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THE EMBEDDING OF $\mathcal{O}_{d_1 d_2}$ INTO $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$

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ABSTRACT. As a particular example of a general theorem presented in **[2]**, there is a conditional expectation from the tensor product of Cuntz algebras, $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$, onto the Cuntz algebra $\mathcal{O}_{d_1 d_2}$. Motivated by this example, we examine the embedding of $\mathcal{O}_{d_1 d_2}$ in $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$, first by examining the index of the conditional expectation mentioned, and then by expressing $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ as a concrete Paschke crossed product by an endomorphism and then abstractly as the image of a faithful representation of the Stacey crossed product of $\mathcal{O}_{d_1 d_2}$ by the same endomorphism.

1. Introduction. The aim of this paper is to study in more detail an embedding of a copy of $\mathcal{O}_{d_1 d_2}$ in $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ that was studied in greater generality in **[2]**. In that paper, it was shown that, if E_1 and E_2 are directed graphs, then there is a conditional expectation from the tensor product of the graph algebras $C^*(E_1)$ and $C^*(E_2)$ onto the C^*-algebra $\mathcal{D}$, where

$$\mathcal{D} = \overline{\operatorname{span}}\, \{S_\mu S_\nu^* \otimes \widetilde{S_\alpha}\widetilde{S_\beta}^* : s(\mu) = s(\nu), s(\alpha) = s(\beta), \\ |\mu| - |\nu| = |\alpha| - |\beta|\}.$$

We form a directed graph $\mathcal{E}$ by constructing the Cartesian product of E_1 and E_2 as follows: $\mathcal{E} = (\mathcal{E}^0, \mathcal{E}^1, r, s)$ where $\mathcal{E}^0 = \{(v, w) : v \in E_1^0, w \in E_2^0\}$ and $\mathcal{E}^1 = \{(e, f) : e \in E_1^1, f \in E_2^1\}$ with $s(e, f) = (s(e), s(f))$ and $r(e, f) = (r(e), r(f))$. If E_1 and E_2 are row-finite, then so is $\mathcal{E}$. If $\mathcal{E}$ is the directed graph formed by constructing the Cartesian product of E_1 and E_2, with the additional hypothesis that $\mathcal{E}$ has no sources, we have that $\mathcal{D} \cong C^*(\mathcal{E})$. The following theorem summarizes this.

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Theorem 1.1. *Let E_1 and E_2 be row-finite directed graphs such that every cycle has an entry. Let $\{P_v, S_e\}$ be the Cuntz-Krieger E_1-system corresponding to $C^*(E_1)$, and let $\{\widetilde{P_w}, \widetilde{S_f}\}$ be the Cuntz-Krieger E_2-system corresponding to $C^*(E_2)$. Further, assume that $P_v \neq 0$ for every $v \in E_1^0$ and $\widetilde{P_w} \neq 0$ for every $w \in E_2^0$. If $\mathcal{E}$ has no sources, there exists a conditional expectation from $C^*(E_1) \otimes C^*(E_2)$ to a subalgebra $\mathcal{D}$ isomorphic to $C^*(\mathcal{E})$ for $\mathcal{E}$ defined as the graph formed by constructing the Cartesian product of E_1 and E_2. If $\mathcal{E}$ has sources, then we can still find a conditional expectation from $C^*(E_1) \otimes C^*(E_2)$ onto $\mathcal{D} = \overline{\operatorname{span}}\,\{S_\mu S_\nu^* \otimes \widetilde{S_\alpha}\widetilde{S_\beta}^* : s(\mu) = s(\nu),\ s(\alpha) = s(\beta),\ |\mu| - |\nu| = |\alpha| - |\beta|\}$, but $\mathcal{D}$ need not be isomorphic to $C^*(\mathcal{E})$.*

As a particular application of this theorem, taking E_1 and E_2 to be the graphs with one vertex and d_1 and d_2 loops, respectively, we obtain that there exists a conditional expectation from the tensor product of Cuntz algebras, $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$, onto the Cuntz algebra $\mathcal{O}_{d_1 d_2}$. In addition, it is possible to realize $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ as the Paschke crossed product of $\mathcal{O}_{d_1 d_2}$ by a canonical endomorphism. We examine this particular example further and study in greater depth the embedding of the subalgebra isomorphic to $\mathcal{O}_{d_1 d_2}$ in $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$. In Section 2, we describe briefly how for $d_1,\ d_2 \geq 2$, it is impossible for $\mathcal{O}_{d_1} d_2$ to be isomorphic to the tensor product $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$. In Section 3 we give details of a proof, first sketched to us by Watatani, that the conditional expectation associated to this particular embedding of $\mathcal{O}_{d_1 d_2}$ in $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ is not of index finite type. We conclude in Section 4 by first establishing that $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ can be viewed as a Paschke crossed product of $\mathcal{O}_{d_1 d_2}$ by an endomorphism, and finally by establishing a much more delicate fact: that $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ can be expressed as the image of a faithful representation of the Stacey crossed product of $\mathcal{O}_{d_1 d_2}$ by the same endomorphism. We thank Astrid an Huef and Iain Raeburn who allowed us access to their preprint at an early stage. It is our hope that by establishing that this representation of the Stacey crossed product is faithful, the manner in which $\mathcal{O}_{d_1 d_2}$ embeds in $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ is made more clear and that there will be generalizations to different types of graph algebras.

