$= \sum_{j,k} \int_X (f^*)_{kj} \,\mathrm{d}E_{jk} = \int f^* \,\mathrm{d}E.$$

This completes the proof of (a).

We now turn to (b). We assume that (4.3) and (4.4) are fulfilled for any $f, g \in C^*(X, .)$. We know from (a) that actually (4.3) is satisfied for any $f \in \mathfrak{B}C^*(X, .)$. The proof of (4.4) is divided into three steps, stated below.

Step 1. If $\xi \in \mathfrak{B}C^*(X,.)$ is such that

$$\int g \cdot \xi \,\mathrm{d}E = \int g \,\mathrm{d}E \cdot \int \xi \,\mathrm{d}E \tag{4.6}$$

for any $g \in C^*(X,.)$, then

$$\int f \cdot \xi \,\mathrm{d}E = \int f \,\mathrm{d}E \cdot \int \xi \,\mathrm{d}E \text{ for any } f \in \mathfrak{B}C^*(X,.).$$

Proof of Step 1. Fix $f \in \mathfrak{B}C^*(X,.)$, $h, w \in \mathcal{H}$ and $\varepsilon > 0$. Let $M \geqslant 1$ be such that $\sup_{x\in X} \|\xi(x)\| \leqslant M$. Put

$$v = \Big(\int \xi \,\mathrm{d}E\Big)h$$

and

$$\mu = \sum_{j,k} (|E_{jk}^{(h,w)}| + |E_{jk}^{(v,w)}|).$$

Since μ is finite and regular, Lemma 4.7 gives us a map $g \in C^*(X,.)$ such that

$$\int_X \|f(x) - g(x)\| \,\mathrm{d}\mu(x) \leqslant \frac{\varepsilon}{M}.$$

Then (4.6) holds and, therefore, (remember that $M \geqslant 1$):

$$\begin{aligned}
&\Big|\Big\langle\Big(\int f \cdot \xi \,\mathrm{d}E - \int f \,\mathrm{d}E \cdot \int \xi \,\mathrm{d}E\Big)h, w\Big\rangle\Big| \\
&\leqslant \Big|\Big\langle\Big(\int f \cdot \xi \,\mathrm{d}E - \int g \cdot \xi \,\mathrm{d}E\Big)h, w\Big\rangle\Big| \\
&\quad + \Big|\Big\langle\Big(\int g \,\mathrm{d}E \cdot \int \xi \,\mathrm{d}E - \int f \,\mathrm{d}E \cdot \int \xi \,\mathrm{d}E\Big)h, w\Big\rangle\Big| \\
&= \Big|\sum_{j,k} \int_X ((f-g)\xi)_{jk} \,\mathrm{d}E_{kj}^{(h,w)}\Big| \\
&\quad + \Big|\sum_{j,k} \int_X (g_{jk} - f_{jk}) \,\mathrm{d}E_{kj}^{(v,w)}\Big| \\
&\leqslant \sum_{j,k} \int_X \|(f(x) - g(x))\xi(x)\| \,\mathrm{d}|E_{kj}^{(h,w)}|(x)
\end{aligned}$$

$$+\sum_{j,k}\int_X \|g(x)-f(x)\|\,\mathrm{d}|E_{kj}^{(v,w)}|(x)$$
$$\leqslant M\int_X \|f(x)-g(x)\|\,\mathrm{d}\mu(x)\leqslant \varepsilon.$$

Step 2. For any $f \in \mathfrak{B}C^*(X,.)$ and $g \in \mathcal{C}(X,.)$, (4.4) holds.

Proof of Step 2. It follows from Step 1 and our assumptions in (b) that

$$\int g^*\cdot f^*\,\mathrm{d}E = \int g^*\,\mathrm{d}E\cdot\int f^*\,\mathrm{d}E.$$

Now it suffices to apply (4.3):

$$\int f\cdot g\,\mathrm{d}E = \Big(\int g^*\cdot f^*\,\mathrm{d}E\Big)^* = \Big(\int g^*\,\mathrm{d}E\cdot\int f^*\,\mathrm{d}E\Big)^*$$
$$= \int f\,\mathrm{d}E\cdot\int g\,\mathrm{d}E.$$

Step 3. The condition (4.4) is satisfied for any $f, g \in \mathfrak{B}C^*(X,.)$.

Proof of Step 3. Just apply Step 2 and then Step 1. □

Proof of Theorem 4.6. According to Proposition 4.8 (b), it suffices to show that there exists a regular n-measure $E\colon \mathfrak{B}(X) \to M_n(\mathcal{B}(\mathcal{H}))$ such that (4.5) holds and that such an E is unique. According to Theorem 4.2, for any $h, w \in \mathcal{H}$ there is a unique $\mu^{(h,w)} = [\mu_{jk}^{(h,w)}] \in \mathcal{M}(X,.)$ such that

$$\langle\pi(f)h,w\rangle = \int f\,\mathrm{d}\mu^{(h,w)} \tag{4.7}$$

for each $f \in C^*(X,.)$ ($\langle\cdot,-\rangle$ is the scalar product of $\mathcal{H}$). Now, for any $j, k \in \{1,\dots,n\}$ and each $A \in \mathfrak{B}(X)$, there is a unique bounded operator on $\mathcal{H}$, denoted by $E_{jk}(A)$, for which $\mu_{jk}^{(h,w)}(A) = \langle(E_{jk}(A))h,w\rangle$ $(h, w \in \mathcal{H})$. We put $E(A) = [E_{jk}(A)] \in M_n(\mathcal{B}(\mathcal{H}))$. We want to show that $E(\mathfrak{u}.A) = \mathfrak{u}.E(A)$. Since $\mu^{(h,w)} \in \mathcal{M}(X,.)$, we obtain:

