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STATISTICAL CONVERGENCE OF FUNCTIONAL SEQUENCES

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ABSTRACT. Statistical convergence in Lebesgue spaces is considered in this paper. A criterion for statistical convergence is given. It is shown that the known Tauberian theorems for scalar case are valid in this case, too.

1. Introduction. Apparently, the concept of statistical convergence of a sequence of numbers, as the generalization of the classical concept of a limit of a sequence, was first introduced by Fast in [**5**]. In [**6, 7, 16, 17**], the basic properties of statistically convergent sequences have been studied. Later a lot of research appeared with various generalizations of this concept (see [**1, 2, 3, 4, 8, 9, 10, 11, 12, 13, 14, 15, 18**]). In [**10, 11, 12**], this concept was used in the theory of approximation by positive operators.

In this paper, we consider the statistical convergence in Lebesgue spaces L_p. We introduce the concept of statistical fundamentality in L_p and prove its equivalence to the one of statistical convergence. We also prove that the Tauberian theorems of [**6**] stay valid in our case.

2. Preliminaries. We will use the standard notation. $\mathbb{N}$ will be the set of all positive integers; $\mathbb{R}$ is the set of all real numbers; $\chi_M(\cdot)$ is the characteristic function of M. Throughout this paper, we will denote by $|M|$ the cardinality of the set M. $C(M)$ denotes the space of continuous functions on M.

Recall the definition of the concept of statistical convergence of sequences of numbers. Let $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ be some sequence and $a \in \mathbb{R}$

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a number. For all $\varepsilon > 0$, put

$$a^{\varepsilon} \equiv \{n \in \mathbb{N} : |a_n - a| \geq \varepsilon\}.$$

The value

$$\delta(M) = \lim_{n\to\infty} \frac{\sum_{k=1}^{n} \chi_M(k)}{n},$$

is called a *statistical density* (stat density) of M.

Definition 2.1. The sequence $\{a_n\}_{n\in\mathbb{N}}$ is called *statistically convergent* to a, if $\delta(a^{\varepsilon}) = 0$, for all $\varepsilon > 0$, and this is denoted as

$$\text{st} \lim_{n\to\infty} a_n = a.$$

Many properties of statistically convergent sequences have been studied, and this concept is generalized in various directions. More details about these and related facts can be found in **[1]**–**[18]**.

We will need the Tauberian theorems on statistically convergent sequences. Let $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ be a sequence. Let $\Delta a_n = a_n - a_{n+1}$. The following theorem is true.

Theorem 2.2. ([**6**]). *Let* $\text{st} \lim_{n\to\infty} a_n = a$ *and* $\Delta a_n = \bar{\bar{o}}(1/n)$. *Then there exist* $\lim_{n\to\infty} a_n$ *and* $\lim_{n\to\infty} a_n = a$.

The converse statement is not always true, i.e., the following theorem is true.

Theorem 2.3. ([**6**]). *Let* $\{r_k\}_{k\in\mathbb{N}}$ *be a decreasing sequence of positive numbers such that* $\{kr_k\}_{k\in\mathbb{N}}$ *is unbounded. Then there exist* $\{a_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$: $\text{st} \lim_{k\to\infty} a_k = 0$ *and* $\Delta a_k = \bar{\bar{o}}(r_k)$, *while* $\lim_{k\to\infty} a_k$ *does not exist.*

3. Statistical convergence in L_p. Let $\{f_n(x)\}_{n\in\mathbb{N}}$ be some sequence of functions $f_n : M \to \mathbb{R}$ and $M \subset \mathbb{R}$ some set. This sequence is called statistically convergent to A at the point $x_0 \in M$, if the sequence $\{f_n(x_0)\}_{n\in\mathbb{N}}$ statistically converges to A, i.e., $\text{st} \lim_{n\to\infty} f_n(x_0) = A$. This sequence is called statistically convergent to $f(x)$ on M if

$$st \lim_{n\to\infty} f_n(x) = f(x), \quad \text{for all } x \in M. \tag{3.1}$$

Definition 3.1. We say that $\{f_n\}_{n\in\mathbb{N}}$ statistically uniformly converges to f on M if

$$\text{st}\lim_{n\to\infty}\sup_{x\in M}|f_n(x)-f(x)|=0. \tag{3.2}$$

This kind of convergence is denoted as $f_n \overset{st}{\rightrightarrows} f$ on M.

