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COMPUTATION OF FIXED POINT INDEX AND ITS APPLICATIONS

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ABSTRACT. In this paper, we make the nonlinear double integral equation of Hammerstein type the background of the research. Computation for the fixed point index of operators such as $A = K_1F_1K_2F_2$ is given. As applications of the main results, we investigate the existence of positive solutions to the nonlinear double integral equation of Hammerstein type and the boundary value problem for the system of elliptic partial differential equations.

1. Introduction. Let E be a real Banach space with norm $\|\cdot\|$, and let $P \subset E$ be a cone of E. We define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. P is said to be normal if there exists a positive constant N such that $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$, where θ represents the zero element in E and the smallest N is called the normal constant of P. P is called solid if it contains interior points, i.e., $P \neq \emptyset$. For the concepts and the properties about the cone we refer to [**2, 4, 17**].

In this paper, we make the nonlinear double integral equation of Hammerstein type as the background of the research and study the computation of the fixed point index of the operators such as $A = K_1F_1K_2F_2$ more systematically. As an application, the following nonlinear double integral equation of Hammerstein type

$$(1.1) \qquad u(x) = \int_0^1 k_1(x,y)f_1\left(y, \int_0^1 k_2(y,z)f_2(z,u(z))\,dz\right)dy,$$

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is investigated, where $f_i(x,u) : [0,1] \times [0,+\infty) \to [0,+\infty)$, $k_i(x,y) : [0,1] \times [0,1] \to [0,+\infty)$, $i = 1,2$. The theory of the fixed point index is one of the main methods in studying the existence of solutions to the nonlinear operator equation. The study of the nonlinear Hammerstein integral equations was initiated by Hammerstein **[7]** in 1929. Subsequently, a great number of papers dealing with the existence of nontrivial solutions of Hammerstein integral equations have been published, see for example, **[9, 10, 18, 19, 20]** and the references therein. Yang and O'Regan **[20]** studied a system of nonlinear Hammerstein integral equations:

$$\begin{cases} u(x) = \displaystyle\int_0^1 k(x,y)f(y,u(y),v(y))dy, \\ v(x) = \displaystyle\int_0^1 k(x,y)g(y,u(y),v(y))dy. \end{cases} \tag{1.2}$$

The existence of positive solutions of (1.2) was obtained by using topological methods and cone theory. Generally speaking, in order to compute the fixed point index we usually change the systems of differential equations into the nonlinear double integral equation of Hammerstein type, see **[8, 14]**.

In this paper, broader results are obtained when the nonlinear operator is controlled by α homogeneous operator (which is broader than the linear operator). To the best of our knowledge the papers dealing with (1.1) are few when the nonlinear operator is controlled by the α homogeneous operator. Many authors studied computation for the fixed point index when the nonlinear operator is controlled by the homogeneous operator or the linear operator **[5, 8, 11]**, which is different from the results of this paper. Motivated by the work **[5, 7, 8, 11, 14, 15, 18, 3, 19, 20]**, we obtained the results of this paper.

The organization of this paper is as follows. In Section 2, some useful preliminaries are presented. In Section 3, we give the results of computation of the fixed point index of the operators such as $A = K_1F_1K_2F_2$ more systematically. In Sections 4 and 5, we use the main results of Section 3 to establish the existence of positive solutions to the nonlinear double integral equation of Hammerstein type and the positive solutions of the system of elliptic partial differential equations.

2. Preliminaries. Suppose that X is a retract in Banach space E and U is a bounded open set in X. Let $A : \overline{U} \to X$ be a completely continuous operator, which has no fixed point on ∂U (the boundary of U with respect to X). Then we can define the fixed point index $i(A, U, X)$ of A over U with respect to X. (For example, every closed convex subset is a retract.) One can refer to [**4**] for the definition and properties of $i(A, U, X)$.

2.1. Definitions.

Definition 2.1. [**13**]. Let P be a solid cone of E. The operator $A : P \to P$ is a positive α homogeneous operator, if A satisfies $A(tx) = t^\alpha Ax$, for any $t > 0$ and $\alpha \in R^+$. In particular, when $\alpha = 1$, the operator A is called homogeneous operator.

Definition 2.2. [**1**]. Let P be a cone of E. The operator $T : E \to E$ is called *e-positive* if there exists an element $e \in P \setminus \{\theta\}$, such that, for every $u \in P \setminus \{\theta\}$, there are numbers $\alpha = \alpha(u), \beta = \beta(u) > 0$, such that

$$\alpha e < Tu < \beta e.$$

2.2. Lemmas.

Lemma 2.3. [**12**]. *Let a, b be real numbers. Then*

(i) $|a+b|^\mu \le |a|^\mu + |b|^\mu$, *for* $0 \le \mu \le 1$;
(ii) $|a+b|^\mu \le 2^{\mu-1}(|a|^\mu + |b|^\mu)$, *for* $\mu > 1$.

Lemma 2.4. [**17**]. *Let Ω be a bounded open set of E with $\theta \in \Omega$. Assume that $A : P \cap \overline{\Omega} \to P$ is a completely continuous operator satisfying*

$$Ax \ne \mu x, \quad x \in P \cap \partial\Omega, \ \mu \ge 1.$$

Then we have $i(A, \Omega \cap P, P) = 1$.

Lemma 2.5. [**6**]. *Let $\Omega \subset E$ be a bounded open set with $\theta \in \Omega$. Suppose $A : \overline{\Omega} \cap P \to P$ is a completely continuous operator and has no fixed points on $\partial\Omega \cap P$. If $\|Aw\| \le \|w\|$, for all $w \in \partial\Omega \cap P$, then* $i(A, \Omega \cap P, P) = 1$.