$$\langle(E_{pq}(\mathfrak{u}.A))h,w\rangle = (\mu^{(h,w)}(\mathfrak{u}.A))_{pq} = (\mathfrak{u}.\mu^{(h,w)}(A))_{pq}$$
$$= \sum_{j,k}\mathfrak{u}_{pj}\overline{\mathfrak{u}}_{qk}\cdot\mu_{jk}^{(h,w)}(A) = \sum_{j,k}\mathfrak{u}_{pj}\overline{\mathfrak{u}}_{qk}\cdot\langle(E_{jk}(A))h,w\rangle$$

$$= \langle (\mathfrak{u}.E(A))_{pq} h, w \rangle ,$$

which shows that indeed $E(\mathfrak{u}.A) = \mathfrak{u}.E(A)$. Further, observe that $E_{jk}^{(h,w)} = \mu_{jk}^{(h,w)}$, and thus E is an operator-valued regular n-measure and

$$\left\langle \left(\int f \, \mathrm{d}E \right) h, w \right\rangle = \langle \pi(f) h, w \rangle$$

(thanks to (4.7)). Consequently,

$$\int f \, \mathrm{d}E = \pi(f),$$

and we are done.

The uniqueness of E follows from the above construction, and its proof is left to the reader. □

Example 4.9. Let $(X, .)$ be an n-space, and let $E = [E_{jk}] \colon \mathfrak{B}(X) \to M_n(\mathcal{B}(\mathcal{H}))$ be a spectral n-measure. We denote by $\mathfrak{B}_{inv}(X)$ the σ-algebra of all invariant Borel subsets of X (that is, $A \in \mathfrak{B}(X)$ belongs to $\mathfrak{B}_{inv}(X)$ if and only if $\mathfrak{u}.A = A$ for any $\mathfrak{u} \in \mathfrak{U}_n$). Let

$$F \colon \mathfrak{B}_{inv}(X) \ni A \longmapsto \sum_j E_{jj}(A) \in \mathcal{B}(\mathcal{H}).$$

Then, for every $A \in \mathfrak{B}_{inv}(X)$, one has:

(E1) $E_{jk}(A) = 0$ whenever $j \neq k$,
(E2) $E_{11}(A) = \ldots = E_{nn}(A) = \frac{1}{n} F(A)$,

and F is a spectral measure (possibly with $F(X) \neq I_{\mathcal{H}}$ where $I_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$). Let us briefly prove these claims. Since $E(A) = E(\mathfrak{u}.A) = \mathfrak{u}.E(A)$ for any $\mathfrak{u} \in \mathfrak{U}_n$, conditions (E1)–(E2) are fulfilled. Further, if $j_A \colon X \to M_n$ is given by $j_A(x) = \chi_A(x) \cdot I$ where $I \in M_n$ is the unit matrix, then $j_A \in \mathfrak{B}C^*(X, .)$ and, for $B \in \mathfrak{B}_{inv}(X)$,

$$F(A \cap B) = \int j_{A \cap B} \, \mathrm{d}E = \int j_A \cdot j_B \, \mathrm{d}E$$
$$= \int j_A \, \mathrm{d}E \cdot \int j_B \, \mathrm{d}E = F(A) F(B).$$

What is more, Proposition 4.8 (a) implies that $F(A)$ is self-adjoint, and hence F is indeed a spectral measure. One may also easily check that $F(X) = I_{\mathcal{H}}$ if and only if the representation $\pi_E\colon C^*(X,.) \ni f \mapsto \int f \, \mathrm{d}E \in \mathcal{B}(\mathcal{H})$ is nondegenerate.

The spectral measure F defined above corresponds to the representation of the center Z of $C^*(X,.)$. It is a simple exercise that Z consists precisely of all $f \in \mathcal{C}_0(X, \mathbb{C} \cdot I)$ which are constant on the sets of the form $\mathfrak{U}_n.\{x\}$ ($x \in X$). Thus, $\mathfrak{B}_{inv}(X)$ may naturally be identified with the Borel σ-algebra of the spectrum of Z, and consequently, F is the spectral measure induced by the representation $\pi_E\big|_Z$ of Z.

Conditions (E1)–(E2) show that a nonzero spectral n-measure E for $n > 1$ *never* satisfies the condition of a spectral measure—that $E(A \cap B) = E(A)E(B)$. Indeed, $E(X) \neq (E(X))^2$.

The next result is well known. For the reader's convenience, we give its short proof.

Lemma 4.10. *Let $\mathcal{A}$ be a C^*-algebra, and let $\pi\colon \mathcal{A} \to M_n$ (where $n \geqslant 1$ is finite) be a nonzero irreducible representation of $\mathcal{A}$. Then π is surjective.*

Proof. Let $J = \pi(\mathcal{A})$. Since π is irreducible, $J' = \mathbb{C} \cdot I$, and consequently, $J'' = M_n$. But it follows from von Neumann's double commutant theorem that $J'' = J + \mathbb{C} \cdot I$ (here we use the fact that n is finite). So, the facts that J is a $*$-algebra and $M_n = J + \mathbb{C} \cdot I$ imply that J is a two-sided ideal in M_n. Consequently, $J = \{0\}$ or $J = M_n$. But $\pi \neq 0$, and hence $J = M_n$. □

With the aid of the above lemma and Theorem 4.6, we shall now characterize all irreducible representations of $C^*(X,.)$.