It is clear that, if $f_n \overset{st}{\rightrightarrows} f$ on M, then the relation (3.1) holds. But the converse of this statement is not true.

Suppose that the relation (3.2) is true. Put $\alpha_n=\sup_{x\in M}|f_n(x)-f(x)|$, for all $n\in\mathbb{N}$. Consequently, $\text{st}\lim_{n\to\infty}\alpha_n=0$. It is known that $\exists K\equiv\{n_k\}_{k\in\mathbb{N}}$: $n_1<n_2<\cdots$, $\delta(K)=1$ and $\lim_{k\to\infty}\alpha_{n_k}=0$. Thus, the sequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ converges uniformly to f in M. Hence, if $f_n(x)$ is continuous on M and $f_n \overset{st}{\rightrightarrows} f$ on M, then f is also continuous on M. Moreover, for $M\equiv[a,b]$, we have

$$\text{st}\lim_{n\to\infty}\int_a^b f_n(x)\,dx=\int_a^b f(x)\,dx.$$

So, the following statement is true.

Statement 3.2. Let $\{f_n\}_{n\in\mathbb{N}}\subset C[a,b]$ and $f_n \overset{st}{\rightrightarrows} f$ on $[a,b]$. Then $f\in C[a,b]$ and

$$\text{st}\lim_{n\to\infty}\int_a^b f_n(x)\,dx=\int_a^b f(x)\,dx.$$

Consider the L_p-case. Let $f_n,\ f\in L_p(a,b)$, $1\le p<+\infty$.

Definition 3.3. We say that $f_n \overset{st}{\to} f$ in L_p if

$$\text{st}\lim_{n\to\infty}\int_a^b|f_n(x)-f(x)|^p\,dx=0,\quad 1\le p<+\infty. \tag{3.3}$$

If the relation (3.3) is true, then there exists $K\equiv\{n_k\}_{k\in\mathbb{N}}$, $n_1<n_2<\cdots$, $\delta(K)=1$:

$$\lim_{n\to\infty}\int_a^b|f_{n_k}(x)-f(x)|^p\,dx=0.$$

This implies that there exists a subsequence $\{f_{k_n}\}_{n\in\mathbb{N}}$ of the sequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ such that $f_{k_n}(x) \to f(x)$ for almost every $x \in [a,b]$. Let

$$\mathscr{K} \equiv \{K \subset \mathbb{N} : \delta(K) = 1\}.$$

In the sequel, we will need the following easily provable:

Lemma 3.4. *Let $K_j \in \mathscr{K}$, $j = 1,2 \Rightarrow K_1 \bigcap K_2 \in \mathscr{K}$.*

In fact, let $I_n \equiv \{1;\ldots;n\}$. We have:

$$K_1 \bigcap K_2 = \left(K_1 \bigcup K_2\right) \Big\backslash \left[(K_2 \backslash K_1) \bigcup (K_1 \backslash K_2)\right].$$

Consequently,

$$K_1 \bigcap K_2 \bigcap I_n = \left[\left(K_1 \bigcup K_2\right) \bigcap I_n\right] \Big\backslash \left[\left((K_2 \backslash K_1) \bigcup (K_1 \backslash K_2)\right) \bigcap I_n\right]. \tag{3.4}$$

As

$$\left((K_2 \backslash K_1) \bigcup (K_1 \backslash K_2)\right) \bigcap I_n = \left((K_2 \backslash K_1) \bigcap I_n\right) \bigcup \left((K_1 \backslash K_2) \bigcap I_n\right),$$

taking into account

$$(K_2 \backslash K_1) \bigcap I_n \subset K_1^c \bigcap I_n \Longrightarrow \frac{|(K_2 \backslash K_1) \bigcap I_n|}{|I_n|} \leq \frac{|K_1^c \bigcap I_n|}{|I_n|} \longrightarrow 0, \quad n \to \infty,$$

$$\frac{|(K_1 \backslash K_2) \bigcap I_n|}{|I_n|} \longrightarrow 0, \quad n \to \infty,$$

we get

$$\frac{|((K_2 \backslash K_1) \bigcup (K_1 \backslash K_2)) \bigcap I_n|}{|I_n|} \longrightarrow 0, \quad n \to \infty.$$