2. Comments on $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ and $\mathcal{O}_{d_1 d_2}$.

2.1. $\mathcal{O}_{d_1 d_2}$ as a subalgebra of $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$. Let $\mathcal{O}_{d_1}$ and $\mathcal{O}_{d_2}$ be Cuntz algebras, that is, $\mathcal{O}_{d_1} = C^*(\{S_i : 0 \leq i \leq d_1 - 1\})$ and $\mathcal{O}_{d_2} = C^*(\{\widetilde{S_j} : 0 \leq j \leq d_2 - 1\})$ where the S_i's and $\widetilde{S_j}$'s are isometries satisfying the Cuntz relations. In **[3]**, Cuntz showed that $\mathcal{O}_n = \overline{\text{span}}\,\{S_\mu S_\nu^* : \mu, \nu \text{ are words in } \{0, \ldots, n-1\}\}$. Using this and the fact that $\mathcal{O}_n$ is nuclear,

$$\begin{aligned}\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2} = \overline{\text{span}}\,\{S_\mu S_\nu^* \otimes \widetilde{S_\alpha}\widetilde{S_\beta}^* : \mu, \nu &\text{ are words in } \{0, \ldots, d_1 - 1\} \\ &\text{and } \alpha, \beta \text{ are words in } \{0, \ldots, d_2 - 1\}\}.\end{aligned}$$

It can also be shown that

$$\begin{aligned}\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2} = C^*(&\{S_i \otimes \text{Id} : 0 \leq i \leq d_1 - 1\} \\ &\cup \{\text{Id} \otimes \tilde{S}_j : 0 \leq j \leq d_2 - 1\}).\end{aligned}$$

Since this paper is centered on investigating $\mathcal{O}_{d_1 d_2}$ as a subalgebra of $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$, it is important to determine what a subalgebra of $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ that is isomorphic to $\mathcal{O}_{d_1 d_2}$ might look like. From **[2]**, we see that

$$\begin{aligned}\mathcal{O}_{d_1 d_2} \cong \overline{\text{span}}\,\{S_\mu S_\nu^* \otimes \widetilde{S_\alpha}\widetilde{S_\beta}^* &: s(\mu) = s(\nu), \\ &s(\alpha) = s(\beta),\ |\mu| - |\nu| = |\alpha| - |\beta|\} \\ &= C^*(\{S_i \otimes \widetilde{S_j} : 0 \leq i \leq d_1 - 1,\ 0 \leq j \leq d_2 - 1\}).\end{aligned}$$

2.2. Structure of $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ and $\mathcal{O}_{d_1 d_2}$. For $d_1, d_2 \geq 2$, we would like to show that $\mathcal{O}_{d_1 d_2}$ is not isomorphic to $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$. The easiest way to see this is to use some results from K-theory for C^*-algebras. Due to the Kunneth formula, which can be found in **[11]**, we know there is a short exact sequence that helps us determine the structure of $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$. Also, from sources such as **[10, 15]** we know that $K_0(\mathcal{O}_{d_1}) = \mathbf{Z}/(d_1 - 1)$, $K_0(\mathcal{O}_{d_2}) = \mathbf{Z}/(d_2 - 1)$, $K_1(\mathcal{O}_{d_1}) = 0$, and $K_1(\mathcal{O}_{d_2}) = 0$. We have the short exact sequence:

$$\begin{aligned}0 \longrightarrow K_0(\mathcal{O}_{d_1}) \otimes K_0(\mathcal{O}_{d_2}) \longrightarrow &K_0(\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}) \\ &\longrightarrow \text{Tor}\,(K_1(\mathcal{O}_{d_1}), K_0(\mathcal{O}_{d_2})) \longrightarrow 0.\end{aligned}$$

But, $K_1(\mathcal{O}_{d_1}) = 0$, and so $\mathrm{Tor}\,(K_1(\mathcal{O}_{d_1}), K_0(\mathcal{O}_{d_2})) = \mathrm{Tor}\,(0, K_0(\mathcal{O}_{d_2})) = 0$. Then, we have the short exact sequence

$$0 \to \mathbf{Z}/(d_1 - 1) \otimes \mathbf{Z}/(d_2 - 1) \to K_0(\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}) \to 0.$$