Lemma 2.6. [**17**]. *Suppose that $A : P \cap \overline{\Omega} \to P$ is a completely continuous operator. If there exist $u^* \in P$ and $u^* \neq \theta$ such that*

$$x - Ax \neq \lambda u^*, \quad x \in P \cap \partial\Omega,\ \lambda \geq 0,$$

then we have $i(A, \Omega \cap P, P) = 0$.

Lemma 2.7. [**20**]. *If $p : R^+ \to R^+$ is concave, then $p(a + b) \leq p(a) + p(b)$, for $a \in R^+$ and $b \in R^+$.*

3. Main results. In order to compute the fixed point index of operators such as $A = K_1F_1K_2F_2$, here we list the hypotheses to be used later:

(H) Let $K_1, K_2 : P \to P$ be positive linear operators, $F_1, F_2 : P \to P$ nonlinear operators, and $A = K_1F_1K_2F_2$ a completely continuous operator.

Theorem 3.1. *Assume that condition* (H) *holds. Let P be a normal cone in E, the normal constant of P equal to 1. Suppose that $G_1 : P \to P$ is a positive α homogeneous increasing operator with $0 < \alpha \leq 1$ and $G_2 : P \to P$ is a positive β homogeneous increasing operator with $0 < \alpha\beta \leq 1$. If there exists $w_i > \theta$, $i = 1, 2$, such that*:

(i) *$F_ix \leq G_ix + w_i$, for all $x \in P$;*
(ii) *$G_1(x + y) \leq G_1x + G_1y$, for all $x, y \in P$;*
(iii) *$C = K_1G_1K_2G_2$, and for all $x \in P\backslash\{\theta\}$,*

$$\sup_{\|x\|=1} \|Cx\| < \begin{cases} \infty, & 0 < \alpha\beta < 1, \\ 1, & \alpha\beta = 1, \end{cases} \tag{3.1}$$

then $i(A, P \cap T_R, P) = 1$, where $T_R = \{x \in E : \|x\| \leq R\}$ for sufficiently large R.

Remark. Let $T_R = \{x \in E : \|x\| \leq R\}$ be a closed ball of center θ and radius R. For sufficiently large $R \geq R_0$, Theorem 3.1 holds, where R_0 is defined in the following proof.

Proof. By Definition 2.1, we have $G_1(tx) = t^\alpha G_1 x$, $G_2(tx) = t^\beta G_2 x$, and

$$\begin{aligned} C(tx) &= K_1G_1K_2G_2(tx) = K_1G_1K_2t^\beta G_2x \\ &= K_1G_1t^\beta(K_2G_2x) = K_1(t^\beta)^\alpha G_1K_2G_2x \\ &= (t^\beta)^\alpha K_1G_1K_2G_2x = t^{\alpha\beta}K_1G_1K_2G_2x \\ &= t^{\alpha\beta}Cx. \end{aligned} \tag{3.2}$$

When $0 < \alpha\beta < 1$, C is an $\alpha\beta$ homogeneous operator. By condition (iii), there exists a constant M such that $\sup_{\|x\|=1} \|Cx\|/\|x\| < M$. Letting $0 < \delta < \min\{1, M\}$, we have $M/\delta > 1$. For sufficiently large $t > (M/\delta)^{1/(1-\alpha\beta)}$, by (3.2), we have

$$\sup_{\|x\|=1} \frac{\|Ctx\|}{\|tx\|} = t^{\alpha\beta-1} \cdot \sup_{\|x\|=1} \frac{\|Cx\|}{\|x\|} < t^{\alpha\beta-1} \cdot M < \delta. \tag{3.3}$$

Letting $u = tx$ in (3.3), when t is large enough, we have

$$\sup_{u\in P\setminus\{\theta\}} \frac{\|Cu\|}{\|u\|} < \delta. \tag{3.4}$$

When $\alpha\beta = 1$, C is a homogeneous operator. By condition (iii), we have $\sup_{\|x\|=1} \|Cx\|/\|x\| < 1$. It follows from (3.2) that

$$\sup_{\|x\|=1} \frac{\|Csx\|}{\|sx\|} = \sup_{\|x\|=1} \frac{s\|Cx\|}{s\|x\|} < 1. \tag{3.5}$$

In (3.5), when s is large enough, letting $u = sx$, then

$$\sup_{u\in P\setminus\{\theta\}} \frac{\|Cu\|}{\|u\|} < 1. \tag{3.6}$$

By conditions (i), (ii) and (iii), we have

$$\begin{aligned} Ax &= K_1F_1K_2F_2x \le K_1G_1K_2F_2x + K_1w_1 \\ &\le K_1G_1K_2(G_2x + w_2) + K_1w_1 \\ &= K_1G_1(K_2G_2x + K_2w_2) + K_1w_1 \\ &\le K_1G_1K_2G_2x + K_1G_1K_2w_2 + K_1w_1 \\ &= Cx + w_0, \end{aligned} \tag{3.7}$$

where $w_0 = K_1G_1K_2w_2 + K_1w_1$. Select

$$R_0 > \frac{\|w_0\|}{1-\delta}.$$

Let

$$T_{R_0} = \{x \in E : \|x\| < R_0\}.$$

Set $H(t,u) = u - tAu$. Next, we prove

$$H(t,u) = u - tAu \neq \theta, \quad \text{for all } t \in [0,1], \quad u \in \partial T_{R_0} \cap P. \tag{3.8}$$