From $(K_1 \bigcap I_n) \subset (K_1 \bigcup K_2) \bigcap I_n$ and $K_1 \in \mathscr{K}$, it follows that

$$\frac{|(K_1 \bigcup K_2) \bigcap I_n|}{|I_n|} \longrightarrow 1, \quad n \to \infty,$$

and hence, $K_1 \bigcup K_2 \in \mathscr{K}$. Then, from (3.4), we directly obtain

$$\frac{|K_1 \bigcap K_2 \bigcap I_n|}{|I_n|} = \frac{|(K_1 \bigcup K_2) \bigcap I_n|}{|I_n|} - \frac{|((K_2 \backslash K_1) \bigcup (K_1 \backslash K_2)) \bigcap I_n|}{|I_n|} \longrightarrow 1, \quad n \to \infty,$$

i.e., $K_1 \bigcap K_2 \in \mathscr{K}$. Lemma 3.4 is proved.

Definition 3.5. We say that $\{f_n\}_{n\in\mathbb{N}}$ is statistically fundamental (st-fundamental) in L_p if, for all $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N} : \delta(\Delta_\varepsilon) = 0$, where

$$\Delta_\varepsilon \equiv \left\{ n \in \mathbb{N} : \|f_n - f_{n_\varepsilon}\|_p \geq \varepsilon \right\},$$

$$\|f\|_p = \left(\int_a^b |f(t)|^p \, dt \right)^{1/p}.$$

It is absolutely clear that, if $f_n \overset{st}{\to} f$ in L_p, then the sequence $\{f_n\}_{n\in\mathbb{N}}$ is st-fundamental in L_p. In fact, let $\varepsilon > 0$ be an arbitrary number. Put $A_\varepsilon \equiv \{n : \|f_n - f\|_p \geq \varepsilon\}$. It is clear that $\delta(A^c_{\varepsilon/2}) = 1$, where $M^c \equiv \mathbb{N} \backslash M$. Take for all $n_\varepsilon \in A^c_{\varepsilon/2} : \|f_{n_\varepsilon} - f\| < \varepsilon/2$. We have

$$\{n : \|f_n - f\| < \varepsilon/2\} \subset \{n : \|f_n - f_{n_\varepsilon}\| < \varepsilon\},$$

i.e., $A^c_{\varepsilon/2} \subset A^c_\varepsilon$. Hence, $\delta(\Delta^c_\varepsilon) = 1 \Rightarrow \delta(\Delta_\varepsilon) = 0$.

Now, vice versa, let $\{f_n\}_{n\in\mathbb{N}}$ be st-fundamental in L_p. Denote by $O_r(x_0)$ the ball in L_p, i.e., $O_r(x_0) \equiv \{x \in L_p : \|x - x_0\| < r\}$. From st-fundamentality, it follows that there exists $n_j \in \mathbb{N} : \delta(K_j) = 1$, where $K_j \equiv \{n : \|f_n - f_{n_j}\|_p \leq 2^{1-j}\}$, $j = 1, 2$. By Lemma 3.4, we obtain $K_1 \bigcap K_2 \in \mathscr{K}$. Put $M_1 \equiv \overline{O_1(f_{n_1})} \bigcap \overline{O_{2^{-1}}(f_{n_2})}$ ($\overline{M}$ is a closure of M in L_p). It is obvious that $f_n \in M_1$, for all $n \in (K_1 \bigcap K_2) \equiv K_{(1)}$. Thus, there exists $n_3 \in \mathbb{N} : K_3 \in \mathscr{K}$, where $K_3 \equiv \{n : \|f_n - f_{n_3}\|_p \leq 2^{-2}\}$. Let $K_{(2)} = K_{(1)} \bigcap K_3$. It is clear that $K_{(2)} \in \mathscr{K}$. Now, let $M_2 \equiv M_1 \bigcap \overline{O_{2^{-2}}(f_{n_3})}$. Denote by $d_p(M)$ the diameter of the set M, i.e., $d_p(M) = \sup\limits_{x,y\in M} \|x - y\|_p$. Continuing in the same way, we obtain a sequence of closed sets $\{M_n\}_{n\in\mathbb{N}}$ in L_p, whose diameters tend to zero: $d_p(M_n) \leq 2^{-n+1} \to 0$, $n \to \infty$. Moreover, $K_{(n)} \in \mathscr{K}$, where $K_{(n)} \equiv \{n : f_n \in M_n\}$. It is absolutely clear that $M_n \supset M_{n+1} \supset \cdots$.

