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NON-SYNTHETIC DIAGONAL OPERATORS ON THE SPACE OF FUNCTIONS ANALYTIC ON THE UNIT DISK

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ABSTRACT. Examples are given of continuous operators of functions analytic on the unit disk having the monomials as eigenvectors which fail spectral synthesis (that is, which have closed invariant subspaces which are not the closed linear span of collections of eigenvectors). Examples include the diagonal operator having as eigenvalues an enumeration of $\mathbb{Z}\times i\mathbb{Z} \equiv \{m+in : m,n \in \mathbb{Z}\}$ and diagonal operators having as eigenvalues enumerations of $\{n^{1/p}e^{2\pi ij/3p} : 0 \leq j < p\}$ where p is an integer at least 2.

1. Introduction. A vector x in a complete metrizable topological vector space X is said to be *cyclic* for a continuous linear operator $T : X \to X$ if the closed linear span of the orbit $\{T^n x : n \geq 0\}$ of x under T is all of X. Operators which have a cyclic vector are said to be *cyclic*. For example, the function $f(t) \equiv 1$ on $[0,1]$ is cyclic for the operator $T : g(t) \to tg(t)$ of multiplication by t on the Banach space $C([0,1])$ of continuous functions on $[0,1]$ by virtue of the Weierstrass approximation theorem.

A closed subspace M of X is *invariant* for $T : X \to X$ if $Tx \in M$ whenever $x \in M$. Examples of invariant subspaces include the closed linear span of eigenvectors for T, if any exist, and more generally, the closed linear spans $\overline{\vee\{T^n x : n \geq 0\}}$ of orbits $\{T^n x : n \geq 0\}$ of vectors $x \in X$. Hence, a vector x is cyclic for T if and only if the smallest closed invariant subspace for T containing x is all of X. The importance of cyclic vectors derives from the long-standing study of

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invariant subspaces of operators and the approximation theorems they yield.

If H is a Hilbert space with an orthonormal basis $\{e_n\}$ and $D : H \to H$ is a continuous linear operator, we say D is a *diagonal operator with eigenvalues* $\{\lambda_n\} \subset \mathbb{C}$ *if* $D(e_n) = \lambda_n e_n$ *for all* $n \geq 0$. That is, D is a diagonal operator if every basis element e_n is an eigenvector for D. It is well known that a diagonal operator on a Hilbert space is cyclic if and only if its eigenvalues are distinct. In this case, a necessary condition for a vector $\sum_{n=0}^{\infty} a_n e_n$ in H to be cyclic for D is that $a_n \neq 0$ for all $n \geq 0$. It might seem reasonable to expect that the converse is true; however, this need not always be the case as seen in Theorem 1.1.

A cyclic diagonal operator $T : X \to X$ acting on a complete metrizable topological vector space is said to *admit spectral synthesis* or to be *synthetic* if the closed invariant subspaces for T consist only of spaces spanned by the eigenvectors they contain. A necessary condition for T to admit spectral synthesis is that the eigenvectors for T have closed linear span equalling all of X. The following is a list of some conditions equivalent to a cyclic diagonal operator acting on a separable Hilbert space to admit spectral synthesis. For a more complete list see, for example, [**16,** Lemma 6].

Theorem 1.1. *Let $\mathcal{H}$ be a separable complex Hilbert space and let D be any bounded linear operator on $\mathcal{H}$ for which there exists an orthonormal basis $\{e_n\}$ for $\mathcal{H}$ and a sequence $\{\lambda_n\}$ of complex numbers for which $De_n = \lambda_n e_n$ for all $n \geq 0$. Then $\{\lambda_n\}$ is bounded. Moreover, D is cyclic if and only if $\lambda_m \neq \lambda_n$ for all $m \neq n$, and in this case, the following are equivalent*:

