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JORDAN DERIVATIONS OF INCIDENCE ALGEBRAS

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ABSTRACT. Let $\mathcal{R}$ be a commutative ring with identity and $I(X,\mathcal{R})$ the incidence algebra of a locally finite pre-ordered set X. In this note, we characterize the derivations of $I(X,\mathcal{R})$ and prove that every Jordan derivation of $I(X,\mathcal{R})$ is a derivation, provided that $\mathcal{R}$ is 2-torsion free.

1. Introduction. Let $\mathcal{R}$ be a commutative ring with identity, A an algebra over $\mathcal{R}$. An $\mathcal{R}$-linear mapping $D : A \to A$ is called a derivation if $D(xy) = D(x)y + xD(y)$ for all $x, y \in A$, and is called a *Jordan derivation* if

$$D(x^2) = D(x)x + xD(x)$$

for all $x \in A$. There has been a great interest in the study of Jordan derivations of various algebras in the last decades. The standard problem is to find out whether a Jordan derivation degenerate to a derivation. Jacobson and Rickart **[8]** proved that every Jordan derivation of the full matrix algebra over a 2-torsion free unital ring is a derivation by relating the problem to the decomposition of Jordan homomorphisms. In **[7]**, Herstein showed that every Jordan derivation from a 2-torsion free prime ring into itself is also a derivation. These results have been extended to different rings and algebras in various directions (see **[2, 3, 6, 11, 20]** and the references therein). We would like to refer the reader to Brešar's paper **[4]** for a comprehensive and more detailed understanding of this topic.

We now recall the definition of incidence algebras. Let $(X, \leqslant)$ be a locally finite pre-ordered set. This means $\leqslant$ is a reflexive and transitive binary relation on the set X, and, for any $x \leqslant y$ in X, there are only finitely many elements z satisfying $x \leqslant z \leqslant y$. The *incidence algebra*

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$I(X,\mathcal{R})$ of X over $\mathcal{R}$ is defined on the set

$$I(X,\mathcal{R}) := \{f : X \times X \longrightarrow \mathcal{R} \mid f(x,y) = 0 \text{ if } x \nleqslant y\}$$

with algebraic operation given by

$$\begin{aligned}(f+g)(x,y) &= f(x,y) + g(x,y),\\ (rf)(x,y) &= rf(x,y),\\ (fg)(x,y) &= \sum_{x\leqslant z\leqslant y} f(x,z)g(z,y)\end{aligned}$$

for all $f, g \in I(X,\mathcal{R})$, $r \in \mathcal{R}$ and $x, y, z \in X$. The product (fg) is usually called *convolution* in function theory. It would be helpful to point out that the full matrix algebra $\mathrm{M}_n(\mathcal{R})$ and the upper (or lower) triangular matrix algebras $\mathrm{T}_n(\mathcal{R})$ are special examples of incidence algebras

Incidence algebras were first considered by Ward [**18**] as generalized algebras of arithmetic functions. Rota and Stanley developed incidence algebras as the fundamental structures of enumerative combinatorial theory and allied areas of arithmetic function theory (see [**16**]). Motivated by the results of Stanley [**17**], automorphisms and other algebraic mappings of incidence algebras have been extensively studied (see [**1, 5, 9, 10, 12, 13, 14, 15**] and the references therein). Baclawski [**1**] studied the automorphisms and derivations of incidence algebras $I(X,\mathcal{R})$ when X is a locally finite partially ordered set. In particular, he proved that every derivation of $I(X,\mathcal{R})$ with X a locally finite partially ordered set can be decomposed as a sum of an inner derivation and a transitive induced derivation (see the definition in Section 2). Koppinen [**10**] has extended these results to the incidence algebras $I(X,\mathcal{R})$ with X a locally finite pre-ordered set. In the present paper, we first characterize the derivations of $I(X,\mathcal{R})$ by a direct computation. Based on such characterization of derivations, we prove that every Jordan derivation of $I(X,\mathcal{R})$ is a derivation provided that $\mathcal{R}$ is 2-torsion free.