Proposition 4.11. *Every nonzero irreducible representation π of $C^*(X,.)$ (where $(X,.)$ is an n-space) is n-dimensional and has the form $\pi(f) = f(x)$ (for some $x \in X$).*

Proof. Let $\pi\colon C^*(X,.) \to \mathcal{B}(\mathcal{H})$ be a nonzero irreducible representation. It follows from Theorem 4.6 that there is a spectral n-measure $E\colon \mathfrak{B}(X) \to \mathcal{B}(\mathcal{H})$ such that (4.5) holds. Let $\mathfrak{B}_{inv}(X)$ and $F\colon \mathfrak{B}_{inv}(X) \to \mathcal{B}(\mathcal{H})$ be as in Example 4.9. Then F is a (regular) spectral measure (with $F(X) = I_{\mathcal{H}}$ because π is nondegenerate) and, for any $A \in \mathfrak{B}_{inv}(X)$ and $f \in C^*(X,.)$, we have

$$\int f \,\mathrm{d}E \cdot \int \chi_A I \,\mathrm{d}E = \int \chi_A I \,\mathrm{d}E \cdot \int f \,\mathrm{d}E$$

(where I is the unit $n \times n$-matrix). Since π is irreducible, we deduce that, for every $A \in \mathfrak{B}_{inv}(X)$, $\int \chi_A I \,\mathrm{d}E$ is a scalar multiple of the identity operator on $\mathcal{H}$. This implies that we may think of F as a complex-valued (spectral) measure. But $\mathfrak{B}_{inv}(X)$ is naturally 'isomorphic' to the σ-algebra of all Borel sets of $X/\mathfrak{U}_n$ (which is locally compact) and thus F is supported on a set $S := \mathfrak{U}_n.a$ for some $a \in X$. But then

$$\int \chi_X I \,\mathrm{d}E = \int \chi_S I \,\mathrm{d}E,$$

and consequently, $\pi(f) = \int f\big|_S \,\mathrm{d}E_S$, where E_S is the restriction of E to $\mathfrak{B}(S)$. Since the vector space $\{f\big|_S \colon\ f \in C^*(X,.)\}$ is finite dimensional (and its dimension is equal to n^2), we infer that $\mathcal{A} := \pi(C^*(X,.))$ is finite dimensional as well and $\dim \mathcal{A} \leqslant n^2$. So, the irreducibility of π implies that $\mathcal{H}$ is finite dimensional, while Lemma 4.10 shows that $\dim \mathcal{H} \leqslant n$. Finally, Proposition 3.3 (b) completes the proof. □

5. Homogeneous C^*-algebras.

Definition 5.1. A C^*-algebra is said to be *n-homogeneous* (where n is finite) if and only if every nonzero irreducible representation of it is n-dimensional.

Our version of Fell's characterization of n-homogeneous C^*-algebras **[9]** reads as follows.

Theorem 5.2. *For a C^*-algebra $\mathcal{A}$ and finite $n \geqslant 1$, the following conditions are equivalent*:

- (i) *$\mathcal{A}$ is an n-homogeneous C^*-algebra*;
- (ii) *there is an n-space $(X,.)$ such that $\mathcal{A}$ is isomorphic* (*as a C^*-algebra*) *to $C^*(X,.)$.*

What is more, if $\mathcal{A}$ is n-homogeneous, the n-space $(X,.)$ appearing in (ii) *is unique up to isomorphism.*

Proof of Theorem 5.2. We infer from Proposition 3.3 (c) that the n-space $(X, .)$ appearing in (ii) is unique up to isomorphism. In addition, it easily follows from Proposition 4.11 that $C^*(X, .)$ is n-homogeneous for any n-space $(X, .)$. So, it remains to show that (i) implies (ii). To this end, assume $\mathcal{A}$ is n-homogeneous, and let $\mathfrak{X}$ be the set of all representations (including the zero one) $\pi\colon \mathcal{A} \to M_n$, equipped with the topology of pointwise convergence. Since each representation is a bounded linear operator of norm not greater than 1, $\mathfrak{X}$ is compact. Consequently, $X := \mathfrak{X} \setminus \{0\}$ is locally compact. We define an action of $\mathfrak{U}_n$ on X by the formula:

$$(\mathfrak{u}.\pi)(a) = \mathfrak{u}.\pi(a) \quad (a \in \mathcal{A},\ \pi \in X,\ \mathfrak{u} \in \mathfrak{U}_n).$$

It is easily seen that the action is continuous. What is more, Lemma 4.10 ensures us that it is free as well. So, $(X, .)$ is an n-space. The next step of construction is very common. For any $a \in \mathcal{A}$, let $\widehat{a}\colon X \to M_n$ be given by $\widehat{a}(\pi) = \pi(a)$. It is clear that $\widehat{a} \in \mathcal{C}_0(X, M_n)$ (indeed, if X is noncompact, then $\mathfrak{X} = X \cup \{0\}$ is a one-point compactification of X and $\widehat{a}$ extends to a map on $\mathfrak{X}$ which vanishes at 0). We also readily have $\widehat{a}(\mathfrak{u}.\pi) = \mathfrak{u}.\widehat{a}(\pi)$ for any $\mathfrak{u} \in \mathfrak{U}_n$. So, we have obtained a $*$-homomorphism $\Phi\colon \mathcal{A} \ni a \mapsto \widehat{a} \in C^*(X, .)$. It follows from (i) (and the fact that all irreducible representations separate points of a C^*-algebra) that Φ is one-to-one and, consequently, Φ is isometric. So, to end the proof, it suffices to show that $\mathcal{E} = \Phi(\mathcal{A})$ is dense in $C^*(X, .)$. To this end, we involve Theorem 2.2. It follows from Lemma 4.10 that condition (AX0) is fulfilled. Further, let π_1 and π_2 be arbitrary members of X.