Assume on the contrary that there exists a $u_0 \in \partial T_{R_0} \cap P$ with $\|u_0\| = R_0$ and $t_0 \in [0,1]$ such that $u_0 - t_0Au_0 = \theta$. For $0 < \alpha\beta < 1$, by (3.7), we have

$$u_0 \leq t_0Cu_0 + t_0w_0 \leq Cu_0 + w_0, \tag{3.9}$$

since P is the normal cone with normal constant 1. By (3.4) and (3.9), we have

$$\|u_0\| \leq \|Cu_0\| + \|w_0\| \leq \delta\|u_0\| + \|w_0\|. \tag{3.10}$$

By (3.10), we have $\|u_0\| \leq \|w_0\|/(1-\delta)$, which contradicts $\|u_0\| = R_0 > \|w_0\|/(1-\delta)$. By Lemma 2.4, for sufficiently large $R > R_0$, we have $i(A, P \cap T_R, P) = 1$. When $\alpha\beta = 1$, since P is a normal cone with the normal constant 1, by (3.6) and (3.7), we have

$$\|Au\| \leq \|Cu\| + \|w_0\| \leq \|u\|.$$

For sufficiently large $R \geq \|u\| \geq \|Au\|$, by Lemma 2.5, we have $i(A, P \cap T_R, P) = 1$. The proof has been completed. □

Corollary 3.2. *Let P be a normal cone of E, the normal constant of P equal to* 1 *and $A : P \to P$ a completely continuous operator. If there exists a positive α homogeneous increasing operator $B : P \to P$ with $0 < \alpha \leq 1$ and there exists $u_0 \in P$ such that*:

(i) *$Ax \leq Bx + u_0$, for all $x \in P$;*
(ii) *for all $x \in P\backslash\{\theta\}$,*

$$\sup_{\|x\|=1} \|Bx\| < \begin{cases} \infty, & 0 < \alpha\beta < 1, \\ 1, & \alpha\beta = 1, \end{cases} \tag{3.11}$$

then $i(A, P\cap T_R, P) = 1$, *where* $T_R = \{x \in E : \|x\| \le R\}$ *for sufficiently large* R.

Proof. In the proof of Theorem 3.1, set $C = B$, $w_0 = u_0$, which completes the proof. $\square$

Theorem 3.3. *Assume that condition* (H) *holds and the operator* A *has no fixed point on* $P\cap\partial T_r$ *for* $r > 0$, *where* $T_r = \{x \in E : \|x\| \le r\}$. *Suppose that* $G_1 : P \to P$ *is a positive* α *homogeneous increasing operator and* $G_2 : P \to P$ *is a positive* β *homogeneous increasing operator such that:*

(i) $F_i x \ge G_i x$, $i = 1, 2$, *for all* $x \in P \cap \overline{T}_r$;
(ii) *for* $\alpha\beta \in (0, 1)$, *there exist* $u^* \in P\backslash\{\theta\}$ *and a real number* $\delta > 0$ *such that* $Cu^* \ge \delta u^*$, *where* $C = K_1G_1K_2G_2$,

then $i(A, P \cap T_r, P) = 0$ *for sufficiently small* r.

Remark. Let $T_r = \{x \in E : \|x\| \le r\}$ be a closed ball of center θ and radius r. For $0 < r \le r_0$, Theorem 3.3 holds, where r_0 is introduced in the following proof.

Proof. We divide the proof into two steps.

Step 1: It follows from (3.2) that C is a positive $\alpha\beta$ homogeneous operator. By conditions (i) and (ii), we have

$$Ax = K_1F_1K_2F_2x \ge K_1F_1K_2G_2x \ge K_1G_1K_2G_2x = Cx. \tag{3.12}$$

Step 2: By condition (ii), when $\alpha\beta < 1$, $Cu^* \ge \delta u^*$. When $0 < \delta < 1$, $\delta^{1/(1-\alpha\beta)} < 1$ is obvious. Let $0 < t < \delta^{1/(1-\alpha\beta)}$, we have $Ctu^* = t^{\alpha\beta}Cu^* \ge t^{\alpha\beta}\delta u^* \ge tu^*$. Setting $tu^* = v^*$, we have $Cv^* \ge v^*$. When $\delta \ge 1$, we have $Cu^* \ge \delta u^* \ge u^*$. Setting $u^* = v^*$, we have $Cv^* \ge v^*$. Since $u^* \in P\backslash\{\theta\}$, when $\delta > 0$, there exists a $v^* \in P\backslash\{\theta\}$ such that $Cv^* \ge v^*$. Letting $D = \{u \in E : u \ge v^*\}$, we have $d = d(\theta, D) > 0$. When $0 < r_0 < d$, for $x \in P$, $\|x\| \le r_0$, we have that $x \notin D$ and $x \ngeq v^*$. Let $T_{r_0} = \{x \in E : \|x\| \le r_0\}$. Next, we prove

$$x - Ax \ne \lambda v^*, \quad \text{for all } x \in P \cap \partial T_{r_0},\ \lambda \ge 0. \tag{3.13}$$

We assume on the contrary that there exists $x_0 \in P \cap \partial T_{r_0}$ with $\lambda_0 > 0$ (since A has no fixed point on $P \cap \partial T_{r_0}$), such that

$$x_0 - Ax_0 = \lambda_0 v^*. \tag{3.14}$$

By (3.14), we have $x_0 = Ax_0 + \lambda_0 v^* \geq \lambda_0 v^*$. Setting $\lambda^* = \sup\{\lambda : x_0 \geq \lambda v^*\}$, we have that $\lambda^* \geq \lambda_0 > 0$ and $x_0 \geq \lambda^* v^*$. Since $x_0 \ngeq v^*$, we have $0 < \lambda^* < 1$. By (3.12) and (3.14), we have $x_0 \geq Ax_0 \geq Cx_0$. C is an increasing operator, therefore, $Cx_0 \geq C\lambda^* v^*$. Since C is a positive $\alpha\beta$ homogeneous operator and $0 < \alpha\beta < 1$, we have

$$\begin{aligned} x_0 &= Ax_0 + \lambda_0 v^* \geq Cx_0 + \lambda_0 v^* \\ &\geq C\lambda^* v^* + \lambda_0 v^* = (\lambda^*)^{\alpha\beta} Cv^* + \lambda_0 v^* \\ &\geq \lambda^* Cv^* + \lambda_0 v^* \geq \lambda^* v^* + \lambda_0 v^* \\ &= (\lambda^* + \lambda_0) v^*. \end{aligned} \tag{3.15}$$