2. Derivations on incidence algebras.

In this section, we characterize the derivations of the incidence algebras $I(X,\mathcal{R})$ where X is a locally finite pre-ordered set. We first single out some properties of $I(X,\mathcal{R})$ for later use. One can find more information in [**10, 16**].

The identity element δ of $I(X,\mathcal{R})$ is given by $\delta(x,y)=\delta_{xy}$ for $x\leqslant y$, where $\delta_{xy}\in\{0,1\}$ is the Kronecker delta. If $x,y\in X$ with $x\leqslant y$, let e_{xy} be defined by $e_{xy}(u,v)=1$ if $(u,v)=(x,y)$, and $e_{xy}(u,v)=0$ otherwise. Then $e_{xy}e_{uv}=\delta_{yu}e_{xv}$ by the definition of convolution. Moreover, the set $\mathfrak{B}:=\{e_{xy}\mid x\leqslant y\}$ forms an $\mathcal{R}$-linear basis of $I(X,\mathcal{R})$. Note that incidence algebras allow infinite summation, and hence, the $\mathcal{R}$-linear mapping here means a mapping preserving infinite sum and scalar multiplication. The following lemma, to some extent, is well-known, and we give the proof here for completeness.

Lemma 2.1. *Let A be an $\mathcal{R}$-algebra with an $\mathcal{R}$-linear basis Y. Then an $\mathcal{R}$-linear operator $D:A\to A$ is a derivation if and only if*

$$D(xy)=D(x)y+xD(y),\quad \textit{for all } x,y\in Y.$$

Proof. We only need to prove the sufficiency. Let $a,b\in A$ and

$$a=\sum_{x\in Y}C_x x,\quad b=\sum_{y\in Y}C'_y y,$$

where $C_x,C'_y\in\mathcal{R}$. Then

$$\begin{aligned}
D(ab)&=D\bigg(\sum_{x,y\in Y}C_xC'_yxy\bigg)\\
&=\sum_{x,y\in Y}C_xC'_y(D(x)y+xD(y))\\
&=D\bigg(\sum_{x\in Y}C_xx\bigg)\bigg(\sum_{y\in Y}C'_yy\bigg)+\bigg(\sum_{x\in Y}C_xx\bigg)D\bigg(\sum_{y\in Y}C'_yy\bigg)\\
&=D(a)b+aD(b).
\end{aligned}$$

□

Let $D:I(X,\mathcal{R})\to I(X,\mathcal{R})$ be an $\mathcal{R}$-linear operator. Motivated by the above lemma, we denote, for all $i,j\in X$ with $i\leqslant j$,

$$D(e_{ij})=\sum_{e_{xy}\in\mathfrak{B}}C^{ij}_{xy}e_{xy}.$$

We will use this notation for the rest of this paper. For any $x\in X$, define L_x and R_x to be the subsets of X by

$$L_x:=\{i\in X\mid i\leqslant x, i\neq x\}$$

and

$$R_x := \{j \in X \mid x \leqslant j, j \neq x\}.$$

Note that the set $L_x \cap R_x$ may not be empty since X is a pre-ordered set.

Theorem 2.2. *Let $D : I(X, \mathcal{R}) \to I(X, \mathcal{R})$ be an $\mathcal{R}$-linear operator. Then D is a derivation if and only if D satisfies*

$$D(e_{ij}) = \sum_{x \in L_i} C_{xi}^{ii} e_{xj} + C_{ij}^{ij} e_{ij} + \sum_{y \in R_j} C_{jy}^{jj} e_{iy} \tag{1}$$

for all $e_{ij} \in \mathfrak{B}$, where the coefficients C_{xy}^{ij} are subject to the following relations

$$\begin{cases} C_{jk}^{jj} + C_{jk}^{kk} = 0, & \text{if } j \leqslant k; \\ C_{ij}^{ij} + C_{jk}^{jk} = C_{ik}^{ik}, & \text{if } i \leqslant j,\ j \leqslant k. \end{cases}$$