Therefore, $K_0(\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2})$ is isomorphic to $\mathbf{Z}/(d_1 - 1) \otimes \mathbf{Z}/(d_2 - 1)$. Now, we know that a Cuntz algebra and a tensor product of Cuntz algebras are simple, purely infinite, separable and classifiable. Then, by [**9**], a Cuntz algebra is isomorphic to a tensor product of two other Cuntz algebras if and only if they have isomorphic k-theory. But, $K_0(\mathcal{O}_{d_1 d_2}) = \mathbf{Z}/(d_1 d_2 - 1)$ and $K_0(\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}) = \mathbf{Z}/(d_1 - 1) \otimes \mathbf{Z}/(d_2 - 1)$. From [**4**], $\mathbf{Z}/m\mathbf{Z} \otimes \mathbf{Z}/n\mathbf{Z}$ is isomorphic to $\mathbf{Z}/d\mathbf{Z}$ where $d = gcd(m, n)$. In general, it is not the case that $d_1 d_2 - 1 = \mathrm{gcd}\,(d_1 - 1, d_2 - 1)$. So, in general, $\mathcal{O}_{d_1 d_2}$ is not isomorphic to $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$.

It is also interesting to note that $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ is not even necessarily a Cuntz algebra. We stated above that the K_1-group of a Cuntz algebra is 0. This is not in general the case with $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$. By the Kunneth formula, we have the exact sequence

$$0 \to 0 \otimes \mathbf{Z}/(d_2 - 1) \to K_1(\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}) \to \mathrm{Tor}\,(\mathbf{Z}/(d_1 - 1), \mathbf{Z}/(d_2 - 1)) \to 0.$$

By [**5**], $\mathrm{Tor}\,(\mathbf{Z}/(d_1 - 1), \mathbf{Z}/(d_2 - 1))$ is isomorphic to $\mathbf{Z}/q\mathbf{Z}$ where $q = \mathrm{gcd}\,(d_1 - 1, d_2 - 1)$. So, we have the exact sequence

$$0 \longrightarrow 0 \longrightarrow K_1(\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}) \longrightarrow \mathbf{Z}/q\mathbf{Z} \longrightarrow 0.$$

Thus, $K_1(\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2})$ is isomorphic to $\mathbf{Z}/q\mathbf{Z}$ where $q = gcd(d_1 - 1, d_2 - 1)$. Now, $\mathbf{Z}/q\mathbf{Z}$ is isomorphic to 0 if and only if $q = 1$. So, $K_1(\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}) = 0$ if and only if $\mathrm{gcd}\,(d_1 - 1, d_2 - 1) = 1$. Then, $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ need not even be a Cuntz algebra unless $\mathrm{gcd}\,(d_1 - 1, d_2 - 1) = 1$. In this case, the deep results of Kirchberg and Rordam (see [**9**]) show that if $\mathrm{gcd}\,(d_1 - 1, d_2 - 1) = 1$, then $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ is isomorphic to $\mathcal{O}_{\mathrm{gcd}\,(d_1 - 1, d_2 - 1) + 1}$. From this, we see that $\mathcal{O}_2 \otimes \mathcal{O}_2$ is a Cuntz algebra since $\mathrm{gcd}\,(2 - 1, 2 - 1) = 1$ implies that $\mathcal{O}_2 \otimes \mathcal{O}_2$ is isomorphic to $\mathcal{O}_2$. More generally, $\mathcal{O}_n \otimes \mathcal{O}_2$ is isomorphic to $\mathcal{O}_2$ for any n. This certainly makes sense because $K_1(\mathcal{O}_n \otimes \mathcal{O}_2) = \mathbf{Z}/\mathbf{Z} \cong 0$ since $\mathrm{gcd}\,(n - 1, 2 - 1) = 1$. But, in general, it is not the case that $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ is isomorphic to a Cuntz algebra.

3. Index of the conditional expectation.

3.1. Definition of the index of a conditional expectation. In [**2**], we define an action α of $\mathbf{T}$ on $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ whose fixed point subalgebra is isomorphic to $\mathcal{O}_{d_1 d_2}$. This action α acts as follows on the generators of $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$:

$$\alpha_z(S_\mu {S_\nu}^* \otimes \widetilde{S_\alpha}\widetilde{S_\beta}^*) = z^{|\mu|-|\nu|-(|\alpha|-|\beta|)} S_\mu {S_\nu}^* \otimes \widetilde{S_\alpha}\widetilde{S_\beta}^*.$$

Then

$$\Phi(a) = \int_{\mathbf{T}} \alpha_z(a)\, dz$$

defines a conditional expectation from $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ to a subalgebra isomorphic to $\mathcal{O}_{d_1 d_2}$. We will now examine the topic of the index of this conditional expectation, a concept first defined by Watatani in [**14**]. We thank Yasuo Watatani for indicating to us the key elements of the proof of the main result in this section and for allowing us to reproduce it here ([**13**]).