This contradicts the above definition of λ^*. For sufficiently small r such that $0 < r < r_0$, we have $i(A, P \cap T_r, P) = 0$. □

Theorem 3.4. *In Theorem* 3.3, *condition* (ii) *is substituted for the following condition*:

(ii$'$) *for $\alpha\beta = 1$, there exists $u^* \in P \backslash \{\theta\}$ such that $Cu^* \geq u^*$, where $C = K_1 G_1 K_2 G_2$.*

Then the conclusion of Theorem 3.3 *still holds.*

Proof. It follows from the proof of Theorem 3.3 that $Ax \geq Cx$. By condition (ii$'$), when $\alpha\beta = 1$, C is a positive homogeneous operator with $Cu^* \geq u^*$. Let $T_r = \{x \in E : \|x\| \leq r\}$. Next, we prove

$$x - Ax \neq \lambda u^*, \quad x \in P \cap \partial T_r,\ \lambda \geq 0. \tag{3.16}$$

By the same argument as Step 2 of the proof of Theorem 3.3, we have $i(A, P \cap T_r, P) = 0$. □

Corollary 3.5. *Let Ω be a bounded open set in E with $\theta \in \Omega$ and $A : P \cap \overline{\Omega} \to P$ a completely continuous operator. If there exist another Banach space E_1, a cone P_1 in E_1, a positive α homogeneous operator $B : P \to P$ with $0 < \alpha < 1$, a linear operator $N : P \to P_1$, an increasing operator $L : P_1 \to P_1$ and a real number $\delta > 0$ such that*:

(i) *$NAx \geq NBx$, for all $x \in P \cap \overline{\Omega}$;*
(ii) *there exist $u^* \in P\backslash\{\theta\}$ and a non-negative integer n such that $NB^n u^* \geq \delta N u^*, Nu^* \neq \theta$;*
(iii) *$NBx \equiv LNx$, for all $x \in P$;*
(iv) *A has no fixed point on $P \cap \partial T_r$.*

Then $i(A, P \cap T_r, P) = 0$, where $T_r = \{x \in E : \|x\| \leq r\}$ for sufficiently small r.

Remark. Let $T_r = \{x \in E : \|x\| \leq r\}$ be a closed ball of center θ and radius r. For sufficiently small $r > 0$ and $r < r_0$, Theorem 3.3 holds, where r_0 is introduced in the following proof.

Proof. Since B is a positive α $(0 < \alpha < 1)$ homogeneous operator, it follows from $0 < \delta < 1$ that $\delta^{1/(1-\alpha^n)} < 1$. Setting $0 < t < \delta^{1/(1-\alpha^n)}$, by condition (ii), we have

$$\begin{aligned} NB^n tu^* &= NB^{n-1} t^\alpha B u^* \\ &= NB^{n-2} t^{\alpha^2} B^2 u^* = N t^{\alpha^n} B^n u^* \\ &= t^{\alpha^n} NB^n u^* \geq t^{\alpha^n} N\delta u^* \\ &\geq tNu^* = Ntu^*. \end{aligned} \tag{3.17}$$

Set $tu^* = v^*$. By (3.17), we have $NB^n v^* \geq Nv^*$. When $\delta \geq 1$, $NB^n u^* \geq N\delta u^* \geq Nu^*$. Set $u^* = v^*$, we have $NB^n v^* \geq Nv^*$. Since $u^* \in P\backslash\{\theta\}$, when $\delta > 0$, there exists $v^* \in P\backslash\{\theta\}$, such that $NB^n v^* \geq Nv^*$. Let $D = \{u \in E : u \geq v^*\}$ and $d = d(\theta, D) > 0$. When $0 < r_0 < d$, $x \in P$, $\|x\| \leq r_0$, we have that $x \notin D$; hence, $x \ngeq v^*$.

Next, we will prove

$$x - Ax \neq \lambda v^*, \quad \text{for all } x \in P \cap \partial T_{r_0},\ \lambda \geq 0. \tag{3.18}$$

We assume on the contrary that there exist $x_0 \in P \cap \partial T_{r_0}$ and $\lambda_0 > 0$ (since A has no fixed point on $P \cap \partial T_{r_0}$) such that

$$x_0 - Ax_0 = \lambda_0 v^*. \tag{3.19}$$

By (3.19), we have $Nx_0 = NAx_0 + \lambda_0 Nv^* \geq \lambda_0 Nv^*$. Letting $\lambda^* = \sup\{\lambda : Nx_0 \geq \lambda Nv^*\}$, it is evident that $\lambda^* \geq \lambda_0 > 0$ and $Nx_0 \geq \lambda^* Nv^*$. Since $x_0 \ngeq v^*$ and $Nx_0 \ngeq Nv^*$, we have $0 < \lambda^* < 1$.