Proof. ($\Rightarrow$). For any $i \in X$, there is $D(e_{ii}) = D(e_{ii})e_{ii} + e_{ii}D(e_{ii})$. Thus,

$$\begin{aligned} \sum_{e_{xy} \in \mathfrak{B}} C_{xy}^{ii} e_{xy} &= \Big(\sum_{e_{xy} \in \mathfrak{B}} C_{xy}^{ii} e_{xy}\Big) e_{ii} + e_{ii} \Big(\sum_{e_{xy} \in \mathfrak{B}} C_{xy}^{ii} e_{xy}\Big) \\ &= \sum_{x, x \leqslant i} C_{xi}^{ii} e_{xi} + \sum_{y, i \leqslant y} C_{iy}^{ii} e_{iy}. \end{aligned} \tag{2}$$

Taking $x = i$ and $y = i$ in (2), we have $C_{ii}^{ii} = 0$. Hence,

$$D(e_{ii}) = \sum_{x \in L_i} C_{xi}^{ii} e_{xi} + \sum_{y \in R_i} C_{iy}^{ii} e_{iy}. \tag{3}$$

For any $e_{ij} \in \mathfrak{B}$ with $i \neq j$, we deduce from equation (3) that

$$\begin{aligned} D(e_{ij}) &= D(e_{ii} e_{ij} e_{jj}) \\ &= D(e_{ii}) e_{ij} + e_{ii} D(e_{ij}) e_{jj} + e_{ij} D(e_{jj}) \\ &= D(e_{ii}) e_{ij} + C_{ij}^{ij} e_{ij} + e_{ij} D(e_{jj}) \\ &= \sum_{x \in L_i} C_{xi}^{ii} e_{xj} + C_{ij}^{ij} e_{ij} + \sum_{y \in R_j} C_{jy}^{jj} e_{iy}. \end{aligned} \tag{4}$$

Note that the identities (3) and (4) proves (1).

In order to determine the coefficients C_{xy}^{ij}, we need to apply the derivation D to the identity $e_{ij}e_{kl} = \delta_{jk}e_{il}$. There are two subcases.

Case 1. When $j \neq k$, then

$$\begin{aligned}
0 &= D(e_{ij})e_{kl} + e_{ij}D(e_{kl}) \\
&= \Bigg(\sum_{x\in L_i} C_{xi}^{ii}e_{xj} + C_{ij}^{ij}e_{ij} + \sum_{y\in R_j} C_{jy}^{jj}e_{iy}\Bigg)e_{kl} \\
&\quad + e_{ij}\Bigg(\sum_{x\in L_k} C_{xk}^{kk}e_{xl} + C_{kl}^{kl}e_{kl} + \sum_{y\in R_l} C_{ly}^{ll}e_{ky}\Bigg) \\
&= C_{jk}^{jj}e_{il} + C_{jk}^{kk}e_{il}.
\end{aligned}$$

Therefore, $C_{jk}^{jj} + C_{jk}^{kk} = 0$.

Case 2. When $j = k$, then

$$\begin{aligned}
D(e_{il}) &= D(e_{ij})e_{jl} + e_{ij}D(e_{jl}) \\
&= \Bigg(\sum_{x\in L_i} C_{xi}^{ii}e_{xj} + C_{ij}^{ij}e_{ij} + \sum_{y\in R_j} C_{jy}^{jj}e_{iy}\Bigg)e_{jl} \qquad (5)\\
&\quad + e_{ij}\Bigg(\sum_{x\in L_j} C_{xj}^{jj}e_{xl} + C_{jl}^{jl}e_{jl} + \sum_{y\in R_l} C_{ly}^{ll}e_{jy}\Bigg) \\
&= \sum_{x\in L_i} C_{xi}^{ii}e_{xl} + C_{ij}^{ij}e_{il} + C_{jl}^{jl}e_{il} + \sum_{y\in R_l} C_{ly}^{ll}e_{iy}.
\end{aligned}$$

Taking (1) into account, we obtain $C_{ij}^{ij} + C_{jl}^{jl} = C_{il}^{il}$, which proves the necessity.