(i) *D admits spectral synthesis,*
(ii) *a vector x is cyclic for D if and only if $\langle x, e_n \rangle \neq 0$ for all n,*
(iii) *there does not exist a sequence $\{\omega_n\}$ of complex numbers in ℓ^1, not all zero, for which $\sum_{n=0}^{\infty} \omega_n \lambda_n^k \equiv 0$ for all $k \geq 0$,*
(iv) *there does not exist a sequence $\{\omega_n\}$ of complex numbers in ℓ^1, not all zero, for which the exponential series $\sum_{n=0}^{\infty} \omega_n e^{\lambda_n z} \equiv 0$ on the complex plane,*
(v) *the weakly closed algebra generated by D and the identity is the commutant of D, and*
(vi) *there does not exist a bounded complex domain Ω such that* $\sup\{|f(z)| : z \in \Omega\} = \sup\{|f(z)| : z \in \Omega \cap \{\lambda_n\}\}$ *for all f bounded*

and analytic on Ω.

If, in addition, the λ_n *lie in the open unit disk and accumulate only on the unit circle, then conditions (i)–(vi) are equivalent to*

(vii) *not almost every point of the unit circle is in the non-tangential cluster set of* $\{\lambda_n\}$.

Regarding condition (iii) of Theorem 1, in 1921 Wolff [**28**] gave the first example of a bounded sequence $\{\lambda_n\}$ of distinct complex numbers and a sequence $\{\omega_n\}$ of complex numbers in ℓ^1, not all zero, for which $0 \equiv \sum \omega_n \lambda_n^k$ for all $k \geq 0$. Wolff's example generates cyclic diagonal operators on ℓ^2 which fail spectral synthesis. Since 1921, the study of cyclic operators and invariant subspaces of diagonal operators on a separable Hilbert space has enjoyed a long and rich history (see, for example, [**2, 17, 18, 20, 21, 25, 26, 28**]). This study has been extended to include cyclic diagonal operators acting on spaces other than a Hilbert space or Banach space. Two such examples are the space $H(\mathbb{C})$ of entire functions and the space $H(\mathbb{D})$ of functions analytic on the open unit disk $\mathbb{D} \equiv \{z \in \mathbb{C} : |z| < 1\}$ (see, for example, the papers of Deters, Marin, Seubert and Wade [**3, 4, 5, 16, 23, 24**]). When endowed with the topology of uniform convergence on compacta, $H(\mathbb{C})$ and $H(\mathbb{D})$ are examples of complete metrizable topological vector spaces. We say that a continuous linear operator D on either $H(\mathbb{C})$ or $H(\mathbb{D})$ is a *diagonal operator with eigenvalues* $\{\lambda_n\}$ *if* $D(z^n) = \lambda_n z^n$ *for all* $n \geq 0$, that is, D is a diagonal operator if each monomial is an eigenvector for D.

The purpose of this paper is to provide classes of examples of cyclic diagonal operators acting on the space $H(\mathbb{D})$ which fail spectral synthesis.

The background information, details of which can be found in [**4**], is as follows. By the radius of convergence formula, a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is in $H(\mathbb{D})$ if and only if $\limsup_{n\to\infty} |a_n|^{1/n} \leq 1$. It follows that a linear map D for which $D(z^n) = \lambda_n z^n$ for all $n \geq 0$ defines a continuous linear operator on $H(\mathbb{D})$ if and only if $\limsup_{n\to\infty} |\lambda_n|^{1/n} \leq 1$. In particular, there exist diagonal operators on $H(\mathbb{D})$ whose eigenvalues are unbounded.

An analogue of Theorem 1.1 (i)–(iv) for diagonal operators on $H(\mathbb{D})$

to admit spectral synthesis was obtained by Deters and Seubert in [**4,** Theorem 3]. In this paper, the analogue of condition (iv) will be used extensively; thus, we state it here.

Theorem 1.2. *Let D be the cyclic diagonal operator on $H(\mathbb{D})$ having distinct eigenvalues $\{\lambda_n\}$. If $\{\lambda_n/n : n \geq 1\}$ is bounded, then D admits spectral synthesis if and only if there does not exist a sequence $\{\omega_n\}$ of complex numbers, not identically zero, for which* $\limsup |\omega_n|^{1/n} < 1$ *and* $0 \equiv \sum_{n=0}^{\infty} \omega_n e^{\lambda_n z}$ *for all z in the open ball $B(0, \epsilon)$ where* $\epsilon \equiv [\ln (1/\limsup |\omega_n|^{1/n})]/[\sup\{|\lambda_n|/n\}]$.