We consider two cases. First assume that $\pi_2 = \mathfrak{u}.\pi_1$ for some $\mathfrak{u} \in \mathfrak{U}_n$. Then $\widehat{a}(\pi_2) = \mathfrak{u}.\widehat{a}(\pi_1)$, and consequently, $\sigma(\widehat{a}(\pi_1)) = \sigma(\widehat{a}(\pi_2))$ $(a \in \mathcal{A})$. So, in that case (AX2) holds. Now assume that there is no $\mathfrak{u} \in \mathfrak{U}_n$ for which $\pi_2 = \mathfrak{u}.\pi_1$. We shall show that, in that case:

$$\pi_1(a) = 0 \quad \text{and} \quad \pi_2(a) = I \text{ for some } a \in \mathcal{A}. \tag{5.1}$$

Let $\mathcal{M} \subset M_{2n}$ consist of all matrices of the form

$$\begin{pmatrix} \pi_1(x) & 0 \\ 0 & \pi_2(x) \end{pmatrix} \qquad \text{with } x \in \mathcal{A}.$$

Since $\mathcal{M}$ is a finite-dimensional C^*-algebra, it is singly generated (see, e.g., [**22**]) and unital (cf., [**26,** subsection 1.11]). Thanks to

Lemma 4.10, $\mathcal{M}$ contains matrices of the form

$$\begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} B & 0 \\ 0 & I \end{pmatrix} \quad (\text{for some } A, B \in M_n).$$

We conclude that the unit of $\mathcal{M}$ coincides with the unit of M_{2n}. This, combined with the fact that $\mathcal{M}$ is singly generated, yields that there is $z \in \mathcal{A}$ such that, for $A_j = \pi_j(z)$ $(j = 1, 2)$, we have

$$\mathcal{M} = \left\{ \begin{pmatrix} p(A_1, A_1^*) & 0 \\ 0 & p(A_2, A_2^*) \end{pmatrix} :\ p \in \mathcal{P} \right\}$$

where $\mathcal{P}$ is the free algebra of all polynomials in two noncommuting variables. Observe that then $M_n = \pi_j(\mathcal{A}) = \{p(A_j, A_j^*) :\ p \in \mathcal{P}\}$ $(j = 1, 2)$, which means that A_1 and A_2 are irreducible matrices. What is more, A_1 and A_2 are not unitarily equivalent, that is, there is no $\mathfrak{u} \in \mathfrak{U}_n$ for which $A_2 = \mathfrak{u}.A_1$ (indeed, if $A_2 = \mathfrak{u}.A_1$, then $\pi_2 = \mathfrak{u}.\pi_1$ since, for every $x \in \mathcal{A}$, there is a $p \in \mathcal{P}$ such that $\pi_j(x) = p(A_j, A_j^*)$). These two remarks imply that

$$\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \in \mathcal{M},$$

because the $*$-commutant in M_{2n} of the matrix

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

consists of matrices of the form

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$$

(this follows from the so-called Schur's lemma on intertwining transformations; see [**6,** Theorem 1.5, Corollary 1.8]; cf., also [**19,** Proposition 5.2.1]), and consequently,

$$\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \in \mathcal{M}'' = \mathcal{M}.$$

So, there is an $a \in \mathcal{A}$ such that

$$\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} \pi_1(a) & 0 \\ 0 & \pi_2(a) \end{pmatrix},$$

which gives (5.1). Replacing a by $(a + a^*)/2$, we may assume that a is self-adjoint. Then $f = \widehat{a} \in \mathcal{E}$ is self-adjoint (and hence normal) and

$\sigma(f(\pi_1)) \cap \sigma(f(\pi_2)) = \varnothing$, which shows that π_1 and π_2 are spectrally separated by $\mathcal{E}$. According to Theorem 2.2, it therefore suffices to check that each $g \in C^*(X, .)$ belongs to $\Delta_2(\mathcal{E})$. To this end, we fix $\pi_1, \pi_2 \in X$ and consider the same two cases as before. If $\pi_2 = \mathfrak{u}.\pi_1$, it follows from Lemma 4.10 that there is an $x \in \mathcal{A}$ for which $\pi_1(x) = g(\pi_1)$. Then $\widehat{x}(\pi_1) = g(\pi_1)$ and $\widehat{x}(\pi_2) = \mathfrak{u}.\widehat{x}(\pi_1) = \mathfrak{u}.g(\pi_1) = g(\pi_2)$, and we are done.

Finally, if $\pi_2 \neq \mathfrak{u}.\pi_1$ for any $\mathfrak{u} \in \mathfrak{U}_n$, (5.1) implies that there are points a_1 and a_2 in $\mathcal{A}$ such that $\pi_1(a_1) = I = \pi_2(a_2)$ and $\pi_1(a_2) = 0 = \pi_2(a_1)$. Moreover, there are points $x, y \in \mathcal{A}$ such that $\pi_1(x) = g(\pi_1)$ and $\pi_2(y) = g(\pi_2)$ (by Lemma 4.10). Put $z = xa_1 + ya_2 \in \mathcal{A}$ and note that $\widehat{z}(\pi_j) = g(\pi_j)$ for $j = 1, 2$, which means that $g \in \Delta_2(\mathcal{E})$. The whole proof is complete. □

Definition 5.3. Let $\mathcal{A}$ be an n-homogeneous C^*-algebra. By an *n-spectrum* of $\mathcal{A}$ we mean any n-space $(X, .)$ such that $\mathcal{A}$ is isomorphic to $C^*(X, .)$. It follows from Theorem 5.2 that an n-spectrum of $\mathcal{A}$ is unique up to isomorphism of n-spaces. By *concrete n-spectrum* of $\mathcal{A}$ we mean the n-space of all nonzero representations $\pi\colon \mathcal{A} \to M_n$ endowed with the pointwise convergence topology and the natural action of $\mathfrak{U}_n$.

The trivial algebra $\{0\}$ is n-homogeneous and its n-spectrum is the empty n-space.

The reader interested in general ideas of operator spectra should consult [**6,** subsection 2.5]; [**7, 8, 9**]; [**4**] as well as [**12, 13**]; [**15, 16**]; [**20**].

