ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 45, Number 1, 2015

CHARACTERIZATIONS OF LINEAR WEINGARTEN SPACELIKE HYPERSURFACES IN LORENTZ SPACE FORMS

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ABSTRACT. In this article, we deal with complete linear Weingarten spacelike hypersurfaces (that is, complete spacelike hypersurfaces whose mean and scalar curvatures are linearly related) immersed in a Lorentz space form. By assuming that the mean curvature attains its maximum and supposing appropriated restrictions on the norm of the traceless part of the second fundamental form, we apply Hopf's strong maximum principle in order to prove that such a spacelike hypersurface must be either totally umbilical or isometric to a hyperbolic cylinder of the ambient space.

1. Introduction and statements of the main results. Let L_1^{n+1} be an $(n+1)$-dimensional Lorentz space, that is, a semi-Riemannian manifold of index 1. When the Lorentz space L_1^{n+1} is simply connected and has constant sectional curvature c, it is called a Lorentz space form, and we will denote it by $L_1^{n+1}(c)$. The Lorentz-Minkowski space $\mathbb{L}^{n+1}$, the de Sitter space $\mathbb{S}_1^{n+1}$ and the anti-de Sitter space $\mathbb{H}_1^{n+1}$ are the standard Lorentz space forms of constant sectional curvature 0, 1 and -1, respectively. We also recall that a hypersurface M^n immersed in a Lorentz space L_1^{n+1} is said to be *spacelike* if the metric on M^n induced from that of the ambient space L_1^{n+1} is positive definite.

The last few decades have seen a steadily growing interest in the study of spacelike hypersurfaces of Lorentz manifolds. Apart from

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2010 AMS *Mathematics subject classification.* Primary 53C42, Secondary 53A10, 53C20, 53C50.

Keywords and phrases. Lorentz space forms, linear Weingarten spacelike hypersurfaces, second fundamental form, totally umbilical hypersurfaces; hyperbolic cylinders.

The second author is partially supported by CNPq, Brazil, grant No. 300769/2012-1. The second and third authors are partially supported by CAPES/CNPq, Brazil, grant Casadinho/Procad 552.464/2011-2. The second author is the corresponding author.

Received by the editors on January 4, 2013.

DOI:10.1216/RMJ-2015-45-1-13

physical motivations, from the mathematical point of view this is mostly due to the fact that such hypersurfaces exhibit nice Bernstein-type properties, and one can truly say that the first remarkable results in this direction were the rigidity theorems of Calabi in [**5**] and Cheng and Yau in [**9**], who showed (the former for $n \leq 4$, and the latter for general n) that the only maximal (that is, with zero mean curvature) complete, noncompact, spacelike hypersurfaces of the Minkowski space $\mathbb{L}^{n+1}$ are the spacelike hyperplanes. Later on, Nishikawa [**16**] obtained similar results for other Lorentz manifolds. For instance, he proved that a complete maximal spacelike hypersurface in de Sitter space $\mathbb{S}_1^{n+1}$ must be totally geodesic (that is, its second fundamental form vanishes identically).

As for the case of the de Sitter space, Goddard [**11**] conjectured that every complete spacelike hypersurface with constant mean curvature H in $\mathbb{S}_1^{n+1}$ should be totally umbilical. Although the conjecture turned out to be false in its original statement, it motivated a great deal of work by several authors trying to find a positive answer to the conjecture under appropriate additional hypotheses. For instance, in [**2**] Akutagawa showed that Goddard's conjecture is true when $H^2 \leq 1$ in the case $n = 2$, and when $H^2 < 4(n-1)/n^2$ in the case $n \geq 3$. Afterwards, Montiel [**15**] solved Goddard's problem in the compact case, proving that the only closed spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$ with constant mean curvature are the totally umbilical hypersurfaces.

Another Goddard-like problem is to study complete spacelike hypersurfaces immersed in a Lorentz space with constant scalar curvature. Many authors, such as Brasil, Colares and Palmas [**4**], Camargo, Chaves and Sousa Jr. [**6**], Caminha [**7**] and Hu, Scherfner and Zhai [**13**], have worked on this direction.