By condition (i) and (3.19), we have $Nx_0 \geq NAx_0 \geq NBx_0$. By condition (iii), we have

$$NBx_0 = LNx_0 \geq LNBx_0 \geq NB^2x_0. \tag{3.20}$$

For any non-negative integer i, by (3.20) we have $NB^ix_0 \geq NB^{i+1}x_0$. Therefore, $NBx_0 \geq NB^nx_0$. By condition (iii), for $x \in P$, we have $NB^nx = L^nNx$. Since B is a positive α homogeneous operator with $0 < \alpha < 1$, we have $0 < \alpha^n < 1$ and $NB^nv^* \geq Nv^*$. Hence, we have

$$\begin{aligned} Nx_0 &= NAx_0 + \lambda_0 Nv^* \geq NBx_0 + \lambda_0 Nv^* \\ &\geq NB^nx_0 + \lambda_0 Nv^* = L^nNx_0 + \lambda_0 Nv^* \\ &\geq L^nN\lambda^*v^* + \lambda_0 Nv^* = NB^n\lambda^* + \lambda_0 Nv^* \\ &= (\lambda^*)^{\alpha^n} NB^nv^* + \lambda_0 Nv^* \geq \lambda^* NB^nv^* + \lambda_0 Nv^* \\ &\geq \lambda^* Nv^* + \lambda_0 Nv^* = (\lambda^* + \lambda_0)Nv^*. \end{aligned} \tag{3.21}$$

This contradicts the above definition of λ^*. For sufficiently small $r \leq r_0$, by Lemma 2.6, we have $i(A, P \cap T_r, P) = 0$. $\square$

Corollary 3.6. *Let Ω be a bounded open set in E with $\theta \in \Omega$ and $A : P \cap \overline{\Omega} \to P$ a completely continuous operator. If there exist a positive α homogeneous increasing operator $B : P \to P$ with $0 < \alpha < 1$, $u^* \in P \backslash \{\theta\}$ and real number $\delta > 0$ such that:*

(i) *$Ax \geq Bx$, for all $x \in P \cap \overline{\Omega}$;*
(ii) *$Bu^* \geq \delta u^*$;*
(iii) *A has no fixed points on $P \cap \partial T_r$, where $T_r = \{x \in E : \|x\| \leq r\}$.*

Then $i(A, P \cap T_r, P) = 0$ for sufficiently small r.

Proof. In the proof of Theorem 3.3, let $C = B$, which completes the proof. $\square$

4. The application for the nonlinear double integral equation of Hammerstein type. Next, we use the main results of Section 3 to study the nonlinear double integral equation of Hammerstein type

$$u(x) = \int_0^1 k_1(x,y) f_1\left(y, \int_0^1 k_2(y,z) f_2(z, u(z))\, dz\right) dy = Au(x). \tag{4.1}$$

Let $E = C[0,1]$, $f_i(x,u) : [0,1] \times [0,+\infty) \to [0,+\infty), k_i(x,y) : [0,1] \times [0,1] \to [0,+\infty)$, $i = 1,2$. $P = \{u \in C[0,1] : u(x) \geq 0$, for all $x \in [0,1]\}$. Next we investigate the existence of the solution to the nonlinear integral equation (4.1). For convenience, we make the following assumptions:

(H_1) There exist $\alpha \in (0,1]$ and $\beta \in (0,\infty)$ satisfying $\alpha\beta \in (0,1)$, $a_1(x) > 0$, $a_2(x) > 0$, $b_1(x) \geq 0$, $b_2(x) \geq 0$ and $a_1(x)$, $a_2(x)$, $b_1(x)$ and $b_2(x) \in C[0,1]$ such that

$$\begin{aligned} f_1(x,u) &\leq a_1(x)u^\alpha + b_1(x), \quad \text{for all } x \in [0,1],\ u \geq 0, \\ f_2(x,u) &\leq a_2(x)u^\beta + b_2(x), \quad \text{for all } x \in [0,1],\ u \geq 0. \end{aligned}$$

We define the Nemytskii operators $G_1, G_2 : P \to P$, which are determined by u^α and u^β respectively, and

$$\begin{aligned} G_1 u^\alpha(x) &= g_1(x, u^\alpha(x)), \quad x \in [0,1], \\ G_2 u^\beta(x) &= g_2(x, u^\beta(x)), \quad x \in [0,1], \end{aligned}$$

where $g_1, g_2 : [0,1] \times [0,+\infty) \to [0,+\infty)$.

(H_2) There exist $\alpha_1 \in (0,+\infty)$, $d_1(x)$, $d_2(x) \in C[0,1]$ and $d_1(x) \geq 0$, $d_2(x) \geq 0$ such that

$$\begin{aligned} f_1(x,u) &\geq d_1(x)u^{\alpha_1}, \qquad \text{for all } x \in [0,1],\ 0 \leq u \leq s, \\ f_2(x,u) &\geq d_2(x)u^{1/\alpha_1}, \quad \text{for all } x \in [0,1],\ 0 \leq u \leq r, \end{aligned}$$

where r and s are sufficiently small positive constants with $s \leq r$ as long as the operator A has no fixed point on $P \cap \partial T_r$ for $r > 0$, where $T_r = \{x \in E : \|x\| \leq r\}$.

(H_3) There exist $\alpha_2 \in (0,+\infty)$, $\alpha_2\beta_2 \in (0,1)$, $m_1(x)$, $m_2(x) \in C[0,1]$ and continuous functions $m_1(x) \geq 0$ and $m_2(x) \geq 0$ such that

$$\begin{aligned} f_1(x,u) &\geq m_1(x)u^{\alpha_2}, \quad \text{for all } x \in [0,1],\ 0 \leq u \leq s, \\ f_2(x,u) &\geq m_2(x)u^{\beta_2}, \quad \text{for all } x \in [0,1],\ 0 \leq u \leq r, \end{aligned}$$

where r and s are sufficiently small positive constants, and $0 < s \leq r < d = d(\theta, D)$. Here $D = \{u \in E : u \geq v^* = v^*(\alpha_2, \beta_2, \delta, u^*), v^* \in P \backslash \{\theta\}\}$ and δ, u^* are defined in the following condition (H_5).