Our approach to n-homogeneous C^*-algebras allows us to prove briefly the following

Proposition 5.4. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two n-homogeneous C^*-algebras such that $\mathcal{A}_1 \subset \mathcal{A}_2$.*

(a) *Every representation $\pi_1\colon \mathcal{A}_1 \to M_n$ is extendable to a representation $\pi_2\colon \mathcal{A}_2 \to M_n$.*
(b) *If every n-dimensional representation (including the zero one) of $\mathcal{A}_1$ has a unique extension to an n-dimensional representation of $\mathcal{A}_2$, then $\mathcal{A}_1 = \mathcal{A}_2$.*

Proof. We begin with (a). We may and do assume that π_1 is nonzero. For $j = 1, 2$, let $(X_j, .)$ denote an n-spectrum of $\mathcal{A}_j$, and let $\Psi_j \colon \mathcal{A}_j \to C^*(X_j, .)$ be a $*$-isomorphism of C^*-algebras. Let $j \colon \mathcal{A}_1 \to \mathcal{A}_2$ be the inclusion map. Then $\Phi := \Psi_2 \circ j \circ \Psi_1^{-1} \colon C^*(X_1, .) \to C^*(X_2, .)$ is a one-to-one $*$-homomorphism. We infer from Proposition 3.3 that there are an invariant open (in X_2) set U and a morphism $\varphi \colon (U, .) \to (X_1, .)$ such that (3.1) holds. We claim that

$$\varphi(U) = X_1. \tag{5.2}$$

Since φ is proper, the set $F := \varphi(U)$ is closed in X_1. It is also invariant. So, if $F \neq X_1$, we may take $b \in X_1 \setminus F$ and apply Lemma 3.2 (c) to get a function $f \in C^*(X_1, .)$ such that $f\big|_F \equiv 0$ and $f(b) = I$. Then $\Phi(f) = 0$, by (3.1), which contradicts the fact that Φ is one-to-one. So, (5.2) is fulfilled.

Further, Proposition 3.3 yields that there is an $x \in X_1$ such that $\pi_1(\Psi_1^{-1}(f)) = f(x)$ for any $f \in C^*(X_1, .)$. It follows from (5.2) that we may find $z \in U$ for which $\varphi(z) = x$. Now define $\pi_2 \colon \mathcal{A}_2 \to M_n$ by $\pi_2(a) = [\Psi_2(a)](z)$ $(a \in \mathcal{A}_2)$. It remains to check that π_2 extends π_1. To see this, for $a \in \mathcal{A}_1$ put $f = \Psi_1(a)$, and note that $\pi_2(a) = [\Psi_2(a)](z) = [\Psi_2(\Psi_1^{-1}(f))](z) = [\Phi(f)](z) = f(\varphi(z)) = f(x) = \pi_1(a)$ (cf., (3.1)).

Now, if the assumption of (b) is satisfied, the above argument shows that φ is one-to-one (since different points of X_2 correspond to different n-dimensional representations of $\mathcal{A}_2$). It may also easily be checked that, for every $z \in X_2 \setminus U$, the representation $\mathcal{A}_2 \ni a \mapsto [\Psi_2(a)](z) \in M_n$ vanishes on $\mathcal{A}_1$ (use (3.1) and the definition of Φ). So, we conclude from the uniqueness of the extension of the zero representation of $\mathcal{A}_1$ that $U = X_2$, and hence, both φ and Φ are isomorphisms. Consequently, $\mathcal{A}_1 = \mathcal{A}_2$, and we are done. □

6. Spectral theorem and n-functional calculus. Whenever $\mathcal{A}$ is a unital C^*-algebra and $x_1, \ldots, x_k$ are arbitrary elements of $\mathcal{A}$, let $C^*(x_1, \ldots, x_k)$ denote the C^*-subalgebra of $\mathcal{A}$ generated by $x_1, \ldots, x_k$, and let $C_1^*(x_1, \ldots, x_k)$ be the smallest C^*-subalgebra of $\mathcal{A}$ which contains $x_1, \ldots, x_k$ as well as the unit of $\mathcal{A}$ (so, $C_1^*(x_1, \ldots, x_k) = C^*(x_1, \ldots, x_k) + \mathbb{C} \cdot 1$ where 1 is the unit of $\mathcal{A}$). We would like to distinguish those systems $(x_1, \ldots, x_k)$ for which one of these two C^*-algebras defined above is n-homogeneous. However, the property of being n-homogeneous is not hereditary for $n > 1$. That is, when $n > 1$, *every* nonzero n-homogeneous C^*-algebra contains a C^*-subalgebra which is not n-homogeneous (namely, a nonzero commutative one). This results in the class of distinguished systems possibly depending on the choice of C^*-algebras related to them. Fortunately, this does not happen, which is explained in the following.

Lemma 6.1. *Let $\mathcal{A}$ be a unital C^*-algebra and $x_1, \ldots, x_k \in \mathcal{A}$. If $C_1^*(x_1, \ldots, x_k)$ is n-homogeneous for some $n > 1$, then*

$$C_1^*(x_1, \ldots, x_k) = C^*(x_1, \ldots, x_k).$$

Proof. Suppose, to the contrary, that the assertion is false. Observe that $\mathcal{I} := C^*(x_1, \ldots, x_k)$ is a two-sided ideal in $\mathcal{B} := C_1^*(x_1, \ldots, x_k)$, since

$$\mathcal{B} = \mathcal{I} + \mathbb{C} \cdot 1 \quad (1 = \text{the unit of } \mathcal{A}).$$

Moreover, $\mathcal{B}/\mathcal{I}$ is isomorphic (as a C^*-algebra) to $\mathbb{C}$, which means that the canonical projection $\pi \colon \mathcal{B} \to \mathcal{B}/\mathcal{I}$ may be considered as a one-dimensional (nonzero) representation. It is obviously irreducible, which contradicts the fact that $\mathcal{B}$ is n-homogeneous (since $n > 1$). □

Taking into account the above result, we may now introduce

Definition 6.2. A system $(x_1, \ldots, x_k)$ of elements of an (unnecessarily unital) C^*-algebra $\mathcal{A}$ is said to be *n-homogeneous* (where $n \geqslant 1$ is finite) if the C^*-subalgebra $C^*(x_1, \ldots, x_k)$ of $\mathcal{A}$ generated by $x_1, \ldots, x_k$ is n-homogeneous.