More recently, Hou and Yang [**12**] have extended the ideas of Li, Suh and Wei [**14**] in order to obtain characterization results for *linear Weingarten spacelike hypersurfaces* into the de Sitter space $\mathbb{S}_1^{n+1}$, that is, spacelike hypersurfaces of $\mathbb{S}_1^{n+1}$ whose mean curvature H and normalized scalar curvature R satisfy the relation $R = aH + b$, for some constants $a, b \in \mathbb{R}$.

Here, motivated by the works above described, our aim is to establish characterization theorems concerning complete linear Weingarten spacelike hypersurfaces immersed in a Lorentz space form $L_1^{n+1}(c)$ of constant sectional curvature $c = 1, 0, -1$. Under the assumption that the mean curvature attains its maximum along such a spacelike hypersurface and, supposing appropriated restrictions on the norm of the traceless part Φ of its second fundamental form, we prove the following results:

Theorem 1.1. *Let M^n be a complete linear Weingarten spacelike hypersurface in $\mathbb{S}_1^{n+1}$, $n \geq 3$, such that $R = aH + b$ with $b < 1$. Suppose that $0 < R < 1 - \frac{2}{n}$. If H attains its maximum on M^n and*

$$\sup_M |\Phi|^2 \leq \frac{n(n-1)R^2}{(n-2)((n-2)-nR)}, \tag{1}$$

then

(i) *either $|\Phi| \equiv 0$ and M^n is totally umbilical;*
(ii) *or $|\Phi|^2 \equiv \dfrac{n(n-1)R^2}{(n-2)((n-2)-nR)}$ and M^n is isometric to a hyperbolic cylinder $\mathbb{H}^1(c_1) \times \mathbb{S}^{n-1}(c_2)$, where $c_1 < 0$, $c_2 > 0$ and $\frac{1}{c_1} + \frac{1}{c_2} = 1$.*

Theorem 1.2. *Let M^n be a complete linear Weingarten spacelike hypersurface in $L_1^{n+1}(c)$, $n \geq 3$, such that $R = aH + b$ with $b < c$. Suppose that $R < 0$, when $c = 1$ or $c = 0$, and that $R < (1 - \frac{2}{n})c$, when $c = -1$. If H attains its maximum on M^n and*

$$\inf_M |\Phi|^2 \geq \frac{n(n-1)R^2}{(n-2)((n-2)c-nR)}. \tag{2}$$

Then

$$|\Phi|^2 \equiv \frac{n(n-1)R^2}{(n-2)((n-2)c-nR)}$$

and M^n is isometric to:

(a) $\mathbb{S}^1(c_1) \times \mathbb{H}^{n-1}(c_2)$, *where $c_1 > 0$, $c_2 < 0$ and $\frac{1}{c_1} + \frac{1}{c_2} = 1$, when $c = 1$;*
(b) $\mathbb{R} \times \mathbb{H}^{n-1}(c_2)$, *where $c_2 < 0$, when $c = 0$;*
(c) $\mathbb{H}^1(c_1) \times \mathbb{H}^{n-1}(c_2)$, *where $c_1 < 0$, $c_2 < 0$ and $\frac{1}{c_1} + \frac{1}{c_2} = -1$, when $c = -1$.*

The proofs of Theorems 1.1 and 1.2 are given in Section 3, jointly with a corollary of Theorem 1.1 related to the compact case (cf., Corollary 3.4).

2. Preliminaries. In this section, we will introduce some basic facts and notation that will appear in the paper. In what follows, we will suppose that all spacelike hypersurfaces considered are connected.