Definition 3.1. Let ϕ be a conditional expectation from A onto B. A finite family $\{(u_i, v_i) : i = 1, \dots, n\} \subseteq A \times A$ is a *quasi-basis* for the conditional expectation ϕ if

$$\sum_{i=1}^{n} u_i \phi(v_i a) = a = \sum_{i=1}^{n} \phi(a u_i) v_i \quad \text{for all } a \in A.$$

Definition 3.2. A conditional expectation $\phi : A \to B$ is of *index-finite type* if there exists a quasi-basis for ϕ. Otherwise we say it is *not of index-finite type.*

Definition 3.3. If a conditional expectation ϕ is of index-finite type, then the *index* of ϕ is

$$\operatorname{Index} \phi = \sum_{i=1}^{n} u_i v_i.$$

Using a proof by contradiction, we will show that the conditional expectation defined above is not of index finite-type. Before giving the proof by contradiction, let us state and prove some useful lemmas.

3.2. Useful lemmas. Let the set $\{S_i \otimes \mathrm{Id}, \mathrm{Id} \otimes \widetilde{S_j} : 0 \leq i \leq d_1 - 1,\ 0 \leq j \leq d_2 - 1\}$ be the set of canonical generators of $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$. Let $a = (a_k)_{k \in \mathbf{Z}}$ be a numerical bi-sequence of finite support. Let $f_a \in C(\mathbf{T})$ be the element of the set of continuous functions on $\mathbf{T}$ defined by $f_a(z) = \sum_k a_k z^k$. Further, define $T_a \in \mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ by $T_a = \sum_{j>0} a_{-j}(S_0 \otimes \mathrm{Id})^{*j} + \sum_{j \geq 0} a_j (S_0 \otimes \mathrm{Id})^j$.

Lemma 3.4. $\Phi(T_a^* T_a) = \sum_{k>0} |a_{-k}|^2 (S_0 \otimes \mathrm{Id})^k (S_0 \otimes \mathrm{Id})^{*j} + \sum_{k \geq 0} |a_k|^2 (S_0 \otimes \mathrm{Id})^{*k} (S_0 \otimes \mathrm{Id})^k$.

Proof.

$$\begin{aligned}
T_a^* T_a &= \Big(\sum_{k>0} \overline{a_{-k}} (S_0 \otimes \mathrm{Id})^k + \sum_{k \geq 0} \overline{a_k} (S_0 \otimes \mathrm{Id})^{*k} \Big) \\
&\qquad \Big(\sum_{j>0} a_{-j} (S_0 \otimes \mathrm{Id})^{*j} + \sum_{j \geq 0} a_j (S_0 \otimes \mathrm{Id})^j \Big) \\
&= \sum_{k>0} \sum_{j>0} \overline{a_{-k}} a_{-j} (S_0 \otimes \mathrm{Id})^k (S_0 \otimes \mathrm{Id})^{*j} \\
&\quad + \sum_{k>0} \sum_{j \geq 0} \overline{a_{-k}} a_j (S_0 \otimes \mathrm{Id})^k (S_0 \otimes \mathrm{Id})^j \\
&\quad + \sum_{k \geq 0} \sum_{j \geq 0} \overline{a_k} a_{-j} (S_0 \otimes \mathrm{Id})^{*k} (S_0 \otimes \mathrm{Id})^{*j} \\
&\quad + \sum_{k \geq 0} \sum_{j \geq 0} \overline{a_k} a_j (S_0 \otimes \mathrm{Id})^{*k} (S_0 \otimes \mathrm{Id})^j.
\end{aligned}$$

To find $\Phi(T_a^* T_a)$, consider how Φ acts on $(S_0 \otimes \mathrm{Id})^k (S_o \otimes \mathrm{Id})^{*j}$, $(S_0 \otimes \mathrm{Id})^k (S_0 \otimes \mathrm{Id})^j$, $(S_0 \otimes \mathrm{Id})^{*k} (S_0 \otimes \mathrm{Id})^{*j}$, and $(S_0 \otimes \mathrm{Id})^{*k} (S_0 \otimes \mathrm{Id})^j$. Note:

$$\begin{aligned}
\Phi[(S_0 \otimes \mathrm{Id})^k (S_0 \otimes \mathrm{Id})^{*j}] &= \int_{\mathbf{T}} \alpha_t [(S_0 \otimes \mathrm{Id})^k (S_0 \otimes \mathrm{Id})^{*j}] dt \\
&= \Big(\int_{\mathbf{T}} t^k \bar{t}^j \, dt \Big) (S_0 \otimes \mathrm{Id})^k (S_0 \otimes \mathrm{Id})^{*j}.
\end{aligned}$$

Direct calculation shows that $\Phi[(S_0 \otimes \mathrm{Id})^k (S_0 \otimes \mathrm{Id})^{*j})] = 0$ unless $k = j$. If $k = j$, then $\Phi[(S_0 \otimes \mathrm{Id})^k (S_0 \otimes \mathrm{Id})^{*j})] = (S_0 \otimes \mathrm{Id})^k (S_0 \otimes \mathrm{Id})^{*k}$.