Take for all $x_n \in M_n$. We have

$$\|x_n - x_{n+p}\|_p \leq d_p(M_n) \longrightarrow 0, \quad n \to \infty, \text{ for all } p \in \mathbb{N}.$$

Hence, the sequence $\{x_n\}_{n\in\mathbb{N}}$ is fundamental in L_p, and let $x_n \to f$, $n \to \infty$. It is absolutely clear that $f \in \bigcap_n M_n$, i.e., $\bigcap_n M_n$ is non-empty. From $d_p(M_n) \to 0$, $n \to \infty$, it directly follows that $\{f\} = \bigcap_n M_n$, i.e., $\bigcap_n M_n$ consists of one element. As $K_{(m)} \in \mathscr{K}$, then there exists $\{n_m\}_{m\in\mathbb{N}} \subset \mathbb{N}$; $n_1 < n_2 < \cdots$:

$$\frac{1}{n}\left|\left\{k \in I_n : k \in K^c_{(m)}\right\}\right| < \frac{1}{m}, \quad \text{for all } n > n_m.$$

Assume that

$$\mathbb{N}_0 \equiv \left\{k \in \mathbb{N} :\, n_m < k \leq n_{m+1} \quad \text{and} \quad k \in K^c_{(m)}\right\},$$

and

$$g_k = \begin{cases} f, & \text{if } k \in \mathbb{N}_0 \text{ and } (k > n_1); \\ f_k, & \text{if otherwise.} \end{cases}$$

Take $\varepsilon > 0$. If $k \in \mathbb{N}_0$ and $(k > n_1)$, then $\|g_k - f\|_p = 0 < \varepsilon$. If $k \notin \mathbb{N}_0 \Rightarrow k \in K_{(m)} \Rightarrow f_k \in M_m \Rightarrow \|f_k - f\|_p \leq \|f_k - f_{n_m}\|_p + \|f_{n_m} - f\|_p \leq 2^{-m+2} < \varepsilon$, for sufficiently great values of m. Consequently, $\lim_{k\to\infty} g_k = f$. Let us show that $\delta(\widetilde{K}) = 0$, where $\widetilde{K} \equiv \{k \in \mathbb{N} : f_k \neq g_k\}$. Let $n_m < n < n_{m+1}$. Let us prove that

$$\{k \leq n : f_k \neq g_k\} \subset \left\{k \leq n : k \in K^c_{(m)}\right\}.$$

Let $f_k \neq g_k$, $k \leq n$. Consequently, $k \in \mathbb{N}_0 \Rightarrow k \in K^c_{(m)}$. Thus,

$$\frac{1}{n}\left|\{k \leq n : f_k \neq g_k\}\right| \leq \frac{1}{n}\left|\left\{k \leq n :\, k \in K^c_{(m)}\right\}\right| < \frac{1}{m}.$$

It is clear that, if $n \to \infty$, then $m \to \infty$. Then, from the previous relation, we get

$$\lim_{n\to\infty} \frac{|\{k \leq n : f_k \neq g_k\}|}{n} = 0. \tag{3.5}$$

Consequently, $\{k \leq n : f_k \neq g_k\}^c \in \mathscr{K}$ and $\lim_{n\to\infty} g_n = f$. Let us show that *st* $\lim_{n\to\infty} f_n = f$. Take $\varepsilon > 0$. We have

$$(3.6)\quad \left\{k \le n : \|f_k - f\|_p \ge \varepsilon\right\} \subset \{k \le n : f_k \ne g_k\} \bigcup \left\{k \le n : \|g_k - f\|_p \ge \varepsilon\right\}.$$