($\Leftarrow$). First it follows from the equation $C_{ij}^{ij} + C_{jk}^{jk} = C_{ik}^{ik}$ that $C_{ii}^{ii} = 0$ for all $i \in X$. Hence, by (1), $D(e_{ii})$ is of the form as in equation (3), i.e.,

$$D(e_{ii}) = \sum_{x\in L_i} C_{xi}^{ii}e_{xi} + \sum_{y\in R_i} C_{iy}^{ii}e_{iy}. \qquad (6)$$

Now by Lemma 2.1, to show that D is a derivation, we just need to show that $D(e_{ij}e_{kl}) = D(e_{ij})e_{kl} + e_{ij}D(e_{kl})$ for all $e_{ij}, e_{kl} \in \mathfrak{B}$. On the other hand, a little modification of identities (5) and (6) shows that the equations $D(e_{ij}e_{kl}) = D(e_{ij})e_{kl} + e_{ij}D(e_{kl})$ for all $e_{ij}, e_{kl} \in \mathfrak{B}$ always hold. This completes the proof of the theorem. $\square$

For any $g \in I(X,\mathcal{R})$, the mapping $f \mapsto [g,f]$ is a derivation of $I(X,\mathcal{R})$. We call it an *inner derivation* and denote it as Inn_g. A mapping $f : \leqslant \to \mathcal{R}$ is called *transitive* if

$$f(i,j) + f(j,k) = f(i,k)$$

for all $i, j, k \in X$ such that $i \leqslant j$, $j \leqslant k$. Note that every mapping $\sigma : X \to \mathcal{R}$ determines a transitive mapping $(i,j) \mapsto \sigma(i) - \sigma(j)$. Transitive mappings of this form are called *trivial*. If $f : \leqslant \to \mathcal{R}$ is a transitive mapping, we define an $\mathcal{R}$-linear mapping $\Delta_f : I(X,\mathcal{R}) \to I(X,\mathcal{R})$ by

$$\Delta_f(e_{ij}) = f(i.j)e_{ij}$$

for all $e_{ij} \in \mathfrak{B}$.

Lemma 2.3. *With notation as above, Δ_f is a derivation. Moreover, Δ_f is inner if and only if f is trivial.*

Proof. Note that $\Delta_f(e_{ij})e_{kl} + e_{ij}\Delta_f(e_{kl}) = \delta_{jk}(f(i,j) + f(k,l))e_{il} = \Delta_f(e_{ij}e_{kl})$ for all $e_{ij}, e_{kl} \in \mathfrak{B}$. Then Lemma 2.1 implies that Δ_f is a derivation.

If f is trivial, there exists a mapping $\sigma : X \to \mathcal{R}$ such that $f(i,j) = \sigma(i) - \sigma(j)$. Let $g = \sum_{i \in X} \sigma(i)e_{ii} \in I(X,\mathcal{R})$. Then $[g, e_{ij}] = (\sigma(i) - \sigma(j))e_{ij} = \Delta_f(e_{ij})$ for all $e_{ij} \in \mathfrak{B}$, and hence Δ_f is inner. Conversely, if Δ_f is inner, there exists an element $g = \sum_{e_{ij} \in \mathfrak{B}} C_{ij}e_{ij}$ such that $\Delta_f(e_{ij}) = [g, e_{ij}]$ for all $e_{ij} \in \mathfrak{B}$. Hence, $f(i,j) = C_{ii} - C_{jj}$. Setting $\sigma : X \to \mathcal{R}$ by $\sigma(i) = C_{ii}$, we get that f is trivial. $\square$

Remark 2.4. We will call the derivation Δ_f by *transitive induced derivation* induced by the transitive mapping f. The definition of transitive mappings first appeared in Nowicki's paper **[13]**, where he dealt with finite incidence algebras, i.e., X is a finite pre-ordered set.