(H_4) There exists $u^* \in P\backslash\{\theta\}$ such that $B_1 u^* \geq u^*$, where B_1 is a homogeneous operator

$$B_1 u(x) = \int_0^1 k_1(x,y) d_1(y) \biggl(\int_0^1 k_2(y,z) d_2(z) u^{1/\alpha_1}(z)\, dz \biggr)^{\alpha_1} dy.$$

(H_5) There exists $u^* \in P\backslash\{\theta\}$ and real number $\delta > 0$ such that $B_2 u^* \geq \delta u^*$, where B_2 is a $\alpha_2\beta_2$ homogeneous operator

$$B_2 u(x) = \int_0^1 k_1(x,y) m_1(y) \biggl(\int_0^1 k_2(y,z) m_2(z) u^{\beta_2}(z) dz \biggr)^{\alpha_2} dy.$$

Theorem 4.1. *Suppose that* $(H_1), (H_2)$ *and* (H_4) *are satisfied. Then equation* (4.1) *has at least one positive solution.*

Proof. It is evident that $A : P \to P$ is completely continuous. We divide the proof into the following two steps.

Step 1: By (H_1), Lemma 2.3 (1) and Lemma 2.7, we have

$$\begin{aligned} Au(x) &= \int_0^1 k_1(x,y) f_1 \biggl(y, \int_0^1 k_2(y,z) f_2(z,u(z))\, dz \biggr) dy \\ &\leq \int_0^1 k_1(x,y) \biggl(a_1(y) (\int_0^1 k_2(y,z) f_2(z,u(z)) dz)^{\alpha} + b_1(y) \biggr) dy \\ &= \int_0^1 k_1(x,y) a_1(y) \biggl(\int_0^1 k_2(y,z) f_2(z,u(z))\, dz \biggr)^{\alpha} dy \\ &\qquad + \int_0^1 k_1(x,y) b_1(y))\, dy \qquad (4.2) \\ &\leq \int_0^1 k_1(x,y) a_1(y) \biggl(\int_0^1 k_2(y,z) (a_2(z) u^{\beta}(z) + b_2(z))\, dz \biggr)^{\alpha} dy \\ &\qquad + \int_0^1 k_1(x,y) b_1(y)\, dy \\ &= \int_0^1 k_1(x,y) a_1(y) \biggl(\int_0^1 k_2(y,z) (a_2(z) u^{\beta}(z) + b_2(z))\, dz \biggr)^{\alpha} dy \\ &\qquad + \int_0^1 k_1(x,y) b_1(y)\, dy \\ &\leq \int_0^1 k_1(x,y) a_1(y) \biggl(\int_0^1 k_2(y,z) a_2(z) u^{\beta}(z)\, dz \biggr)^{\alpha} dy \end{aligned}$$

$$
\begin{aligned}
&+ \int_0^1 k_1(x,y)a_1(y)\Big(\int_0^1 k_2(y,z)b_2(z)\,dz\Big)^\alpha dy \\
&+ \int_0^1 k_1(x,y)b_1(y)\,dy \\
&= Cu(x) + u_0,
\end{aligned}
$$

where

$$u_0 = \int_0^1 k_1(x,y)a_1(y)\Big(\int_0^1 k_2(y,z)b_2(z)\,dz\Big)^\alpha dy + \int_0^1 k_1(x,y)b_1(y)\,dy,$$

C is an $\alpha\beta$ homogeneous operator with the following form:

$$Cu(x) = \int_0^1 k_1(x,y)a_1(y)\Big(\int_0^1 k_2(y,z)a_2(z)u^\beta(z)\,dz\Big)^\alpha dy, \tag{4.3}$$

and

(4.4)

$$
\begin{aligned}
G_1(K_2G_2u(y) + K_2w_2) &= a_1(y)\Big(\int_0^1 k_2(y,z)(a_2(z)u^\beta(z) + b_2(z))\,dz\Big)^\alpha \\
&\le a_1(y)\Big(\int_0^1 k_2(y,z)a_2(z)u^\beta(z)\,dz\Big)^\alpha \\
&\quad + a_1(y)\Big(\int_0^1 k_2(y,z)b_2(z)\,dz\Big)^\alpha \\
&= G_1K_2G_2u(y) + G_1K_2w_2.
\end{aligned}
$$

By (4.3), we have

(4.5)

$$
\begin{aligned}
\|Cu\| &= \max_{x\in[0,1]} |Cu(x)| \\
&= \max_{x\in[0,1]} \Big|\int_0^1 k_1(x,y)a_1(y)\Big(\int_0^1 k_2(y,z)a_2(z)u^\beta(z)\,dz\Big)^\alpha dy\Big|.
\end{aligned}
$$

It is evident that $\sup_{\|u\|=1}\|Cu\| < +\infty$. It follows from (4.2), (4.4) and (4.5) that conditions (i), (ii) and (iii) of Theorem 3.1 are satisfied. Therefore, $i(A, P\cap T_R, P) = 1$.

Step 2: By (H_2), there exists $r > 0$, for $0 \le u \le r$ and every $y \in [0,1]$ such that $\int_0^1 k_2(y,z)f_2(z,u(z))\,dz \le s$ and the operator A has no fixed point on $P \cap \partial T_r$, we have

$$
\begin{aligned}
Au(x) &= \int_0^1 k_1(x,y) f_1\Big(y, \int_0^1 k_2(y,z) f_2(z,u(z))\,dz\Big)\,dy \\
&\geq \int_0^1 k_1(x,y)\, d_1(y) \Big(\int_0^1 k_2(y,z) f_2(z,u(z))\,dz\Big)^{\alpha_1} dy \\
&\geq \int_0^1 k_1(x,y)\, d_1(y) \Big(\int_0^1 k_2(y,z) d_2(z) u^{1/\alpha_1}(z)\,dz\Big)^{\alpha_1} dy \\
&= B_1 u(x),
\end{aligned} \tag{4.6}
$$

where

$$
B_1 u(x) = \int_0^1 k_1(x,y)\, d_1(y) \Big(\int_0^1 k_2(y,z) d_2(z) u^{1/\alpha_1}(z)\,dz\Big)^{\alpha_1} dy.
$$

It is evident that B_1 is a homogeneous operator. By (4.6) and condition (H_4), the conditions of Theorem 3.4 are satisfied. Therefore, $i(A, P \cap T_r, P) = 0$. Applying the additivity of the fixed point index, we have

$$
i(A, (T_R \backslash \overline{T}_r) \cap P, P) = i(A, T_R \cap P, P) - i(A, \overline{T}_r \cap P, P) = 1.
$$

Hence, equation (4.1) has at least one positive solution. $\square$

Theorem 4.2. *Suppose that* (H_1), (H_3) *and* (H_5) *are satisfied. Then equation* (4.1) *has at least one positive solution.*

Proof. We divide the proof into the following two steps.