This part of the paper is devoted to studies of (finite) n-homogeneous systems. We begin with:

Proposition 6.3. *Let $(x_1, \ldots, x_k)$ be an n-homogeneous system in a C^*-algebra $\mathcal{A}$. Let $(\mathfrak{X}, .)$ be the concrete n-spectrum of $C^*(x_1, \ldots, x_k)$, and let*

$$\sigma_n(x_1, \ldots, x_k) := \{(\pi(x_1), \ldots, \pi(x_k)) \colon \ \pi \in \mathfrak{X}\} \tag{6.1}$$

be equipped with the topology inherited from $(M_n)^k$ and with the action

$$\mathfrak{u}.(A_1, \ldots, A_k) := (\mathfrak{u}.A_1, \ldots, \mathfrak{u}.A_k)$$

(*where* $\mathfrak{u} \in \mathfrak{U}_n$ *and* $(A_1, \dots, A_k) \in \sigma_n(x_1, \dots, x_k)$).

(Sp1) *The pair* $(\sigma_n(x_1, \dots, x_k), .)$ *is an* n*-space.*

(Sp2) *The function*

$$H \colon (\mathfrak{X}, .) \ni \pi \longmapsto (\pi(x_1), \dots, \pi(x_k)) \in (\sigma_n(x_1, \dots, x_k), .)$$

is an isomorphism of n*-spaces.*

(Sp3) *Every member of* $\sigma_n(x_1, \dots, x_k)$ *is irreducible*; *that is, if*

$$(A_1, \dots, A_k) \in \sigma_n(x_1, \dots, x_k)$$

and $T \in M_n$ *commutes with each of* $A_1, A_1^*, \dots, A_k, A_k^*$, *then* T *is a scalar multiple of the unit matrix.*

(Sp4) *The set* $\sigma_n(x_1, \dots, x_k)$ *is either compact or its closure in* $(M_n)^k$ *coincides with* $\sigma_n(x_1, \dots, x_k) \cup \{0\}$.

Proof. Let $\pi_0 \colon \mathcal{A} \to M_n$ be the zero representation, and let $\Omega = \mathfrak{X} \cup \{\pi_0\}$ be equipped with the pointwise convergence topology. Then Ω is compact (cf., the proof of Theorem 5.2). If $\pi_1, \pi_2 \in \Omega$, then the set $\{x \in C^*(x_1, \dots, x_k)\colon\ \pi_1(x) = \pi_2(x)\}$ is a C^*-subalgebra of $C^*(x_1, \dots, x_k)$. This implies that the function $\widetilde{H} \colon \Omega \ni \pi \mapsto (\pi(x_1), \dots, \pi(x_k)) \in \sigma_n(x_1, \dots, x_k) \cup \{0\}$ is one-to-one. It is obviously seen that $\widetilde{H}$ is surjective and continuous. Consequently, $\widetilde{H}$ is a homeomorphism (since Ω is compact). This proves (Sp4) and shows that $\sigma_n(x_1, \dots, x_k)$ is locally compact. It is also clear that $H(\mathfrak{u}.\pi) = \mathfrak{u}.H(\pi)$, which is followed by (Sp1) and (Sp2). Finally, for any $\pi \in \mathfrak{X}$, $C^*(\pi(x_1), \dots, \pi(x_k)) = \pi(C^*(x_1, \dots, x_k)) = M_n$ (see Lemma 4.10), which yields (Sp3) and completes the proof. □

Definition 6.4. Let $(x_1, \dots, x_k)$ be an n-homogeneous system in a C^*-algebra. The n-space $(\sigma_n(x_1, \dots, x_k), .)$ defined by (6.1) is said to be the *n-spectrum* of $(x_1, \dots, x_k)$. According to Proposition 6.3, the n-spectrum of $(x_1, \dots, x_k)$ is an n-spectrum of $C^*(x_1, \dots, x_k)$.

Proposition 6.5. *Let* $\mathbf{x} = (x_1, \dots, x_k)$ *be an* n*-homogeneous system in a* C^**-algebra. There exists a unique* $*$*-homomorphism*

$$\Phi_{\mathbf{x}} \colon C^*(\sigma_n(\mathbf{x}), .) \longrightarrow C^*(\mathbf{x})$$

such that $\Phi_{\mathbf{x}}(p_j) = x_j$, *where* $p_j \colon \sigma_n(\mathbf{x}) \ni (A_1, \dots, A_k) \mapsto A_j \in M_n$ $(j = 1, \dots, k)$. *Moreover,* $\Phi_{\mathbf{x}}$ *is a* $*$*-isomorphism of* C^**-algebras.*

Proof. Let $(\mathfrak{X}, .)$ be the concrete n-spectrum of $C^*(\mathbf{x})$, and let $H\colon (\mathfrak{X}, .) \to (\sigma_n(\mathbf{x}), .)$ be the isomorphism as in point (Sp2) of Proposition 6.3. For $x \in C^*(\mathbf{x})$ let $\widehat{x} \in \mathcal{C}(\mathfrak{X}, .)$ be given by $\widehat{x}(\pi) = \pi(x)$. The proof of Theorem 5.2 shows that the function $C^*(\mathbf{x}) \ni x \mapsto \widehat{x} \in C^*(\mathfrak{X}, .)$ is a $*$-isomorphism of C^*-algebras. Consequently, $\Psi\colon C^*(\mathbf{x}) \ni x \mapsto \widehat{x} \circ H^{-1} \in C^*(\sigma_n(\mathbf{x}), .)$ is a $*$-isomorphism as well. A direct calculation shows that $\Psi(x_j) = p_j$ $(j = 1, \ldots, k)$. This implies that $C^*(p_1, \ldots, p_k) = C^*(\sigma_n(\mathbf{x}), .)$, from which we infer the uniqueness of $\Phi_{\mathbf{x}}$. To convince about its existence, just put $\Phi_{\mathbf{x}} = \Psi^{-1}$. □