Similar calculations yield that

$$\Phi[(S_0\otimes \mathrm{Id})^k(S_0\otimes \mathrm{Id})^j] = \Phi[(S_0\otimes \mathrm{Id})^{*k}(S_0\otimes \mathrm{Id})^{*j}] = 0$$

for any values of k and j, and that $\Phi[(S_0\otimes \mathrm{Id})^{*k}(S_0\otimes \mathrm{Id})^j] = 0$ unless $k = j$. If $k = j$, then $\Phi[(S_0\otimes \mathrm{Id})^{*k}(S_0\otimes \mathrm{Id})^j] = (S_0\otimes \mathrm{Id})^{*k}(S_0\otimes \mathrm{Id})^k$.

Now, let us calculate $\Phi(T_a^*T_a)$. When calculating

$$\Phi\left[\sum_{k>0}\sum_{j>0}\overline{a_{-k}}a_{-j}(S_0\otimes \mathrm{Id})^k(S_0\otimes \mathrm{Id})^{*j}\right],$$

we see that the image under ϕ of each of the terms in the sum equals zero unless $k = j$. Then the image of the double summation is just the image of the sum over $k > 0$, that is, the image of $\sum_{k>0}\overline{a_{-k}}a_{-k}(S_0\otimes \mathrm{Id})^k(S_0\otimes \mathrm{Id})^{*k} = \sum_{k>0}|a_{-k}|^2(S_0\otimes \mathrm{Id})^k(S_0\otimes \mathrm{Id})^{*k}$. Also, $\Phi[\sum_{k>0}\sum_{j\geq 0}\overline{a_{-k}}a_j(S_0\otimes \mathrm{Id})^k(S_0\otimes \mathrm{Id})^j] = 0$ and $\Phi[\sum_{k\geq 0}\sum_{j\geq 0}\overline{a_k}a_{-j}(S_0\otimes \mathrm{Id})^{*k}(S_0\otimes \mathrm{Id})^{*j}] = 0$. Now, consider $\Phi[\sum_{k\geq 0}\sum_{j\geq 0}\overline{a_k}a_j(S_0\otimes \mathrm{Id})^{*k}(S_0\otimes \mathrm{Id})^j]$. Again, the image under ϕ of each of the terms is zero unless $k = j$, giving the image of the double sum to be the image of $\sum_{k\geq 0}|a_k|^2(S_0\otimes \mathrm{Id})^{*k}(S_0\otimes \mathrm{Id})^k$. Putting all of this together, we see that

$$\begin{aligned}\Phi(T_a^*T_a) &= \sum_{k>0}|a_{-k}|^2(S_0\otimes \mathrm{Id})^k(S_0\otimes \mathrm{Id})^{*k}\\ &\quad + \sum_{k\geq 0}|a_k|^2(S_0\otimes \mathrm{Id})^{*k}(S_0\otimes \mathrm{Id})^k. \qquad \square\end{aligned}$$

Lemma 3.5. *Let $a = (a_k)_{k\in\mathbf{Z}}$ be a bi-sequence of finite support. Then* $\|\Phi(T_a^*T_a)\| = \|f_a\|_2^2$.

Proof. Using the Cuntz relations and properties of projections, it can be shown that $\|\Phi(T_a^*T_a)\| \leq \|f_a\|_2^2$.

We show that $\|\Phi(T_a^*T_a)\| \geq \|f_a\|_2^2$. For any j, we know that $(S_0\otimes \mathrm{Id})^j(S_0\otimes \mathrm{Id})^{*j}$ is a projection. Also, if $j > k$,

$$\begin{aligned}&[(S_0\otimes \mathrm{Id})^j(S_0\otimes \mathrm{Id})^{*j}][(S_0\otimes \mathrm{Id})^k(S_0\otimes \mathrm{Id})^{*k}]\\ &\qquad = (S_0\otimes \mathrm{Id})^j(S_0\otimes \mathrm{Id})^{*(j-k)}(S_0\otimes \mathrm{Id})^{*k}\\ &\qquad = (S_0\otimes \mathrm{Id})^j(S_0\otimes \mathrm{Id})^{*j}.\end{aligned}$$

Note also that $\mathrm{Id} \geq (S_0 \otimes \mathrm{Id})^k (S_0 \otimes \mathrm{Id})^{*k}$ for all $k > 0$.