Step 1: Step 1 of the proof of Theorem 4.1 still holds.

Step 2: By (H_3) and (H_5), there exist $r > 0$ and $v^* \in P \backslash \{\theta\}$ for $0 \leq u \leq r < d = d(\theta, D)$ and every $y \in [0,1]$ such that $\int_0^1 k_2(y,z) f_2(z,u(z))\,dz \leq s$. We have

$$
\begin{aligned}
Au(x) &= \int_0^1 k_1(x,y) f_1\Big(y, \int_0^1 k_2(y,z) f_2(z,u(z))\,dz\Big)\,dy \\
&\geq \int_0^1 k_1(x,y) m_1(y) \Big(\int_0^1 k_2(y,z) f_2(z,u(z))\,dz\Big)^{\alpha_2} dy \\
&\geq \int_0^1 k_1(x,y) m_1(y) \Big(\int_0^1 k_2(y,z) m_2(z) u^{\beta_2}(z)\,dz\Big)^{\alpha_2} dy \\
&= B_2 u(x).
\end{aligned} \tag{4.7}
$$

$$B_2u(x) = \int_0^1 k_1(x,y)m_1(y)\bigg(\int_0^1 k_2(y,z)m_2u^{\beta_2}(z)\,dz\bigg)^{\alpha_2} dy.$$

B_2 is an $\alpha_2\beta_2$ homogeneous operator. By (4.7) and (H_5), the conditions of Theorem 3.3 are satisfied. Letting $T_r = \{x \in E : \|x\| \le r\}$, we have $i(A, P \cap T_r, P) = 0$. Applying the additivity of the fixed point index, we have

$$i(A, (T_R \backslash \overline{T}_r) \cap P, P) = i(A, T_R \cap P, P) - i(A, \overline{T}_r \cap P, P) = 1.$$

Hence, equation (4.1) has at least one positive solution. $\square$

5. The application for the coupled system of elliptic partial differential equations. We study the following system of elliptic partial differential equations

$$\begin{cases} Lu = f(x,v) & \text{in } \Omega, \\ Lv = g(x,u) & \text{in } \Omega, \\ Bu = 0, Bv = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where Ω is a bounded convex open domain in R^n and $\partial\Omega \in C^{2+\mu}$ with $0 < \mu < 1$, and L is a uniformly elliptic operator in Ω and defined as follows:

$$Lu = -\sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial u}{\partial x_i} + c(x)u,$$

where $a_{ij}(x) = a_{ji}(x)$, $a_{ij}(x)$, $b_i(x)$, $c(x) \in C^\mu(\overline{\Omega})$ and $c(x) \ge 0$. There exists a constant $\lambda > 0$ such that

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \ge \lambda|\xi|^2, \quad \text{for all } x \in \overline{\Omega},$$

where $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in R^n$. B is the boundary operator and has the following form:

$$Bu = b(x)u + d\frac{\partial u}{\partial \nu},$$

where $b(x) \in C^{1+\mu}(\partial\Omega)$ and $\nu \in C^{1+\mu}$ is an outward vector of $\partial\Omega$, and suppose that one of the following occurs:

(i) $d = 0$ and $b(x) \equiv 1$;

(ii) $d = 1$ and $b(x) \equiv 0$;

(iii) $d = 1$ and $b(x) > 0$.

Let $f(x,v) : \overline{\Omega}\times R^+ \to R^+$ and $g(x,u) : \overline{\Omega}\times R^+ \to R^+$ be continuous, $f(x,0) \equiv 0$, $g(x,0) \equiv 0$. We suppose that

$$(A_1) \qquad \lim_{v\to+\infty} \sup_{x\in\Omega} \frac{f(x,v)}{v^\alpha} = 0, \qquad \lim_{u\to+\infty} \sup_{x\in\Omega} \frac{g(x,u)}{u^\beta} = 0,$$

$$(A_2) \qquad \lim_{v\to 0^+} \inf_{x\in\Omega} \frac{f(x,v)}{v^{\alpha_2}} = \infty, \qquad \lim_{u\to 0^+} \inf_{x\in\Omega} \frac{g(x,u)}{u^{\beta_2}} = \infty,$$

where α and β are defined by (H_1), α_2 and β_2 are defined by (H_3) in Section 4.

Remark 5.1. In order to study the existence of solutions of (5.1), we consider the boundary value problem of the linear elliptic partial differential equation

$$\left\{ \begin{array}{ll} Lu(x) = v(x) & x \in \Omega, \\ Bu(x) = 0 & x \in \partial\Omega. \end{array} \right. \tag{5.2}$$

For every $v \in C^\mu(\overline{\Omega})$, we denote by Kv the unique solution of the linear boundary value problem (5.2). Let $(Kv)(x) = u_v(x)$, $x \in \Omega$, where $K : C^\mu(\overline{\Omega}) \to C^{2+\mu}(\overline{\Omega})$ is a linear completely continuous operator, for details see **[1]**. For the nonlinear elliptic boundary value problem (5.1), we define the Nemytskii operators

$$(F_1 v)(x) = f(x, v(x)), \qquad (F_2 u)(x) = g(x, u(x)), \quad x \in \overline{\Omega}.$$

Remark 5.2. In Section 4, let $E = C(\overline{\Omega})$ and $P = \{u \in C(\Omega) : u(x) \geq 0,\ \text{for all } x \in \overline{\Omega}\}$; Theorem 4.1 and Theorem 4.2 still hold.