Proposition 2.5. *Let D be a derivation of $I(X,\mathcal{R})$. Then there exist an element $g \in I(X,\mathcal{R})$ and a transitive mapping f such that*

$$D = \mathrm{Inn}_g + \Delta_f.$$

Proof. It follows from Theorem 2.2 that

$$D(e_{ij}) = \sum_{x\in L_i} C^{ii}_{xi} e_{xj} + C^{ij}_{ij} e_{ij} + \sum_{y\in R_j} C^{jj}_{jy} e_{iy}$$

for all $e_{ij} \in \mathfrak{B}$, where the coefficients C^{ij}_{xy} are subject to the following relations

$$C^{jj}_{jk} + C^{kk}_{jk} = 0 \qquad \text{and} \quad C^{ij}_{ij} + C^{jk}_{jk} = C^{ik}_{ik}.$$

Define $f :\leqslant \to \mathcal{R}$ by $(i,j) \mapsto C^{ij}_{ij}$. Then f is transitive since $C^{ij}_{ij} + C^{jk}_{jk} = C^{ik}_{ik}$. On the other hand, let $g = \sum_{e_{ij}\in\mathfrak{B}} C^{jj}_{ij} e_{ij}$. Note that $g(i,j) = C^{jj}_{ij} = -C^{ii}_{ij}$. Hence,

$$D(e_{ij}) = \mathrm{Inn}_g(e_{ij}) + \Delta_f(e_{ij})$$

for all $e_{ij} \in \mathfrak{B}$. The proposition follows from the fact that the mappings D, Inn_g and Δ_f are all $\mathcal{R}$-linear. □

We point out that Proposition 2.5 can also be obtained from [**10,** Theorem 6.3], where Koppinen used the coalgebra structure of incidence algebras. Our approach is elementary but is convenient for studying Jordan derivations in the next section. Lemma 2.3 and Proposition 2.5 deduce the following.

Corollary 2.6. *Every derivation of $I(X,\mathcal{R})$ is inner if and only if every transitive mapping is trivial.*

Corollary 2.7. *The incidence algebra $I(X,\mathcal{R})$ is a Kadison algebra, i.e., every local derivation of $I(X,\mathcal{R})$ is a derivation.*

Proof. The proof is similar to that of [**14,** Theorem 3]. □

3. Jordan derivations on incidence algebras. Throughout this section, we assume that $\mathcal{R}$ is 2-torsion free. Let $D : I(X,\mathcal{R}) \to I(X,\mathcal{R})$ be a Jordan derivation. We also denote

$$D(e_{ij}) = \sum_{e_{xy}\in\mathfrak{B}} C^{ij}_{xy} e_{xy}$$

for all $i,j \in X$ with $i \leqslant j$. We assume $C^{ij}_{xy} = 0$, if needed, for $x \nleqslant y$. The following lemma is due to Herstein [**7**].

Lemma 3.1. *Let A be a 2-torsion free ring, $D : A \to A$ be a Jordan derivation. Then for any $a, b, c \in A$, we have*

(i) $D(ab + ba) = D(a)b + aD(b) + D(b)a + bD(a)$,
(ii) $D(aba) = D(a)ba + aD(b)a + abD(a)$,
(iii) $D(abc + cba) = D(a)bc + aD(b)c + abD(c) + D(c)ba + cD(b)a + cbD(a)$.