We are now ready to introduce the following:

Definition 6.6. Let $\mathbf{x} = (x_1, \ldots, x_k)$ be an n-homogeneous system, and let $\Phi_{\mathbf{x}}$ be as in Proposition 6.5. For every $f \in C^*(\sigma_n(x_1, \ldots, x_k), .)$, we denote by $f(x_1, \ldots, x_k)$ the element $\Phi_{\mathbf{x}}(f)$. The assignment $f \mapsto f(x_1, \ldots, x_k)$ is called the *n-functional calculus.*

The reader familiar with functional calculus on normal operators (or normal elements in C^*-algebras) has to be careful with the n-functional calculus, because its main disadvantage is that its values are not n-homogeneous elements in general. Therefore, we cannot speak of the n-spectrum of $f(x_1, \ldots, x_k)$ in general. What is more, it may happen that $\sigma_n(x_1, \ldots, x_k)$ is compact, but $j(x_1, \ldots, x_k)$, where j is constantly equal to the unit matrix, differs from the unit of the underlying C^*-algebra $\mathcal{A}$ from which $x_1, \ldots, x_k$ were taken. This happens precisely when $C^*(x_1, \ldots, x_k)$ has a unit, but this unit is not the unit of $\mathcal{A}$.

As a consequence of Theorem 4.6 and Proposition 6.5 we obtain the *spectral* theorem (for n-homogeneous systems) announced before.

Theorem 6.7. *Let $\boldsymbol{T} = (T_1, \ldots, T_k)$ be an n-homogeneous system of bounded linear operators acting on a Hilbert space $\mathcal{H}$. There exists a unique spectral n-measure $E_{\boldsymbol{T}}\colon \mathfrak{B}(\sigma_n(\boldsymbol{T})) \to M_n(\mathcal{B}(\mathcal{H}))$ such that*

$$\int p_j \, \mathrm{d}E_{\boldsymbol{T}} = T_j \quad (j = 1, \ldots, k)$$

where $p_j\colon \sigma_n(\boldsymbol{T}) \ni (A_1, \ldots, A_k) \mapsto A_j \in M_n$.

Definition 6.8. Let $\boldsymbol{T} = (T_1, \dots, T_k)$ be an n-homogeneous system of bounded Hilbert space operators, and let $E_{\boldsymbol{T}}$ be the spectral n-measure as in Theorem 6.7. $E_{\boldsymbol{T}}$ is called the *spectral n-measure of* $\boldsymbol{T}$, and the assignment

$$\mathfrak{B}C^*(\sigma(\boldsymbol{T}),.) \ni f \longmapsto f(T_1, \dots, T_n) := \int f \, \mathrm{d}E_{\boldsymbol{T}} \in \mathcal{B}(\mathcal{H})$$

is called the *extended n-functional calculus.*

There is nothing surprising in the following

Proposition 6.9. *Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, and let $\boldsymbol{T} = (T_1, \dots, T_k)$ be an n-homogeneous system of operators belonging to $\mathcal{M}$. Let $X = \sigma_n(\boldsymbol{T})$.*

(a) *For any $f \in \mathfrak{B}C^*(X,.)$, $f(\boldsymbol{T}) \in \mathcal{M}$.*
(b) *If $f^{(1)}, f^{(2)}, \dots \in \mathfrak{B}C^*(X,.)$ converge pointwise to $f \colon X \to M_n$ and*

$$\sup_{\substack{m \geqslant 1 \\ x \in X}} \|f^{(m)}(x)\| < \infty,$$

then $f \in \mathfrak{B}C^*(X,.)$ and $\lim_{m\to\infty}(f^{(m)}(\boldsymbol{T}))h = (f(\boldsymbol{T}))h$ for each $h \in \mathcal{H}$.

Proof. We begin with (a). It is clear that $g(\boldsymbol{T}) \in \mathcal{M}$ for $g \in C^*(X,.)$. Let $E_{\boldsymbol{T}} = [E_{pq}]$. Denote by $\langle \cdot, - \rangle$ the scalar product of $\mathcal{H}$, and fix $f = [f_{pq}] \in \mathfrak{B}C^*(X,.)$. We shall show that $f(\boldsymbol{T})$ belongs to the closure of $\{g(\boldsymbol{T})\colon\ g \in C^*(X,.)\}$ in the weak operator topology of $\mathcal{B}(\mathcal{H})$, which will give (a). To this end, we fix $h_1, w_1, \dots, h_r, w_r \in \mathcal{H}$ and $\varepsilon > 0$. Put $\mu = \sum_{s=1}^r \sum_{p,q} |E_{pq}^{(h_s,w_s)}|$. By Lemma 4.7, there is a $g = [g_{pq}] \in C^*(X,.)$ such that

$$\int_X \|f(x) - g(x)\| \, \mathrm{d}\mu(x) \leqslant \varepsilon.$$

But then, for each $s \in \{1, \ldots, r\}$,

$$\begin{aligned}
&\left|\left\langle \left(\int f \,\mathrm{d}E_{\boldsymbol{T}} - \int g \,\mathrm{d}E_{\boldsymbol{T}} \right) h_s, w_s \right\rangle\right| \\
&= \left| \sum_{p,q} \int_X (f_{pq} - g_{pq}) \,\mathrm{d}E_{qp}^{(h_s, w_s)} \right| \\
&\leqslant \sum_{p,q} \int_X |f_{pq} - g_{pq}| \,\mathrm{d}|E_{qp}^{(h_s, w_s)}| \\
&\leqslant \int_X \|f(x) - g(x)\| \,\mathrm{d}\mu(x) \\
&\leqslant \varepsilon,
\end{aligned}$$

and we are done (since $f(\boldsymbol{T}) = \int f \,\mathrm{d}E_{\boldsymbol{T}}$ and $g(\boldsymbol{T}) = \int g \,\mathrm{d}E_{\boldsymbol{T}}$).