Theorem 5.3. *Suppose that the conditions* (A_1) *and* (A_2) *are satisfied. Then* (5.1) *has at least one positive solution* $(u, v) \in C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$.

Proof. It follows from Remark 5.1 that $(u, v) \in C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$ is the solution of boundary value problem (5.1) if and only if $(u, v) \in C(\overline{\Omega}) \times C(\overline{\Omega})$ is the solution of the system of integral equations

$$\begin{cases} u(x) = \displaystyle\int_\Omega k_1(x,y) f(y, v(y))\, dy, \\ v(x) = \displaystyle\int_\Omega k_2(x,y) g(y, u(y))\, dy, \end{cases} \tag{5.3}$$

where $k_i(x,y) : \Omega\times\Omega \to R^+$, $i = 1, 2$, are Green functions. It is evident that the nonlinear integral equations system (5.3) is equivalent to the following nonlinear double integral equation of Hammerstein type:

$$u(x) = \int_\Omega k_1(x,y) f\left(y, \int_\Omega k_2(y,z) g(z,u(z))\, dz\right) dy. \tag{5.4}$$

Let

$$Au(x) = \int_\Omega k_1(x,y) f\left(y, \int_\Omega k_2(y,z) g(z,u(z))\, dz\right) dy. \tag{5.5}$$

By Remark 5.1, we have $A = K_1F_1K_2F_2$, and the maximum principle (e.g., [**16**]) implies that K_1 and K_2 are positive linear operators. It is easily seen that $A : P \to P$ is completely continuous from (5.5) and Remark 5.1. Next we show that the operator A has at least one fixed point. By condition (A_1), there exist $\alpha, \beta \in (0,1)$ and $a_1(x) \geq 0$, $a_2(x) \geq 0$, $b_1(x) \geq 0$, $b_2(x) \geq 0$ such that:

$$\begin{aligned} f(x,v) &\leq a_1(x)v^\alpha + b_1(x), \quad \text{for all } x \in \Omega,\ v \geq 0,\\ g(x,u) &\leq a_2(x)u^\beta + b_2(x), \quad \text{for all } x \in \Omega,\ u \geq 0. \end{aligned}$$

By Step 1 of the proof of Theorem 4.1 and Remark 5.2, we have that $i(A, P \cap T_R, P) = 1$.

It follows from (A_2) that there exist $\alpha_2 \in (0, +\infty)$, $\alpha_2\beta_2 \in (0,1)$, $m_1(x) \geq 0$, $m_2(x) \geq 0$ and sufficiently small $r > 0$ and $s \in (0,r)$ such that

$$\begin{aligned} f(x,v) &\geq m_1(x)v^{\alpha_2}, \quad \text{for all } x \in \Omega,\ 0 \leq v \leq s,\\ g(x,u) &\geq m_2(x)u^{\beta_2}, \quad \text{for all } x \in \Omega,\ 0 \leq u \leq r. \end{aligned}$$

Let $T_r = \{u \in E \mid \|u\| \leq r\}$. Since $g(z,0) \equiv 0$ and the continuity of $g(z,u)$, we have

$$\int_\Omega k_2(y,z) g(z,u(z))\, dz \leq s, \quad \text{for all } 0 \leq u \leq r,\ y \in \Omega.$$

Therefore, we have

$$\begin{aligned} Au(x) &= \int_\Omega k_1(x,y) f(y, \int_\Omega k_2(y,z) g(z,u(z))\,dz)\,dy \\ &\geq \int_\Omega k_1(x,y) m_1(y) \left(\int_\Omega k_2(y,z) g(z,u(z))\,dz \right)^{\alpha_2} dy \\ &\geq \int_\Omega k_1(x,y) m_1(y) \left(\int_\Omega k_2(y,z) m_2(z) u^{\beta_2}(z)\,dz \right)^{\alpha_2} dy \\ &= B_2 u(x), \end{aligned}$$

where

$$B_2 u(x) = \int_\Omega k_1(x,y) m_1(y) \left(\int_\Omega k_2(y,z) m_2(z) u^{\beta_2}(z)\,dz \right)^{\alpha_2} dy.$$

It is easily seen that $B_2 : P \to P$ is an $\alpha_2\beta_2$ homogeneous operator. From [**1,** Lemma 5.3], we have K_1 and K_2 are linear completely continuous operators and e-positive. By Definition 2.2, there exist $\alpha > 0$, $e \in P\backslash\{\theta\}$ such that

$$\begin{aligned} B_2 e &= \int_\Omega k_1(x,y) m_1(y) \left(\int_\Omega k_2(y,z) m_2(z) e^{\beta_2} dz \right)^{\alpha_2} dy \\ &= \int_\Omega k_1(x,y) v^* dy \\ &\geq \alpha e, \end{aligned} \tag{5.6}$$

where

$$v^* = m_1(y) \left(\int_\Omega k_2(y,z) m_2(z) e^{\beta_2}(z)\,dz \right)^{\alpha_2} \not\equiv 0.$$

For condition (H_5) of Theorem 4.2, let $\alpha = \delta$. By (5.6), we have $B_2 e \geq \delta e$. The conditions (H_1), (H_3) and (H_5) are satisfied. We have $i(A, P \cap T_r, P) = 0$. Therefore, equation (5.1) has at least one positive solution $(u,v) \in C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$. $\square$

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