Lemma 3.2. *Let $D : I(X, \mathcal{R}) \to I(X, \mathcal{R})$ be a Jordan derivation. Then*

$$D(e_{ij}) = \sum_{x \in L_i} C^{ii}_{xi} e_{xj} + C^{ij}_{ij} e_{ij} + \sum_{y \in R_j} C^{jj}_{jy} e_{iy} + C^{ij}_{ji} e_{ji}$$

for all $e_{ij} \in \mathfrak{B}$, where the coefficients C^{ij}_{xy} are subject to the following relations

$$\begin{cases} C^{jj}_{jk} + C^{kk}_{jk} = 0, & \text{if } j \leqslant k; \\ C^{ij}_{ij} + C^{jk}_{jk} = C^{ik}_{ik}, & \text{if } i \leqslant j,\ j \leqslant k. \end{cases}$$

Proof. Since e_{ii} is an idempotent element for all $i \in X$, a similar computation as in Theorem 2.2 shows that $C^{ii}_{ii} = 0$ and

$$D(e_{ii}) = \sum_{x \in L_i} C^{ii}_{xi} e_{xi} + \sum_{y \in R_i} C^{ii}_{iy} e_{iy}. \tag{7}$$

For any $e_{ij} \in \mathfrak{B}$ with $i \neq j$, we deduce from Lemma 3.1 (iii) and equation (8) that

$$\begin{aligned} D(e_{ij}) &= D(e_{ii} e_{ij} e_{jj} + e_{jj} e_{ij} e_{ii}) \\ &= D(e_{ii}) e_{ij} + e_{ii} D(e_{ij}) e_{jj} + e_{ij} D(e_{jj}) + e_{jj} D(e_{ij}) e_{ii} \\ &= D(e_{ii}) e_{ij} + C^{ij}_{ij} e_{ij} + e_{ij} D(e_{jj}) + C^{ij}_{ji} e_{ji} \\ &= \sum_{x \in L_i} C^{ii}_{xi} e_{xj} + C^{ij}_{ij} e_{ij} + \sum_{y \in R_j} C^{jj}_{jy} e_{iy} + C^{ij}_{ji} e_{ji}. \end{aligned} \tag{8}$$

Combining equations (8) and (9), we get the form of $D(e_{ij})$. It would be helpful to point out that $C^{ij}_{ji} = 0$ if $j \nleqslant i$ by our assumption.

In order to determine the coefficients C^{ij}_{xy}, we need to apply the Jordan derivation D to the identity $e_{ij} e_{kl} + e_{kl} e_{ij} = \delta_{jk} e_{il} + \delta_{li} e_{kj}$. There are four subcases.

Case 1. When $l \neq i$ and $j \neq k$, we have

$$
\begin{aligned}
0 &= D(e_{ij})e_{kl} + e_{ij}D(e_{kl}) + D(e_{kl})e_{ij} + e_{kl}D(e_{ij}) \\
&= \left(\sum_{x \in L_i} C_{xi}^{ii} e_{xj} + C_{ij}^{ij} e_{ij} + \sum_{y \in R_j} C_{jy}^{jj} e_{iy} + C_{ji}^{ij} e_{ji} \right) e_{kl} \\
&\quad + e_{ij} \left(\sum_{x \in L_k} C_{xk}^{kk} e_{xl} + C_{kl}^{kl} e_{kl} + \sum_{y \in R_l} C_{ly}^{ll} e_{ky} + C_{lk}^{kl} e_{lk} \right) \\
&\quad + \left(\sum_{x \in L_k} C_{xk}^{kk} e_{xl} + C_{kl}^{kl} e_{kl} + \sum_{y \in R_l} C_{ly}^{ll} e_{ky} + C_{lk}^{kl} e_{lk} \right) e_{ij} \\
&\quad + e_{kl} \left(\sum_{x \in L_i} C_{xi}^{ii} e_{xj} + C_{ij}^{ij} e_{ij} + \sum_{y \in R_j} C_{jy}^{jj} e_{iy} + C_{ji}^{ij} e_{ji} \right) \\
&= C_{jk}^{jj} e_{il} + \delta_{ik} C_{ji}^{ij} e_{jl} + C_{jk}^{kk} e_{il} + \delta_{jl} C_{lk}^{kl} e_{ik} \\
&\quad + C_{li}^{ll} e_{kj} + \delta_{ik} C_{lk}^{kl} e_{lj} + C_{li}^{ii} e_{kj} + \delta_{jl} C_{ji}^{ij} e_{ki}.
\end{aligned} \tag{9}
$$