Let M^n be an n-dimensional spacelike hypersurface immersed in a Lorentz space form $L_1^{n+1}(c)$ with constant sectional curvature $c \in \{-1,0,1\}$. We choose a local field of semi-Riemannian orthonormal frame $\{e_1,\ldots,e_{n+1}\}$ in $L_1^{n+1}(c)$, with dual coframe $\{\omega_1,\ldots,\omega_{n+1}\}$, such that, at each point of M^n, $e_1,\ldots,e_n$ are tangent to M^n and e_{n+1} is normal to M^n. We will use the following convention for the indices:

$$1 \le A,B,C,\ldots \le n+1, \quad 1 \le i,\ j,k,\ldots \le n.$$

In this setting, the Lorentz metric of $L_1^{n+1}(c)$ is given by

$$ds^2 = \sum_A \epsilon_A\, \omega_A^2 = \sum_i \omega_i^2 - \omega_{n+1}^2,$$

where $\epsilon_i = 1$ and $\epsilon_{n+1} = -1$. Denoting by $\{\omega_{AB}\}$ the connection forms of $L_1^{n+1}(c)$, we have that the structure equations of $L_1^{n+1}(c)$ are given by:

$$d\omega_A = \sum_B \epsilon_B\, \omega_{AB} \wedge \omega_B, \qquad \omega_{AB} + \omega_{BA} = 0, \tag{3}$$

$$d\omega_{AB} = \sum_C \epsilon_C\, \omega_{AC} \wedge \omega_{CB} - \frac{1}{2}\sum_{C,D} K_{ABCD}\, \omega_C \wedge \omega_D, \tag{4}$$

where

$$K_{ABCD} = \epsilon_A \epsilon_B\, c\,(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}).$$

Next, we restrict all the tensors to M^n. First of all, $\omega_{n+1} = 0$ on M^n; so $\sum_i \omega_{n+1i} \wedge \omega_i = d\omega_{n+1} = 0$ and, by *Cartan's lemma* [**8**] we can write

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j, \quad h_{ij} = h_{ji}. \tag{5}$$

This gives the second fundamental form of M^n, $B = \sum_{ij} h_{ij}\omega_i\omega_j e_{n+1}$. Furthermore, the mean curvature H of M^n is defined by $H = \frac{1}{n}\sum_i h_{ii}$. The structure equations of M^n are given by

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{6}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l, \tag{7}$$

where R_{ijkl} are the components of the curvature tensor of M^n. Moreover, using the previous structure equations, we obtain the Gauss equation

$$R_{ijkl} = c\,(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - (h_{ik}h_{jl} - h_{il}h_{jk}). \tag{8}$$

The Ricci curvature and the normalized scalar curvature of M^n are given, respectively, by

$$R_{ij} = c(n-1)\delta_{ij} - nHh_{ij} + \sum_k h_{ik}h_{kj} \tag{9}$$

and

$$R = \frac{1}{n(n-1)}\sum_i R_{ii}. \tag{10}$$

From (9) and (10), we obtain

$$|B|^2 = n^2H^2 + n(n-1)(R-c), \tag{11}$$

where $|B|^2 = \sum_{i,j} h_{ij}^2$ is the square of the length of the second fundamental form B of M^n.

Set $\Phi_{ij} = h_{ij} - H\delta_{ij}$. We will also consider the following symmetric tensor

$$\Phi = \sum_{i,j} \Phi_{ij}\omega_i \otimes \omega_j.$$

Let $|\Phi|^2 = \sum_{i,j} \Phi_{ij}^2$ be the square of the length of Φ. It is easy to check that Φ is traceless and, from (11), we get

$$|\Phi|^2 = |B|^2 - nH^2 = n(n-1)H^2 + n(n-1)(R-c). \tag{12}$$

Moreover, with respect to a (local) orthonormal frame $\{e_1, \ldots, e_n\}$ on M^n such that $h_{ij} = \lambda_i \delta_{ij}$, we have that $\Phi_{ij} = \mu_i \delta_{ij}$ and, with a straightforward computation, we verify that

$$\begin{aligned} &\sum_i \mu_i = 0, \\ &\sum_i \mu_i^2 = |\Phi|^2, \\ &\sum_i \mu_i^3 = \sum_i \lambda_i^3 - 3H|\Phi|^2 - nH^3. \end{aligned} \tag{13}$$

The components h_{ijk} of the covariant derivative ∇B satisfy

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{ik}\omega_{kj} + \sum_k h_{jk}\omega_{ki}. \tag{14}$$

The *Codazzi equation* and the *Ricci identity* are, respectively, given by

$$h_{ijk} = h_{ikj} \tag{15}$$

and

$$h_{ijkl} - h_{ijlk} = \sum_m h_{mj} R_{mikl} + \sum_m h_{im} R_{mjkl}, \tag{16}$$

where h_{ijk} and h_{ijkl} denote the first and the second covariant derivatives of h_{ij}.