As $\lim_{k\to\infty} g_k = f$ in L_p, then $\|g_k - f\|_p < \varepsilon$, for all $k \ge n_\varepsilon$. Consequently,

$$\left|\left\{k \le n : \|g_k - f\|_p \ge \varepsilon\right\}\right| \le n_\varepsilon \Longrightarrow \frac{1}{n}\left|\left\{k \le n : \|g_k - f\|_p \ge \varepsilon\right\}\right| \to 0, \quad n \to \infty.$$

Then, using (3.5), from (3.6) we obtain $1/n\{k \le n : \|f_k - f\|_p \ge \varepsilon\} \le 1/n\{k \le n : f_k \ne g_k\} + 1/n\{k \le n : \|g_k - f\|_p \ge \varepsilon\} \to 0$, $n \to \infty$. So, *st* $\lim_{n\to\infty} f_n = f$. Thus, we have proved the following theorem.

Theorem 3.6. *Let* $\{f_n\}_{n\in\mathbb{N}} \subset L_p$ *be some sequence. Then the following statements are equivalent to each other*:

- (i) *There exists* st $\lim_{n\to\infty} f_n$;
- (ii) $\{f_n\}_{n\in\mathbb{N}}$ *is* st-*fundamental*;
- (iii) *there exists* $\{g_n\}_{n\in\mathbb{N}} \subset L_p$: *there exists* $\lim_{n\to\infty} g_n$ *and* $\{n : f_n = g_n\} \in \mathscr{K}$.

This theorem immediately implies the following:

Corollary 3.7. *Let* $\{f_n\}_{n\in\mathbb{N}} \subset L_p$ *and* st $\lim_{n\to\infty} f_n = f$. *Then there exists* $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N} : n_1 < n_2 < \cdots$, $\lim_{k\to\infty} f_{n_k} = f$ *and* $\delta(\{n_k\}_{k\in\mathbb{N}}) = 1$.

In the case of a sequence of numbers, there is no regular matrix summation method which would include a statistical convergence (see **[6]**). The same statement stays valid in our case. The following lemma is true.

Lemma 3.8. *Let* $\{t_k\}_{k\in\mathbb{N}}$ *be some sequence of numbers and* $\sum_{k=1}^{\infty}\chi_{A_0}(k) = +\infty$, *where* $A_0 \equiv \{k \in \mathbb{N} : t_k \ne 0\}$. *Then there exists* $\{f_k\}_{k\in\mathbb{N}} \subset L_p : \sum_{k=1}^{\infty} t_k f_k(t) = \infty$ *for all* $t \in [a,b]$.

In fact, let's take a subsequence $\{m_k\}_{k\in\mathbb{N}}$ such that

$$m_k > k^2 \quad \text{and} \quad t_{m_k} \neq 0.$$

Define

$$x_{m_k}(t) \equiv \frac{1}{t_{m_k}} \quad \text{for all } t \in [a,b], \text{ for all } k \in \mathbb{N};$$
$$x_k(t) \equiv 0, \quad \text{for all } t \in [a,b], \text{ for all } k \notin \{m_1; m_2; \ldots\}.$$

We have

$$\sum_{k=1}^{\infty} t_k x_k(t) = \sum_{k=1}^{\infty} t_{m_k} x_{m_k}(t) = \sum_{k=1}^{\infty} 1 = \infty, \quad \text{for all } t \in [a,b].$$

On the other hand, it is easy to see that $x_k \xrightarrow{st} 0$ in L_p. Lemma 3.8 is proved.

The following theorem is true.