Throughout this paper, the same technique will be used to establish all of the main results. A general outline of the technique is as follows. The details for each result will be provided in the respective proofs. Let D be a diagonal operator having eigenvalues $\{\lambda_n\}$. In each example presented in this paper, $\{\lambda_n\}$ will satisfy $\inf\{\alpha : \sum_{n=1}^{\infty} 1/|\lambda_n|^{\alpha} < \infty\} > 1$; hence, any entire function $S(\lambda)$ having simple zeros at λ_n has order $\rho > 1$ [**1**]. It then follows that there exist constants $\alpha, \beta > 0$, for which $|S(\lambda)| > \alpha e^{\beta|\lambda|^{\rho}}$ whenever λ avoids a disjoint collection of balls $B(\lambda_n, r_n)$. Invoking the residue theorem yields

$$\sum_{n=0}^{\infty} e^{\lambda_n z} \Big/ S'(\lambda_n) = \lim_{r \to \infty} \int_{C_r} (e^{\lambda z}/S(\lambda))\, d\lambda = 0$$

for appropriately chosen contours C_r which avoid the balls $B(\lambda_n, r_n)$. Applying Lemma 1.3, stated below, it will follow that

$$\limsup_{n \to \infty} |\omega_n|^{1/n} < 1 \quad \text{where } \omega_n \equiv 1/S'(\lambda_n).$$

Hence, $0 \equiv \sum_{n=0}^{\infty} \omega_n e^{\lambda_n z}$ for all z near the origin. By Theorem 1.2, D fails to admit spectral synthesis on $H(\mathbb{D})$.

Lemma 1.3. *Let f and g be functions analytic on an open set containing the closed ball $\overline{B(\omega, r)} = \{z : |z - \omega| \leq r\}$. If f has a unique zero in $\overline{B(\omega, r)}$ at ω, if g has no zeros in $\overline{B(\omega, r)}$, and if $|f(z)| \geq |g(z)|$ whenever $|z - \omega| = r$, then $|f'(\omega)| \geq |g(\omega)|/r$.*

This lemma follows immediately by applying the maximum principle to the function $z \to (z - \omega)g(z)/[rf(z)]$.

2. Example of a non-synthetic diagonal operator on $H(\mathbb{D})$. In this section, we show that the diagonal operator D having as eigenvalues an enumeration of the integer lattice points $\mathbb{Z} \times i\mathbb{Z} \equiv \{m + in : m, n \in \mathbb{Z}\}$ fails to admit spectral synthesis as an operator acting on $H(\mathbb{D})$, the space of functions analytic on the unit disk $\mathbb{D}$.

We let C_j denote the square with vertices $\pm(j + ij)$ and $\pm i(j + ij)$, for all $j \geq 0$, and define $\{\lambda_k\}$ to be the enumeration of $\mathbb{Z}\times i\mathbb{Z}$ defined by beginning on the positive real line and moving counterclockwise around larger and larger squares C_j. In this way,

$$\begin{array}{lll} \lambda_0 = 0; & \lambda_1 = 1; & \lambda_2 = 1 + i; \\ \lambda_3 = i; & \lambda_4 = -1 + i; & \lambda_5 = -1; \\ \lambda_6 = -1 - i; & \lambda_7 = -i; & \lambda_8 = 1 - i; \\ \lambda_9 = 2; \dots & \lambda_{24} = 2 - i; & \lambda_{25} = 3 \dots . \end{array}$$

There are a total of $(2j + 1)^2$ integer lattice points on or inside the square C_j. Hence, there are exactly $8j$ points on C_j. Therefore, λ_k is on the square C_j if and only if $1 + 4j(j - 1) \leq k \leq 4j(j + 1)$ and $j \leq |\lambda_k| \leq \sqrt{2}j$. From this, it follows that $\{\lambda_k/k : k \geq 1\}$ is bounded and $\limsup |\lambda_k|^{1/k} \leq 1$. Thus, the diagonal operator D having eigenvalues $\{\lambda_k\}$ is continuous on both $H(\mathbb{C})$ and $H(\mathbb{D})$ [**4,** Lemma 1].