Since the sequence of constants is of finite support, let N be the greatest integer such that $a_{-N} \neq 0$.

$$\begin{aligned}
\Phi(T_a^* T_a) &= \sum_{k=1}^{N} |a_{-k}|^2 (S_0 \otimes \mathrm{Id})^k (S_0 \otimes \mathrm{Id})^{*k} \\
&\quad + \sum_{k=0}^{\infty} |a_k|^2 (S_0 \otimes \mathrm{Id})^{*k} (S_0 \otimes \mathrm{Id})^k \\
&= \sum_{k=1}^{N} |a_{-k}|^2 (S_0 \otimes \mathrm{Id})^k (S_0 \otimes \mathrm{Id})^{*k} + \sum_{k=0}^{\infty} |a_k|^2 \\
&\geq \sum_{k=1}^{N} |a_{-k}|^2 (S_0 \otimes \mathrm{Id})^N (S_0 \otimes \mathrm{Id})^{*N} \\
&\quad + \left[\sum_{k=0}^{\infty} |a_k|^2 \right] (S_0 \otimes \mathrm{Id})^N (S_0 \otimes \mathrm{Id})^{*N} \\
&= \left[\sum_{k=1}^{N} |a_{-k}|^2 + \sum_{k=0}^{\infty} |a_k|^2 \right] (S_0 \otimes Id)^N (S_0 \otimes \mathrm{Id})^{*N}.
\end{aligned}$$

Then

$$\begin{aligned}
\|\Phi(T_a^* T_a)\| &\geq \left\| \left[\sum_{k=1}^{N} |a_{-k}|^2 + \sum_{k=0}^{\infty} |a_k|^2 \right] (S_0 \otimes \mathrm{Id})^N (S_0 \otimes \mathrm{Id})^{*N} \right\| \\
&= \sum_{k \in \mathbf{Z}} |a_k|^2.
\end{aligned}$$

With inequality in both directions, we have that

$$\|\Phi(T_a^* T_a)\| = \sum_{k \in \mathbf{Z}} |a_k|^2. \qquad \square$$

Recall that the Toeplitz algebra is the C^*-algebra generated by the unilateral shift S on the Hilbert space $\mathcal{H} = \ell^2(\mathbf{N})$. Let $\mathcal{K}(\mathcal{H})$ be the algebra of compact operators on $\mathcal{H}$, and consider the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. Let $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ be the natural homomorphism that takes each element in $\mathcal{B}(\mathcal{H})$ to its corresponding coset in $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. Let $U = \pi(S)$. It can be shown that U is unitary in $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$.

Lemma 3.6. $\|T_a^* T_a\| \geq \|\sum_{k\in\mathbf{Z}} a_k U^k\|^2$.

Proof. $S_0 \otimes Id$ is a nonunitary isometry on $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$, and it generates a copy of the Toeplitz algebra inside of $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$. Inside of this copy of the Toeplitz algebra, we may identify $S_0 \otimes \mathrm{Id}$ with its image S.

$$\begin{aligned}\|T_a^* T_a\| &\geq \|\pi(T_a^* T_a)\| = \|\pi(T_a)^*\pi(T_a)\| = \|\pi(T_a)\|^2 \\ &= \left\|\pi\left(\sum_{k>0} a_{-k} S^{*k} + \sum_{k\geq 0} a_k S^k\right)\right\|^2 \\ &= \left\|\sum_{k\in\mathbf{Z}} a_k U^k\right\|^2. \qquad \square\end{aligned}$$

Lemma 3.7. $\|\sum_{k\in\mathbf{Z}} a_k U^k\|^2 = \|f_a\|_\infty^2$.

Proof. From [**6**], we know that Spectrum$(S') = \mathbf{T}$ and that $C^*(S')$ is isomorphic to $C(\text{Spectrum}\,(S'))$, where S' is the unilateral shift operator. Since U is the image of $S\prime$ under isomorphism, this implies that $C^*(U)$ is isomorphic to $C(\text{Spectrum}\,(U)) = C(\mathbf{T})$. So, there exists an isomorphism $\psi : C^*(U) \to C(\mathbf{T})$ such that $\psi(U)(z) = z$. Then $\psi(U^k)(z) = z^k$ and $\psi(\sum_{k\in\mathbf{Z}} a_k U^k)(z) = \sum_{k\in\mathbf{Z}} a_k \psi(U^k)(z) = \sum_{k\in\mathbf{Z}} a_k z^k$. Then the element $\sum_{k\in\mathbf{Z}} a_k U^k$ in $C^*(U)$ is associated with the element $\sum_{k\in\mathbf{Z}} a_k z^k$ in $C(\mathbf{T})$ by the isomorphism ψ. Since isomorphisms preserve norm, $\|\sum_{k\in\mathbf{Z}} a_k U^k\|_{C^*(U)} = \|\sum_{k\in\mathbf{Z}} a_k z^k\|_{C(\mathbf{T})}$. Therefore,

$$\left\|\sum_{k\in\mathbf{Z}} a_k U^k\right\|^2 = \left\|\sum_{k\in\mathbf{Z}} a_k z^k\right\|^2 = \|f_a\|_\infty^2. \qquad \square$$

Lemma 3.8. [**14,** Proposition 2.1.5] (*We state it here for completeness.*) *Let $E : A \to B$ be a conditional expectation of index finite-type. Let $x \in A$. There exists a positive constant k such that $E(x^*x) \geq kx^*x$.*

Corollary 3.9. *Let a and T_a be as defined in the beginning of this section. For some positive constant k, $\|(\Phi(T_a^* T_a)\| \geq k\|T_a^* T_a\|$.*