Theorem 3.9. *There is no matrix summation method which possesses statistical convergence in L_p.*

Proof. From Lemma 3.8, it follows directly that if there is an appropriate matrix $A \equiv (a_{ij})_{i,j\in\mathbb{N}}$, then for all $i \in \mathbb{N}$, there exists $m_i \in \mathbb{N} : a_{ij} = 0$, for all $j \geq m_i$. It is absolutely clear that A should have an infinite number of nonzero elements. Let $a_{n_1 k_1'} \neq 0$. Assume $k_1 = \max\{j \in \mathbb{N} : a_{n_1 j} \neq 0\}$. It is clear that $a_{n_1 k_1} \neq 0$. Choose the indices $\{n_m; k_m\}_{m\in\mathbb{N}}$ from the following conditions

$$a_{n_m k_m} \neq 0, \quad k_m \geq m^2 \quad \text{and} \quad a_{n_m j} = 0, \text{ for all } j > k_m.$$

Now we define a sequence of functions $\{x_n(t)\}_{n\in\mathbb{N}}$ as follows:

$$x_{k_1}(t) \equiv a_{n_1 k_1}^{-1}, \quad \text{for all } t \in [a,b], \ldots,$$
$$x_{k_m}(t) \equiv a_{n_m k_m}^{-1}\left[m - \sum_{i=1}^{m-1} a_{n_i k_i} x_{k_i}(t)\right], \quad \text{for all } t \in [a,b], \ldots,$$
$$x_k(t) \equiv 0, \quad \text{for all } t \in [a,b], \text{ for all } k \notin \{k_1; k_2; \ldots\}.$$

We have:

$$y_n(t) \equiv (A\overline{x}(t))_n = \sum_{i:n_i \leq n} a_{n_i k_i} x_{k_i}(t),$$

where $\overline{x}(t) \equiv (x_1(t); x_2(t); \ldots)$. Thus,

$$y_n \equiv \sum_{i:n_i \le n} 1, \quad \text{for all } n \in \mathbb{N}.$$

In particular,

$$y_{n_m} = m, \quad m \in \mathbb{N}.$$

Obviously, $y_n(t) \overset{n}{\to} \infty$ in L_p. On the other hand, from $k_m \ge m^2$, it directly follows that $|\{k \le n : x_k \ne 0\}| \le \sqrt{n}$, and, as a result, st $\lim_{n\to\infty} x_n(t) = 0$ in L_p. □

Note that the examples in the scalar case, given in [**6**], show that the statistical convergence does not imply many summation methods. The same statement is true about the statistical convergence in L_p.

4. Tauberian theorems. Let $\{f_n\}_{n\in\mathbb{N}} \subset L_p$ be some sequence. Put $\Delta f_n = f_n - f_{n+1}$. The following analogue of Theorem 2.2 is true in L_p.

Theorem 4.1. *Let* st $\lim_{n\to\infty} f_n = f$ *in* L_p *and* $\|\Delta f_k\|_p = \overline{\overline{o}}(1/k)$. *Then there exists* $\lim_{n\to\infty} f_n$ *in* L_p *and* $\lim_{n\to\infty} f_n = f$.

Proof. We will follow [**6**]. Assume st $\lim_{n\to\infty} f_n = f$. Then, by Theorem 2.3, there exists $\{g_n\}_{n\in\mathbb{N}} \subset L_p : \lim_{n\to\infty} g_n = f$ and $\{n : g_n = f_n\} \in \mathscr{K}$. Every $k \in \mathbb{N}$ can be represented as $k = m_k + p_k$, where

$$m_k = \begin{cases} \max\{i \le k : x_i = y_i\}, & A_k \ne \emptyset, \\ -1, & A_k = \emptyset, \end{cases}$$

$A_k = \{i \le k : x_i = y_i\}$. As proved in [**6**], it holds that

$$\lim_{k\to\infty} \frac{p_k}{m_k} = 0.$$

It is clear that there exists $B > 0 : \|\Delta f_k\|_p \le B/k$, for all $k \in \mathbb{N}$. We have

$$\begin{aligned} \|g_{m_k} - f_k\|_p &= \|f_{m_k} - f_{m_k+p_k}\|_p \\ &\le \sum_{i=m_k}^{m_k+p_k-1} \|\Delta f_i\|_p \le B\frac{p_k}{m_k} \longrightarrow 0, \quad k \to \infty. \end{aligned}$$

As $\lim_{k\to\infty} g_{m_k} = f$ in L_p, it directly follows that $\lim_{k\to\infty} f_k = f$. $\square$

The following analogue of Theorem 2.3 is also true.