Define the function

$$S(z) = z\Pi'\left(1 - \left(\frac{z}{m + in}\right)\right)e^{(z/(m+in)+z^2/(2(m+in)^2))};$$

(in this context, Π' indicates that the product is taken over all integers m and n except when $m = 0 = n$). This is the Weierstrass σ-function having zeros at the integer lattice points $\mathbb{Z} \times i\mathbb{Z}$ (see, [**27,** Chapter XX]). For each $r \in \mathbb{Z}^+$, we let C_r denote the square passing through the points $\pm(r+1/2)$ and $\pm i(r+1/2)$ and having horizontal and vertical sides. For any complex number λ on any square C_r, the distance $d(\lambda) \equiv \inf\{|\lambda - \lambda_k| : \lambda_k \in \mathbb{Z} \times i\mathbb{Z}\}$ from λ to any point in $\mathbb{Z} \times i\mathbb{Z}$ is at least $1/2$. Since $|S(\lambda)| \geq Cd(\lambda)e^{\pi|\lambda|^2/2}$ where C is a positive constant [**22,** page 108], it follows that $\lim_{r\to\infty} \int_{C_r} (e^{\lambda z}/S(\lambda))d\lambda = 0$ for every complex number z. Moreover, by the residue theorem,

$$\int_{C_r} (e^{\lambda z}/S(\lambda))\, d\lambda = \sum_{\{k:|\lambda_k|<r\}} e^{\lambda_k z}/S'(\lambda_k).$$

Thus, $\sum_{k=0}^{\infty} \omega_k e^{\lambda_k z} = 0$ for all $z \in \mathbb{C}$ where $\omega_k \equiv 1/S'(\lambda_k)$.

Whenever $|z - \lambda_k| = 1/2$, we have

$$\begin{aligned}|S(z)| &\geq Cd(\lambda)e^{\pi|z|^2/2} \geq (1/2)Ce^{\pi(|\lambda_k|-1/2)^2/2}\\ &\geq c_1 e^{\pi|\lambda_k|^2} \geq ce^{-\pi|\lambda_k|^2/2}e^{\pi|\lambda_k|^2}e^{\pi|\lambda_k|2}\\ &\geq ce^{-\pi|\lambda_k|^2/2}e^{\pi|z||\overline{\lambda_k}|}\\ &\geq |ce^{\pi(z\overline{\lambda_k}-|\lambda_k|^2/2)}|.\end{aligned}$$

Then, applying Lemma 1.3 with $f(z) = S(z)$, $\omega = \lambda_k$, $r = 1/2$, and $g(z) = ce^{(\pi(z\overline{\lambda_k}-|\lambda_k|^2/2))}$, it follows that

$$|S'(\lambda_k)| \geq 2|g(\lambda_k)| = c_1 e^{\pi|\lambda_k|^2/2}$$

for all points $\lambda_k \in \mathbb{Z} \times i\mathbb{Z}$ where c_1 is a positive constant. Since $j^2 \leq |\lambda_k|^2 \leq 2j^2$ whenever $1 + 4j(j-1) \leq k < 4j(j+1)$, it follows that $\limsup_{k\to\infty} |\omega_k|^{1/k} < 1$. Hence, the diagonal operator D having eigenvalues $\mathbb{Z}\times i\mathbb{Z}$ fails spectral synthesis as an operator acting on $H(\mathbb{D})$ by Theorem 1.2.

However, since $S(z)$ has order 2 and type $\pi/2$, it follows that D admits spectral synthesis as an operator acting on the space of entire functions $H(\mathbb{C})$ [**5,** Theorem 3].