Proof. By Lemma 3.8, $\Phi(T_a^*T_a) \geq kT_a^*T_a$ for some positive constant k. This implies that $\|\Phi(T_a^*T_a)\| \geq k\|T_a^*T_a\|$ since ${T_a}^*T_a$ is a positive operator. □

3.3. Proof that Φ is not of index-finite type.

Theorem 3.10. *Let Φ be the conditional expectation from $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ to $\mathcal{D} \cong \mathcal{O}_{d_1 d_2}$ defined by $\Phi(a) = \int_{\mathbf{T}} \alpha_z(a)\, dz$. Then Φ is not of index-finite type.*

Proof. Suppose that Φ is of index-finite type. From this assumption, we hope to derive a contradiction. We have the following string of inequalities:

$$\begin{aligned}
\|f_a\|_2^2 &= \|\Phi(T_a^*T_a)\| \quad \text{by Lemma 3.5}\\
&\geq k\|T_a^*T_a\| \quad \text{by Corollary 3.9}\\
&\geq k\|\sum_{k\in\mathbf{Z}} a_k U^k\|^2 \quad \text{by Lemma 3.6}\\
&= k\|f_a\|_\infty^2 \quad \text{by Lemma 3.7.}
\end{aligned}$$

But we also know that $|f_a(z)| \leq \sup_{z'\in\mathbf{T}} |f_a(z')| = \|f_a\|_\infty$. Then, $|f_a(z)| \leq \|f_a\|_\infty \Rightarrow |f_a(z)|^2 \leq \|f_a\|_\infty^2 \Rightarrow \int_{\mathbf{T}} |f_a(z)|^2\, dz \leq \int_{\mathbf{T}} \|f_a\|_\infty^2\, dz$. But, $\|f_a\|_2^2 = \int_{\mathbf{T}} |f_a(z)|^2\, dz$ and $\int_{\mathbf{T}} \|f_a\|_\infty^2\, dz = \|f_a\|_\infty^2$. Then, $\|f_a\|_2^2 \leq \|f_a\|_\infty^2$, and we have that $\|f_a\|_2^2 \geq k\|f_a\|_\infty^2$ and $\|f_a\|_2^2 \leq \|f_a\|_\infty^2$. Thus, $\|f_a\|_2 \geq \sqrt{k}\|f_a\|_\infty$ and $\|f_a\|_2 \leq \|f_a\|_\infty$. By the definition of equivalent norms, this implies that $\|\cdot\|_\infty$ and $\|\cdot\|_2$ are equivalent norms on $C(\mathbf{T})$, which we know is not true. We have arrived at our desired contradiction, and therefore our conditional expectation Φ is not of index-finite type. □

4. Expressing $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ as the crossed product of $\mathcal{O}_{d_1 d_2}$ by an endomorphism. In this section, we illustrate how $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ can be shown to be the faithful image of two different types of crossed product C^*-algebras: once as a very concrete Paschke crossed product and then as the image of a faithful representation of the more abstract Stacey crossed product by an endomorphism.

4.1. Key definitions of crossed products by endomorphisms.

Definition 4.1. Let $\mathcal{A}$ be a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is a separable Hilbert space, and let ω be a non-unitary isometry in $\mathcal{B}(\mathcal{H})$ such that $\omega\mathcal{A}\omega^* \subset \mathcal{A}$ and $\omega^*\mathcal{A}\omega \subset \mathcal{A}$. The map $\rho : \mathcal{A} \to \mathcal{A}$ defined by $\rho(a) = \omega a \omega^*$ is an endomorphism of $\mathcal{A}$ into itself, and we call the C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by $\mathcal{A}$ and ω the *spatial* or *Paschke crossed product of* $\mathcal{A}$ *by the endomorphism* ρ ([**7**]).

Definition 4.2. Let $\mathcal{A}$ be a C^*-algebra with an identity element, and let α be an endomorphism of $\mathcal{A}$. A *covariant representation* of $(\mathcal{A}, \alpha)$ is a pair (π, S) where π is a unital representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$, and S is an isometry on $\mathcal{H}$ satisfying $\pi(\alpha(a)) = S\pi(a)S^*$ for $a \in \mathcal{A}$ ([**1**]).

Remark 4.3. In the context of the Hilbert space representation, this definition looks very similar to Definition 4.1, the Paschke definition of a crossed product by an endomorphism. In this case, if π is a Hilbert space representation of A, then $C^*(\pi(A), S)$ is the Paschke crossed product of $\pi(A)$ by the endomorphism defined by $\pi(a) \to S\pi(a)S^*$. Note that, given a covariant representation, $\pi(\mathcal{A})$ and S would generate the Paschke crossed product of $\pi(\mathcal{A})$ by an endomorphism.