The Laplacian Δh_{ij} of h_{ij} is defined by $\Delta h_{ij} = \sum_k h_{ijkk}$. From equations (15) and (16), we obtain that

$$\Delta h_{ij} = \sum_k h_{kkij} + \sum_{k,l} h_{kl} R_{lijk} + \sum_{k,l} h_{li} R_{lkjk}. \tag{17}$$

Since

$$\Delta |B|^2 = 2\Big(\sum_{i,j} h_{ij}\Delta h_{ij} + \sum_{i,j,k} h_{ijk}^2 \Big),$$

from (17), we get

$$\frac{1}{2}\Delta|B|^2 = |\nabla B|^2 + \sum_{i,i,k} h_{ij}h_{kkij} + \sum_{i,j,k,l} h_{ij}h_{lk}R_{lijk} \tag{18}$$

$$+ \sum_{i,j,k,l} h_{ij}h_{il}R_{lkjk}. \tag{19}$$

Consequently, taking a (local) orthonormal frame $\{e_1, \ldots, e_n\}$ on M^n such that $h_{ij} = \lambda_i \delta_{ij}$, from equation (18), we obtain the following Simons-type formula

$$\frac{1}{2}\Delta|B|^2 = |\nabla B|^2 + \sum_i \lambda_i (nH)_{,ii} + \frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2. \tag{20}$$

Now, let $\phi = \sum_{i,j} \phi_{ij}\omega_i \otimes \omega_j$ be a symmetric tensor on M^n defined by

$$\phi_{ij} = nH\delta_{ij} - h_{ij}.$$

Following Cheng and Yau **[10]**, we introduce an operator $\square$ associated to ϕ acting on any smooth function f by

$$\square f = \sum_{i,j} \phi_{ij} f_{ij} = \sum_{i,j} (nH\delta_{ij} - h_{ij}) f_{ij}. \tag{21}$$

Since ϕ_{ij} is divergence-free, it follows from **[10]** that the operator $\square$ is self-adjoint relative to the L^2 inner product of M^n, that is,

$$\int_M f\square g = \int_M g \square f,$$

for any smooth functions f and g on M^n.

Setting $f = nH$ in (21) and taking a local frame field $\{e_1, \ldots, e_n\}$ on

M^n such that $h_{ij} = \lambda_i \delta_{ij}$, from equation (11), we obtain the following:

$$\begin{aligned}\square(nH) &= nH\Delta(nH) - \sum_i \lambda_i (nH)_{,ii} \\ &= \frac{1}{2}\Delta(nH)^2 - \sum_i (nH)_{,i}^2 - \sum_i \lambda_i (nH)_{,ii} \\ &= \frac{1}{2}\Delta|B|^2 - \frac{n(n-1)}{2}\Delta R - n^2|\nabla H|^2 \\ &\quad - \sum_i \lambda_i (nH)_{,ii}.\end{aligned}$$

Consequently, taking into account equation (20), we get

$$\begin{aligned}\square(nH) = |\nabla B|^2 - n^2|\nabla H|^2 - \frac{n(n-1)}{2}\Delta R \\ + \frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2.\end{aligned} \tag{22}$$

3. Proofs of Theorems 1.1 and 1.2. In order to prove our results, we will need some auxiliary lemmas. The first one is a classic algebraic lemma due to Okumura in **[17]**, and completed with the analysis of the equality case on paper **[3]** due to Alencar and do Carmo.

Lemma 3.1. *Let $\mu_1, \ldots, \mu_n$ be real numbers such that $\sum_i \mu_i = 0$ and $\sum_i \mu_i^2 = \beta^2$, with $\beta \geq 0$. Then, we have*

$$-\frac{(n-2)}{\sqrt{n(n-1)}}\beta^3 \leq \sum_i \mu_i^3 \leq \frac{(n-2)}{\sqrt{n(n-1)}}\beta^3. \tag{23}$$

Moreover, equality holds in (23) if, and only if, either at least $(n-1)$ of the numbers μ_i are equal.