3. A class of non-synthetic operators on $H(\mathbb{D})$. In this section, we prove that, for any integer $p \geq 2$, the diagonal operator having as eigenvalues $3p$ copies of the sequence $n^{1/p}$ placed on the $3p$ rays $\{z \in \mathbb{C} : \arg z = 2\pi ij/3p\}$ for $0 \leq j < 3p$, fails to admit spectral synthesis as an operator acting on $H(\mathbb{D})$. This result, combined with the results of the previous section, suggest that the synthesis of a diagonal operator depends not only on the growth of its eigenvalues, but also how they are distributed throughout the complex plane. For instance, the diagonal operator having as eigenvalues $\{\sqrt{n}\}$ admits spectral synthesis [**23**]; however, the main result of this section shows that the diagonal operator having as eigenvalues $\{\sqrt{n}e^{\pi ij/3} : 0 \leq j < 6\}$ consisting of six copies of the sequence $\{\sqrt{n}\}$ on six rays, fails to admit synthesis.

We need to establish a protocol for enumerating the eigenvalues $\{\lambda_k\}$ of the diagonal operator D to ensure that the operator D is continuous, that is, $\limsup |\lambda_k|^{1/k} \leq 1$. Thus, we will require that any enumeration of $\{\lambda_k\}$ of the set of points of the form $\{a_n e^{2\pi ij/p} : 0 \leq j < p\}$ where $\{a_n\}$ is an increasing sequence of positive numbers and p is a positive integer, be such that $\{|\lambda_k|\}$ is non-decreasing. Such an enumeration is always obtained by listing the points of the set by starting on the positive real axis and traversing circles of increasing radii a_n in the counter-clockwise direction.

Theorem 3.1. *The diagonal operator D acting on $H(\mathbb{D})$ having eigenvalues $\{n^{1/p}e^{2\pi ij/3p} : 0 \leq j < p\}$ fails spectral synthesis whenever p is an integer at least* 2*.*

Proof. Let $\{\lambda_k\}$ be an enumeration as established in the previous paragraph. Define the entire function $f(z) \equiv \Pi_{n=1}^{\infty}(1-(z/n^3))$ having simple zeros only at $\{n^3\}$. For every $0 < \epsilon < \pi/\sqrt{3}$, there exists an $R_\epsilon \in \mathbb{R}$ such that $|f(re^{i\theta})| \geq e^{(\pi/\sqrt{3}-\epsilon)r^{1/3}}$ for $r \geq R_\epsilon$ and $re^{i\theta} \notin E \equiv \cup_{n=1}^{\infty}\overline{B(n^3,n^2)}$ [**14,** pages 94–95].

The entire function $S(z) \equiv f(z^{3p})$ has simple zeros only at $\{\lambda_k\}$. Fix k large enough so that $|\lambda_k| \geq R_\epsilon$, and choose r_k so that $B(\lambda_k, r_k) \cap \{\lambda_k : m \in \mathbb{N}\} = \{\lambda_k\}$. Then, $|S(z)| \geq e^{(\pi/\sqrt{3}-\epsilon)|z|^p}$ for all $|z-\lambda_k| = r_k$. Applying Lemma 1.3 with $f(z) = S(z)$, $g(z) = e^{(\pi/\sqrt{3}-\epsilon)z^p}$, $r = r_k$ and $\omega = \lambda_k$, we obtain

$$|S'(\lambda_k)| \geq e^{(\pi/\sqrt{3}-\epsilon)|\lambda_k|^p}/r_k.$$

Consider $|\lambda_k| = m^{1/p}$ for some $m \in \mathbb{N}$ such that $3p(m-1) \leq k < 3pm$. Hence, $|S'(\lambda_k)|^{1/k} \geq e^{(\pi/\sqrt{3}-\epsilon)1/(3p)}/r_k^{1/(3pm)}$. Hence, $\limsup_{k\to\infty} |1/S'(\lambda_k)|^{1/k} \leq 1/e^{(\pi/\sqrt{3}-\epsilon)1/(3p)} < 1$. Thus, if $\omega_k \equiv 1/S'(\lambda_k)$ for all k, $\limsup |\omega_k|^{1/k} < 1$.