We turn to (b). It is clear that $f \in \mathfrak{B}C^*(X, .)$. Replacing $f^{(m)}$ by $f^{(m)} - f$, we may assume $f = 0$. Observe that, then,

$$\lim_{m \to \infty} ((f^{(m)})^* f^{(m)})_{pq}(x) = 0$$

for any $x \in X$ and $p, q \in \{1, \ldots, n\}$, and the functions $((f^{(1)})^* f^{(1)})_{pq}$, $((f^{(2)})^* f^{(2)})_{pq}, \ldots$ are uniformly bounded. Therefore (by Lebesgue's dominated convergence theorem), for any $h \in H$,

$$\begin{aligned}
\|(f^{(m)}(\boldsymbol{T}))h\|^2 &= \left\langle (f^{(m)}(\boldsymbol{T}))^* (f^{(m)}(\boldsymbol{T}))h, h \right\rangle \\
&= \left\langle \left(\int (f^{(m)})^* f^{(m)} \,\mathrm{d}E_{\boldsymbol{T}} \right) h, h \right\rangle \\
&= \sum_{p,q} \int_X ((f^{(m)})^* f^{(m)})_{pq} \,\mathrm{d}E_{qp}^{(h,h)} \longrightarrow 0 \quad (m \to \infty),
\end{aligned}$$

which finishes the proof. □

We end the paper with the note that the above result enables defining the extended n-functional calculus for n-homogeneous systems in W^*-algebras.

REFERENCES

1. R. Bhatia, *Matrix analysis*, Springer, New York, 1997.

2. E. Bishop, *A generalization of the Stone-Weierstrass theorem*, Pac. J. Math. **11** (1961), 777–783.

3. B. Blackadar, *Operator algebras. Theory of C^*-algebras and von Neumann algebras*, Encycl. Math. Sci. **122**: Operator algebras and non-commutative geometry III), Springer-Verlag, Berlin, 2006.

4. W.-M. Ching, *Topologies on the quasi-spectrum of a C^*-algebra*, Proc. Amer. Math. Soc. **46** (1974), 273–276.

5. J. Dixmier, *C^*-algebras*, North-Holland Publishing Company, Amsterdam, 1977.

6. J. Ernest, *Charting the operator terrain*, Mem. Amer. Math. Soc. **171** (1976), 207 pages.

7. J.M.G. Fell, *C^*-algebras with smooth dual*, Illinois J. Math. **4** (1960), 221–230.

8. ______, *The dual spaces of C^*-algebras*, Trans. Amer. Math. Soc. **94** (1960), 365–403.

9. ______, *The structure of algebras of operator fields*, Acta Math. **106** (1961), 233–280.

10. M.I. Garrido and F. Montalvo, *On some generalizations of the Kakutani-Stone and Stone-Weierstrass theorems*, Extr. Math. **6** (1991), 156–159.

11. J. Glimm, *A Stone-Weierstrass theorem for C^*-algebras*, Ann. Math. **72** (1960), 216–244.

12. D.W. Hadwin, *An operator-valued spectrum*, Not. Amer. Math. Soc. **23** (1976), A-163.

13. ______, *An operator-valued spectrum*, Ind. Univ. Math. J. **26** (1977), 329–340.

14. D. Hofmann, *On a generalization of the Stone-Weierstrass theorem*, Appl. Categ. Struct. **10** (2002), 569–592.

15. J.S. Kim, Ch.R. Kim and S.G. Lee, *Reducing operator valued spectra of a Hilbert space operator*, J. Korean Math. Soc. **17** (1980), 123–129.

16. S.G. Lee, *Remarks on reducing operator valued spectrum*, J. Korean Math. Soc. **16** (1980), 131–136.

17. R. Longo, *Solution of the factorial Stone-Weierstrass conjecture. An application of the theory of standard split W^*-inclusions*, Inv. Math. **76** (1984), 145–155.

18. K. Löwner, *Über monotone Matrixfunctionen*, Math. Z. **38** (1934), 177–216.

19. P. Niemiec, *Unitary equivalence and decompositions of finite systems of closed densely defined operators in Hilbert spaces*, Diss. Math. (Rozprawy Mat.) **482** (2012), 1–106.

20. C. Pearcy and N. Salinas, *Finite-dimensional representations of separable C^*-algebras*, Not. Amer. Math. Soc. **21** (1974), A-376.

21. S. Popa, *Semiregular maximal abelian $*$-subalgebras and the solution to the factor state Stone-Weierstrass problem*, Inv. Math. **76** (1984), 157–161.

22. T. Saitô, *Generations of von Neumann algebras*, in *Lecture on operator algebras* Lect. Notes Math. **247**, Springer, Berlin, 1972, 435–531.

23. S. Sakai, *C^*-Algebras and W^*-Algebras*, Springer-Verlag, Berlin, 1971.

24. M.H. Stone, *Application of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc. **41** (1937), 375–481.

25. ______, *The generalized Weierstrass approximation theorem*, Math. Mag. **21** (1948), 167–184.

26. M. Takesaki, *Theory of operator algebras* I, Encycl. Math. Sci. **124**, Springer-Verlag, Berlin, 2002.

27. V. Timofte, *Stone-Weierstrass theorems revisited*, J. Approx. Theory **136** (2005), 45–59.

28. J. Tomiyama and M. Takesaki, *Applications of fibre bundles to the certain class of C^*-algebras*, Tôhoku Math. J. **13** (1961), 498–522.

Instytut Matematyki, Wydział Matematyki i Informatyki, Uniwersytet Jagielloński, ul. Łojasiewicza 6, 30-348 Kraków, Poland
Email address: **piotr.niemiec@uj.edu.pl**