To obtain the second lemma, we will reason as in the proof of Lemma 2.1 of **[14]**.

Lemma 3.2. *Let M^n be a linear Weingarten spacelike hypersurface in a space form $L_1^{n+1}(c)$, such that $R = aH + b$ for some $a, b \in \mathbb{R}$. Suppose that*

$$(n-1)a^2 + 4n(c-b) \geq 0. \tag{24}$$

Then

$$|\nabla B|^2 \geq n^2|\nabla H|^2. \tag{25}$$

Moreover, if the inequality (24) *is strict and the equality holds in* (25) *on* M^n*, then* H *is constant on* M^n*.*

Proof. Since we are supposing that $R = aH + b$, from equation (11), we get

$$2\sum_{i,j} h_{ij}h_{ijk} = \left(2n^2H + n(n-1)a\right) H_{,k}.$$

Thus,

$$4\sum_k \left(\sum_{i,j} h_{ij}h_{ijk}\right)^2 = \left(2n^2H + n(n-1)a\right)^2 |\nabla H|^2.$$

Consequently, using the Cauchy-Schwartz inequality, we obtain that

$$\begin{aligned} 4|B|^2|\nabla B|^2 &= 4\Big(\sum_{i,j} h_{ij}^2\Big)\Big(\sum_{i,j,k} h_{ijk}^2\Big) \\ &\geq 4\sum_k \left(\sum_{i,j} h_{ij}h_{ijk}\right)^2 \\ &= \left(2n^2H + n(n-1)a\right)^2 |\nabla H|^2. \end{aligned} \tag{26}$$

On the other hand, since $R = aH + b$, from equation (11), we easily see that

$$\left(2n^2H + n(n-1)a\right)^2 = n^2(n-1)\left((n-1)a^2 + 4n(c-b)\right) + 4n^2|B|^2.$$

Consequently, from (26), we have

$$|B|^2|\nabla B|^2 \geq n^2|B|^2|\nabla H|^2.$$

Therefore, we obtain either $|B| = 0$ and $|\nabla B|^2 = n^2|\nabla H|^2$ or $|\nabla B|^2 \geq n^2|\nabla H|^2$. Moreover, if $(n-1)a^2 + 4n(c-b) > 0$, from the previous identity we get that $(2n^2H + n(n-1)a)^2 > 4n^2|B|^2$. Consequently, if $|\nabla B|^2 = n^2|\nabla H|^2$ holds on M^n, from (26) we conclude that $\nabla H = 0$ on M^n and, hence, H is constant on M^n. $\square$

In what follows, we will consider Cheng and Yau's modified operator

$$L = \square + \frac{n-1}{2} a\Delta. \tag{27}$$

Related to such an operator, we have the following sufficient criteria of ellipticity for the L operator.

Lemma 3.3. *Let M^n be a linear Weingarten spacelike hypersurface immersed in a Lorentzian space form $L_1^{n+1}(c)$, such that $R = aH + b$ with $b < c$. Then, L is elliptic.*

Proof. From equation (11), since $R = aH + b$ with $b < c$, we easily see that H cannot vanish on M^n and, by choosing the appropriate Gauss mapping, we may assume that $H > 0$ on M^n.

Let us consider the case that $a = 0$. Since $R = b < c$, from equation (11), if we choose a (local) orthonormal frame $\{e_1, \dots, e_n\}$ on M^n such that $h_{ij} = \lambda_i \delta_{ij}$, we have that $\sum_{i<j} \lambda_i \lambda_j > 0$. Consequently,

$$n^2 H^2 = \sum_i \lambda_i^2 + 2 \sum_{i<j} \lambda_i \lambda_j > \lambda_i^2$$

for every $i \in \{1, \dots, n\}$ and, hence, we have that $nH - \lambda_i > 0$ for every i. Therefore, in this case, we conclude that L is elliptic.