For each $r \in \mathbb{R}$, define $C_r \equiv \{z \in \mathbb{C} : |z| = \widehat{r}\}$ where $\widehat{r} = (1/2)((r+1)^{1/p} + r^{1/p})$. Then, for each r such that $C_{\widehat{r}^{3p}} \cap E = \emptyset$ and for each $z \in \mathbb{C}$, by the residue theorem,

$$\frac{1}{2\pi i}\int_{C_r} (e^{\lambda z}/S(\lambda))\, d\lambda = \sum_{\{k:|\lambda_k|\leq \widehat{r}\}} (e^{\lambda_k z}/S'(\lambda_k)).$$

Since the open balls $B(n^3,n^2)$ are pairwise disjoint there exists an increasing sequence $\{r_n\}$ such that $C_{\widehat{r_n}^{3p}} \cap E = \emptyset$. Thus,

$$\left| \int_{C_r} (e^{\lambda z}/S(\lambda))\, d\lambda \right| \leq (2\pi \widehat{r} e^{\widehat{r}|z|})/(e^{(\pi/\sqrt{3}-\epsilon)\widehat{r}^p}) \longrightarrow 0$$

as $r \to \infty$. Hence,

$$\sum_{k=0}^{\infty} \omega_k e^{\lambda_k z} \equiv 0.$$

Therefore, by Theorem 1.2, D fails to admit spectral synthesis on $H(\mathbb{D})$. □

The statement of Theorem 3.1 requires that $p \geq 2$. If $p = 1$, the technique used in this paper for establishing failure to admit spectral synthesis cannot be used. The eigenvalues would be an enumeration $\{\lambda_k\}$ of $\{ne^{2\pi ij/3}\}$; hence,

$$\inf \left\{ \alpha : \sum_{k=1}^{\infty} 1/|\lambda_k|^{\alpha} < \infty \right\} = 1.$$

Thus, any entire function having simple zeros at $\{\lambda_k\}$ would have order equal to one, and appropriate bounds could not be obtained using the techniques of this paper. It is not known by the authors if this diagonal operator admits spectral synthesis or not.

Theorem 3.1 can be easily extended to include diagonal operators having as eigenvalues b copies of $\{n^{a/b}\}$ placed on the b rays $\{z \in \mathbb{C} : \arg z = 2\pi ij/b\}$ for $0 \leq j < b$ for certain rational powers a/b smaller than 1, as stated in the following corollary.

Corollary 3.2. *The diagonal operator D acting on $H(\mathbb{D})$ having eigenvalues $\{n^{a/b}e^{2\pi ij/b} : 0 \leq j < b\}$ fails synthesis whenever a and b are integers for which $b > a > 2$.*

Notice that, in Theorem 3.1, the eigenvalues came from taking roots of the sequence $\{a_n\} \equiv \{n^3\}$, and the proof relied on estimating the entire function with simple zeros at $\{a_n\}$. This result can be generalized by using the zeros $\{a_n\}$ of an entire function $f(z)$ having certain properties. The estimates involved in this case rely heavily on the work of Levin (see **[14, 15]**) involving the growth of entire functions based on the distribution of the zeros. Following Levin's method, we define two conditions guaranteeing that the zeros $\{a_n\}$ are separated. We say $\{a_n\}$ satisfies condition (C) if there exists a $d > 0$ such that the closed balls $\{\overline{B(a_n, d|a_n|^{1-\rho/2})} : n \in \mathbb{N}\}$ are pairwise disjoint, and $\{a_n\}$ satisfies condition (C′) if $\{a_n\}$ is non-decreasing and there exists a $d > 0$ such that $|a_{n+1}| - |a_n| > d|a_n|^{1-\rho}$, where the closed balls $\{\overline{B(a_n, d|a_n|^{1-\rho})} : n \in \mathbb{N}\}$ are pairwise disjoint. In these definitions, ρ represents the order of the entire function having simple zeros at $\{a_n\}$ (see, for example [**1**]).

Theorem 3.3. *Let $f(z)$ be an entire function of order $\rho \in (0, 1/2)$ whose zeros $\{a_n\}$ are all positive real numbers and are all simple. If*

(i) $\{a_n\}$ *satisfies either Condition* (C) *or* (C′),
(ii) $\lim_{r\to\infty} \frac{n(r)}{r^\rho} \in (0, \infty)$, *where* $n(r) = \sum_{\{n:|a_n|\le r\}} 1$,

and q is any integer greater than $1/\rho$, then the diagonal operator D on $H(\mathbb{D})$ with eigenvalues $\{a_n^{1/q} e^{2\pi i j/q} : 0 \le j < q\}$ fails to admit spectral synthesis.