We consider the coefficient of e_{il} in equation (10). If $e_{il} = e_{jl}$, then $C_{ji}^{ij} = C_{ii}^{ii} = 0$. If $e_{il} = e_{ik}$, then $C_{lk}^{kl} = C_{kk}^{kk} = 0$. Therefore, $C_{jk}^{jj} + C_{jk}^{kk} = 0$ if $(i,l) \neq (k,j)$ or $2(C_{jk}^{jj} + C_{jk}^{kk}) = 0$ if $(i,l) = (k,j)$. Hence, $C_{jk}^{jj} + C_{jk}^{kk} = 0$ for $j \leqslant k$ since $\mathcal{R}$ is 2-torsion free.

Case 2. When $l \neq i$ and $j = k$, we have

$$
\begin{aligned}
D(e_{il}) &= D(e_{ij})e_{kl} + e_{ij}D(e_{kl}) + D(e_{kl})e_{ij} + e_{kl}D(e_{ij}) \\
&= \left(\sum_{x \in L_i} C_{xi}^{ii} e_{xj} + C_{ij}^{ij} e_{ij} + \sum_{y \in R_j} C_{jy}^{jj} e_{iy} + C_{ji}^{ij} e_{ji} \right) e_{jl} \\
&\quad + e_{ij} \left(\sum_{x \in L_j} C_{xj}^{jj} e_{xl} + C_{jl}^{jl} e_{jl} + \sum_{y \in R_l} C_{ly}^{ll} e_{jy} + C_{lj}^{jl} e_{lj} \right) \\
&\quad + \left(\sum_{x \in L_j} C_{xj}^{jj} e_{xl} + C_{jl}^{jl} e_{jl} + \sum_{y \in R_l} C_{ly}^{ll} e_{jy} + C_{lj}^{jl} e_{lj} \right) e_{ij} \\
&\quad + e_{jl} \left(\sum_{x \in L_i} C_{xi}^{ii} e_{xj} + C_{ij}^{ij} e_{ij} + \sum_{y \in R_j} C_{jy}^{jj} e_{iy} + C_{ji}^{ij} e_{ji} \right) \\
&= \sum_{x \in L_i} C_{xi}^{ii} e_{xl} + C_{ij}^{ij} e_{il} + \delta_{ij} C_{ji}^{ij} e_{jl} + C_{jl}^{jl} e_{il}
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
&+\sum_{y\in R_l} C^{ll}_{ly}e_{iy}+\delta_{jl}C^{jl}_{lj}e_{ij}+C^{ll}_{li}e_{jj}+\delta_{ij}C^{jl}_{lj}e_{lj}+C^{ii}_{li}e_{jj}+\delta_{lj}C^{ij}_{ji}e_{ji}\\
=&\sum_{x\in L_i} C^{ii}_{xi}e_{xl}+C^{ij}_{ij}e_{il}+C^{jl}_{jl}e_{il}\\
&+\sum_{y\in R_l} C^{ll}_{ly}e_{iy}+\delta_{ij}C^{jl}_{lj}e_{lj}+\delta_{lj}C^{ij}_{ji}e_{ji},
\end{aligned}
$$

where the last identity follows from the facts $\delta_{ij}C^{ij}_{ji}=0$ and $C^{ll}_{li}+C^{ii}_{li}=0$. Therefore, there is

$$
C^{il}_{il}e_{il}+C^{il}_{li}e_{li}=C^{ij}_{ij}e_{il}+C^{jl}_{jl}e_{il}+\delta_{ij}C^{jl}_{lj}e_{lj}+\delta_{lj}C^{ij}_{ji}e_{ji}. \tag{11}
$$

We consider the coefficient of e_{il} in equation (12). Note that $l\neq i$ in this case. We obtain $C^{ij}_{ij}+C^{jl}_{jl}=C^{il}_{il}$.