Now, suppose that $a \neq 0$. From equation (11), we get that

$$a = \frac{1}{n(n-1)H} \left(|B|^2 - n^2 H^2 + n(n-1)(c-b) \right).$$

Consequently, for every $i \in \{1, \dots, n\}$, with a straightforward algebraic computation, we verify that

$$\begin{aligned} nH - \lambda_i + \frac{n-1}{2} a & \\ &= nH - \lambda_i + \frac{1}{2nH} \left(|B|^2 - n^2 H^2 + n(n-1)(c-b) \right) \\ &= \frac{1}{2nH} \left(\sum_{j \neq i} \lambda_j^2 + (\sum_{j \neq i} \lambda_j)^2 + n(n-1)(c-b) \right). \end{aligned}$$

Therefore, since $b<c$, we also conclude in this case that L is elliptic. $\square$

In what follows, we present some computations which are common for the proofs of Theorems 1.1 and 1.2. At this point, we assume that M^n is a complete linear Weingarten spacelike hypersurface immersed in a Lorentzian space form $L_1^{n+1}(c)$, $n \geq 3$, such that $R = aH + b$ with $(n-1)a^2 + 4n(c-b) \geq 0$.

In this setting, let us choose a (local) orthonormal frame $\{e_1, \dots, e_n\}$ on M^n such that $h_{ij} = \lambda_i \delta_{ij}$. Since $R = aH + b$, from (22) and (27), we have that

$$L(nH) = |\nabla B|^2 - n^2|\nabla H|^2 + \frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2. \tag{28}$$

Thus, since from Gauss equation (8) we have that $R_{ijij} = (c - \lambda_i\lambda_j)(1 - \delta_{ij})$, we can rewrite equation (28) in the following way

$$L(nH) = |\nabla B|^2 - n^2|\nabla H|^2 + nc(|B|^2 - nH^2) + |B|^4 - nH\sum_i \lambda_i^3. \tag{29}$$

Consequently, taking into account (13), from (29), we get

$$\begin{aligned} L(nH) = |\nabla B|^2 - n^2|\nabla H|^2 - nH\sum_i \mu_i^3 \\ + |\Phi|^2(|\Phi|^2 - nH^2 + nc). \end{aligned} \tag{30}$$

Thus, by applying Lemmas 3.1 and 3.2, from (30), we have

$$L(nH) \geq |\Phi|^2\left(|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi| - nH^2 + nc\right). \tag{31}$$

On the other hand, from (12), we obtain

$$H^2 = \frac{1}{n(n-1)}|\Phi|^2 - (R-c). \tag{32}$$

As observed at the beginning of the proof of Lemma 3.3, we can assume that $H > 0$ on M^n. Thus,

$$H = \frac{1}{\sqrt{n(n-1)}}\sqrt{|\Phi|^2 - n(n-1)(R-c)}. \tag{33}$$

Hence, from (31), (32) and (33), we get

$$L(H) \geq \frac{1}{n(n-1)}|\Phi|^2 P_R\left(|\Phi|\right), \tag{34}$$

where

$$P_R(x) = (n-2)x^2 - (n-2)x\sqrt{x^2 - n(n-1)(R-c)} + n(n-1)R. \tag{35}$$

Furthermore, we observe that $P_R(x) = 0$ if, and only if,

$$x^2 = \frac{n(n-1)R^2}{(n-2)((n-2)c - nR)}. \tag{36}$$

Now, we proceed with the proof of Theorem 1.1:

Proof of Theorem 1.1.

Initially, since we are supposing that $0 < R < 1-2/n$, from (35) and (36), we obtain that $P_R(0) = n(n-1)R > 0$, and the function $P_R(x)$ is strictly decreasing for $x \geq 0$, with $P_R(\widetilde{x}) = 0$ at

$$\widetilde{x} = R\sqrt{\frac{n(n-1)}{(n-2)((n-2) - nR)}} > 0.$$

Thus, hypothesis (1) ensures that $0 \leq |\Phi| \leq \widetilde{x}$ and $P_R(|\Phi|) \geq 0$. Then, from (34), we have

$$L(H) \geq \frac{1}{n(n-1)}|\Phi|^2 P_R\left(|\Phi|\right) \geq 0. \tag{37}$$

Since we are supposing $b < 1$, Lemma 3.3 guarantees that L is elliptic. So, since we are also assuming that H attains its maximum on M^n, from (37) we can apply Hopf's strong maximum principle in order to conclude that H is constant on M^n.