The proof of Theorem 3.3 follows the same technique outlined in the final paragraph of Section 1 and invoked in the proofs in Section 2 and Theorem 3.1. Therefore, only an outline is provided (complete details can by found in [**19,** pages 50–52]). Define $S(\lambda)$ to be an entire function with zeros only at the points $\{\lambda_k\}$ where $\{\lambda_k\}$ is an enumeration of $\{a_n^{1/q} e^{2\pi i j/q} : 0 \le j < p\}$. Applying the residue theorem, we obtain

$$\begin{aligned}\lim_{r\to\infty} \int_{C_r} (e^{\lambda z}/S(\lambda))\, d\lambda &= \lim \sum_{\{k:\lambda_k \in C_r^0\}} (e^{\lambda_k z}/S'(\lambda_k)) \\ &= \sum_{k=0}^{\infty} \omega_k e^{\lambda_k z},\end{aligned}$$

where C_r are appropriately chosen contours which avoid the pairwise disjoint balls obtained by Condition (C) or (C′). Observing that $S(\lambda) = f(\lambda^q)$ and that $|f(re^{i\theta})| \ge e^{\epsilon r^\rho}$ for r large enough and some $\epsilon > 0$ [**14,** page 96], we obtain that

$$\lim_{r\to\infty} \int_{C_r} (e^{\lambda z}/S(\lambda))\, d\lambda = 0.$$

Applying Lemma 1.3 to $f(z) = S(z)$, $g(z) = e^{\epsilon|z|^{\rho q}}$, $\omega = \lambda_k$, and an appropriately chosen r, we obtain

$$\limsup_{k\to\infty} |\omega_k|^{1/k} = \limsup |1/S'(\lambda_k)|^{1/k} < 1.$$

Thus, by Theorem 1.2, D fails to admit spectral synthesis on $H(\mathbb{D})$.

Observe that Theorem 3.1 is an elementary version of Theorem 3.3; however, we have included both in this paper as Theorem 3.1 provides more concrete and familiar examples.

An example of a diagonal operator that would fail to admit spectral synthesis under the conditions of Theorem 3.3, but not Theorem 3.1, would be one having as eigenvalues q copies of $\{n^p + n^{p-1}\}$ placed symmetrically on q rays, where p is a rational number smaller than 1 and q is an integer greater than $1/p$.

4. Remarks. A sequence $\{\lambda_n\}$ is said to be absolutely representing, if every entire function $f(z)$ can be represented as

$$f(z) = \sum_{n=0}^{\infty} a_n e^{\lambda_n z} \quad \text{for some } \{a_n\}.$$

Numerous necessary and sufficient conditions for $\{\lambda_n\}$ to be absolutely representing can be found in the papers of Leont'ev **[10, 11, 12, 13]** and Korobeǐnik **[6, 7, 8, 9]**, for example. Many of these results require $\{\lambda_n/n\}$ to be bounded, and, in some cases, are equivalent to an operator admitting spectral synthesis.

In Theorem 3.1, it is required that the eigenvalues consist of $3p$ copies of $n^{1/p}$ and in Theorem 3.3 it is required that the eigenvalues consist of q copies of each $a_n^{1/q}$. In fact, it is possible to exclude some of the points from both $\mathbb{Z} \times i\mathbb{Z}$ and $\{n^{1/p}e^{2\pi ij/3p} : 0 \le j < p\}$ and still obtain non-synthetic operators as long as the estimate on the entire function $S(z)$ still has the form $|S(z)| > \alpha e^{\beta|z|^2}$. It may be possible to determine a minimum number of copies needed to obtain an operator which fails to admit spectral synthesis, and, if so, it is likely to be smaller than either of these conditions.

Another avenue for future work on this topic is to determine whether the eigenvalues can be required to be close to the points $a_n^{1/q}$ rather than having to be the exact points. Work by Levin, for example **[14,** page 98], can be used to obtain some results in this regard. However, more general results may be possible.

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