Case 3. When $l=i$ and $j\neq k$, this case is the same as Case 2.

Case 4. When $l=i$ and $j=k$, a direct calculation shows that $C^{ij}_{ij}+C^{ji}_{ji}=0$. Combining with Case 2 or Case 3, we have $C^{ij}_{ij}+C^{jl}_{jl}=C^{il}_{il}$ for all $i\leqslant j$ and $j\leqslant l$. □

We are now in a position to prove the main result of this paper.

Theorem 3.3. *Let $\mathcal{R}$ be a 2-torsion free commutative ring with identity. Then every Jordan derivation of the incidence algebra $I(X,\mathcal{R})$ is a derivation.*

Proof. Let $D: I(X,\mathcal{R})\to I(X,\mathcal{R})$ be a Jordan derivation. By Lemma 3.2,

$$
D(e_{ij})=\sum_{x\in L_i} C^{ii}_{xi}e_{xj}+C^{ij}_{ij}e_{ij}+\sum_{y\in R_j} C^{jj}_{jy}e_{iy}+C^{ij}_{ji}e_{ji}
$$

for all $e_{ij}\in\mathfrak{B}$, where the coefficients C^{ij}_{xy} are subject to the following relations

$$
\begin{cases}
C^{jj}_{jk}+C^{kk}_{jk}=0, & \text{if } j\leqslant k;\\
C^{ij}_{ij}+C^{jk}_{jk}=C^{ik}_{ik}, & \text{if } i\leqslant j,\ j\leqslant k.
\end{cases}
$$

We define an $\mathcal{R}$-linear operator d by

$$
d(e_{ij})=\sum_{x\in L_i} C^{ii}_{xi}e_{xj}+C^{ij}_{ij}e_{ij}+\sum_{y\in R_j} C^{jj}_{jy}e_{iy}
$$

for all $e_{ij} \in \mathfrak{B}$, where the coefficients C_{xy}^{ij} are subject to the following relations

$$\begin{cases} C_{jk}^{jj} + C_{jk}^{kk} = 0, & \text{if } j \leqslant k; \\ C_{ij}^{ij} + C_{jk}^{jk} = C_{ik}^{ik}, & \text{if } i \leqslant j, \ j \leqslant k. \end{cases}$$

Then Theorem 2.2 makes d a derivation. Then the operator $\Delta := D - d$ is a Jordan derivation of $I(X, \mathcal{R})$ satisfying

$$\Delta(e_{ii}) = 0 \quad \text{and} \quad \Delta(e_{ij}) = C_{ji}^{ij} e_{ji}$$

for all $i \leqslant j$. Note that we have assumed $C_{ji}^{xy} = 0$ if $j \nleqslant i$. We restrict Δ to the subalgebra generated by $e_{ii}, e_{jj}, e_{ij}, e_{ji}$ if $j \leqslant i$ which is isomorphic to the full matrix algebra $\mathrm{M}_2(\mathcal{R})$. Then [**8,** Theorem 7 and Theorem 22] implies that Δ is a derivation, and hence $C_{ji}^{ij} = 0$. This completes the proof of the theorem. $\square$

Remark 3.4. If X is a finite partially ordered set (poset), then the incidence algebra $I(X, \mathcal{R})$ (also called bigraph algebra or finite dimensional CSL algebra) is usually a triangular algebra (see [**19,** page 1245]). In this case, Theorem 3.3 can be deduced from [**11,** Theorem 3.2] or [**20,** Theorem 2.1].

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