If $|\Phi| < \widetilde{x}$, then from (37), we have that $|\Phi| = 0$ and, hence, M^n is totally umbilical.

If $|\Phi| = \widetilde{x}$, since the equality holds in (23) of Lemma 3.1, we conclude that M^n is either totally umbilical or an isoparametric spacelike hypersurface with two distinct principal curvatures, one of which is simple.

Therefore, by the classical congruence theorem due to Abe, Koike and Yamaguchi (cf., [**1,** Theorem 5.1]) and, since we are supposing $R > 0$, we conclude that either $|\Phi| = 0$ and M^n is totally umbilical, or $|\Phi| = \widetilde{x}$ and M^n is isometric to a hyperbolic cylinder $\mathbb{H}^1(c_1) \times \mathbb{S}^{n-1}(c_2)$, where $c_1 < 0$, $c_2 > 0$ and $\frac{1}{c_1} + \frac{1}{c_2} = 1$. $\square$

Since the operator L is self-adjoint relative to the L^2 inner product of the spacelike hypersurface M^n, from inequality (37) and taking into account the description of the totally umbilical spacelike hypersurfaces of $\mathbb{S}_1^{n+1}$ given by Montiel in [**15,** Example 1], we also get the following result in the de Sitter space:

Corollary 3.4. *Let M^n be a compact linear Weingarten spacelike hypersurface in $\mathbb{S}_1^{n+1}$, $n \geq 3$, such that $R = aH + b$ with $(n-1)a^2 + 4n(1-b) \geq 0$. Suppose that $0 < R < 1 - \frac{2}{n}$. If*

$$\sup_M |\Phi|^2 < \frac{n(n-1)R^2}{(n-2)((n-2)-nR)},$$

then $|\Phi| \equiv 0$ and M^n is isometric to $\mathbb{S}^n$, up to scaling.

We conclude our paper by presenting the proof of Theorem 1.2.

Proof of Theorem 1.2. First, since we are assuming $R < 0$, we have that $P_R(0) = n(n-1)R < 0$, and the function $P_R(x)$ is strictly increasing for $x \geq 0$, with $P_R(\widehat{x}) = 0$ at

$$\widehat{x} = -R\sqrt{\frac{n(n-1)}{(n-2)((n-2)c - nR)}} > 0.$$

Thus, the hypothesis (2) guarantees that $|\Phi| \geq \widehat{x} > 0$ and $P_R(|\Phi|) \geq 0$. Hence, from (34), we obtain

$$L(H) \geq \frac{1}{n(n-1)}|\Phi|^2 P_R\left(|\Phi|\right) \geq 0. \tag{38}$$

In a similar way as the proof of Theorem 1.1, since we are supposing that $b < c$, we can apply Hopf's strong maximum principle to guarantee that H is constant on M^n. Moreover, since we are assuming that $|\Phi| > 0$, from (38), we obtain that $L(H) \geq 0$ if, and only if, $|\Phi| = \hat{x}$.

Therefore, Lemma 3.1 assures that M^n is an isoparametric spacelike hypersurface with two distinct principal curvatures, one of which is simple and, hence, using again [**1,** Theorem 5.1] and taking into account that $R < 0$, we conclude that $|\Phi| = \widehat{x}$ and M^n must be isometric to:

(a) $\mathbb{S}^1(c_1) \times \mathbb{H}^{n-1}(c_2)$, where $c_1 > 0$, $c_2 < 0$ and $\frac{1}{c_1} + \frac{1}{c_2} = 1$, when $c = 1$;

(b) $\mathbb{R} \times \mathbb{H}^{n-1}(c_2)$, where $c_2 < 0$, when $c = 0$;
(c) $\mathbb{H}^1(c_1) \times \mathbb{H}^{n-1}(c_2)$, where $c_1 < 0$, $c_2 < 0$ and $\frac{1}{c_1} + \frac{1}{c_2} = -1$, when $c = -1$. $\square$

Acknowledgments. The authors would like to thank the referee for having made valuable comments concerning the results of this paper.

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