Theorem 4.2. *Let $\{r_k\}_{k\in\mathbb{N}}$ be a decreasing sequence of positive numbers such that $\{kr_k\}_{k\in\mathbb{N}}$ is unbounded. Then there exists $\{f_k\}_{k\in\mathbb{N}} \subset L_p$: st $\lim_{k\to\infty} f_k = 0$ in L_p, $\|\Delta f_k\|_p = \bar{\bar{o}}(r_k)$, while $\lim_{k\to\infty} f_k$ does not exist in L_p.*

In fact, let $\{x_k\}_{k\in\mathbb{N}}$ be the sequence constructed in the proof of [**6,** Theorem 4]. Let

$$f_k(t) = x_k, \quad \text{for all } t \in [a,b], \text{ for all } k \in \mathbb{N}.$$

This is the sequence which was sought.

Note that the scheme of the proof of [**6,** Theorem 5] remains valid in this case, i.e., the following theorem is true.

Theorem 4.3. *Let $\{k_i\} \subset \mathbb{N}$ be an increasing sequence satisfying*

$$\lim_{j\to\infty} \inf_{i\geq j} \frac{k_{i+1}}{k_i} > 1,$$

and $\{f_k\}_{k\in\mathbb{N}} \subset L_p : \Delta f_k = 0$, if $k \neq k_i$, for all $i \in \mathbb{N}$. If there exists st $\lim_{k\to\infty} f_k = f$ in L_p, then there exists $\lim_{k\to\infty} f_k = f$ in L_p.

Remark 4.4. Similar results can be obtained for other functional spaces such as Sobolev spaces, Morrey-type spaces, etc.

REFERENCES

1. A. Basu and P.D. Srivastava, *Statistical convergence on composite vector valued sequence space*, J. Math. Appl. **29** (2007), 75—90.

2. J. Connor and K.-G. Grosse-Erdmann, *Sequential definitions of continuity for real functions*, Rocky Mountain J. Math. **33** (2003), 93–121.

3. J.S. Connor, *The statistical and p-Cesaro convergence of sequences*, Analysis **8** (1988), 47–63.

4. ______, *R-type summability methods, Cauchy criteria, p-sets, and statistical convergence*, Proc. Amer. Math. Soc. **115** (1992), 319–327.

5. H. Fast, *Sur la convergence statistique*, Colloq. Math. **2** (1951), 241–244.

6. J.A. Fridy, *On statistical convergence*, Analysis **5** (1985), 301–313.

7. J.A. Fridy, *Statistical limit points*, Proc. Amer. Math. Soc. **118** (1993), 1187–1192.

8. J.A. Fridy and M.K. Khan, *Tauberian theorems via statistical convergence*, J. Math. Anal. Appl. **228** (1998), 73–95.

9. J.A. Fridy and H.I. Miller, *A matrix characterization of statistical convergence*, Analysis **11** (1991), 59–66.

10. A.D. Gadjiev, *Simultaneous statistical approximation of analytic functions and their derivatives by k-positive linear operator*s, Azerbaijan J. Math. **1** (2011), 57–66.

11. A.D. Gadjiev and A.M. Ghorbanalizadeh, *On the A-statistical approximation by sequence of k-positive linear operators*, Proc. IMM NAS Azerb. **31** (2009), 41–51.

12. A.D. Gadjiev and C. Orhan, *Some approximation theorems via statistical convergence*, Rocky Mountain J. Math. **32** (2002), 129–138.

13. M. Macaj and T. Salat, *Statistical convergence of subsequence of a given sequence*, Math. Bohem. **126** (2001), 191–208.

14. I.J. Maddox, *Statistical convergence in a locally convex space*, Math. Proc. Camb. Phil. Soc. **104** (1988), 141–145.

15. D. Rath and B.C. Tripathy, *On statistically convergent and statistically Cauchy sequences*, Indian J. Appl. Math. **25** (1994), 381–386.

16. T. Salat, *On statistically convergent sequences of real numbers*, Math. Slov. **30** (1980), 139–150.

17. I.J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Month. **66** (1959), 361–375.

18. A. Zygmund, *Trigonometric series*, Volume II, Cambridge University Press, London, 1979

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