Definition 4.4. The (*Stacey*) *crossed product of* $\mathcal{A}$ *by* α is a triple $(\mathcal{B}, i_{\mathcal{A}}, t)$ in which $\mathcal{B}$ is a C^*-algebra with identity, $i_{\mathcal{A}} : \mathcal{A} \to \mathcal{B}$ is a unital homomorphism, and t is an isometry in $\mathcal{B}$ satisfying

(a) $i_{\mathcal{A}}(\alpha(a)) = t i_{\mathcal{A}}(a) t^*$ for $a \in \mathcal{A}$;
(b) for every covariant representation (π, S) of $(\mathcal{A}, \alpha)$ on $\mathcal{H}$, there is a unital representation $\pi \times S$ of $\mathcal{B}$ on $\mathcal{H}$ such that $(\pi \times S) \circ i_{\mathcal{A}} = \pi$ and $\pi \times S(t) = S$;
(c) t and $\{i_{\mathcal{A}}(a) : a \in \mathcal{A}\}$ generate $\mathcal{B}$.

The notation for the Stacey crossed product of $\mathcal{A}$ by the endomorphism α is $\mathcal{A} \times_\alpha \mathbf{N}$ ([**1**]).

4.2. Expressing $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ as a Paschke crossed product. Recall that, in this particular example,

$$\mathcal{O}_{d_1 d_2} \cong \mathcal{D} = \overline{\operatorname{span}}\,\{S_\mu S_\nu^* \otimes \widetilde{S_\alpha}\widetilde{S_\beta}^* : |\mu| - |\nu| = |\alpha| - |\beta|\}.$$

Let π_1 be a Hilbert space representation of $\mathcal{O}_{d_1}$ on $\mathcal{H}_1$, and let π_2 be a Hilbert space representation of $\mathcal{O}_{d_2}$ on $\mathcal{H}_2$. Then we have a Hilbert space representation $\pi_1 \otimes \pi_2$ of $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$. Writing $\pi_1 \otimes \pi_2(\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2})$ simply as $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ and $\pi_1 \otimes \pi_2(\mathcal{D})$ as $\mathcal{O}_{d_1 d_2}$, it can be shown that $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ is the crossed product of $\mathcal{O}_{d_1 d_2}$ by the endomorphism ρ where $\rho(a) = \omega a \omega^*$, with ω being the simple tensor $S_0 \otimes \mathrm{Id}$, one of the generators of $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$. To show this, it can be shown directly that $C^*(\mathcal{O}_{d_1 d_2}, S_0 \otimes \mathrm{Id}) = \mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$. Therefore, $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ is the Paschke crossed product of $\mathcal{O}_{d_1 d_2}$ by the endomorphism ρ. A complete proof of this can be found in [**2**].

4.3. Expressing $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ by means of a Stacey crossed product. Motivated by a question posed by Iain Raeburn, we investigate here whether or not this same example can be viewed in the context of a universally defined notion of a crossed product by an endomorphism. In [**1**], Boyd, Keswani and Raeburn summarize such a universal definition, first defined by Stacey in [**12**] and provide an example to show the advantage of this universal definition. We establish here the fact that $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$ can be viewed as the image of a faithful representation of the Stacey crossed product of $\mathcal{O}_{d_1 d_2}$ by an endomorphism.

4.3.1. The basic method. As stated in Remark 4.3, $\{\pi(\mathcal{A}), S\}$ generates a Paschke crossed product, where (π, S) is a covariant representation of (A, α). Using the notation from Definition 4.4, $\{i_A(\mathcal{A}), t\}$ generates the universal crossed product, called $\mathcal{B}$. Stacey shows in [**12**] that (A, α) has exactly one crossed product, up to isomorphism. Then there is a unital representation $\pi \times S$ of $\mathcal{B}$ on the Hilbert space $\mathcal{H}$ such that

$$(\pi \times S) \circ i_{\mathcal{A}} = \pi \quad \text{and} \quad \pi \times S(t) = S.$$

So, we have that $(\pi \times S)(i_{\mathcal{A}}(\mathcal{A})) = \pi(\mathcal{A})$. If $\pi \times S$ is faithful, these spaces are isomorphic. Then $C^*(i_A(\mathcal{A}), t) \cong C^*(\pi(\mathcal{A}), S)$ so that we have a faithful representation of the Stacey crossed product onto the Paschke crossed product.

4.3.2. Proof that $\pi \times S$ is faithful. In this case, the pair $(\pi_1 \otimes \pi_2, \pi_1 \otimes \pi_2(S_0 \otimes \mathrm{Id}))$ is a covariant representation of $(\mathcal{D}, \rho)$. For ease of notation, denote this covariant representation by (π, S). Let $\mathcal{B}$ be the Stacey crossed product. Then there is a unital representation $\pi \times S$ of $\mathcal{B}$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$. Using the following proposition from [**8**], we can show that this representation is faithful. Then we know that there is a faithful representation of the Stacey crossed product of $\mathcal{O}_{d_1d_2}$ onto $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$. Many thanks to Astrid an Huef and Iain Raeburn for giving me early notice of this forthcoming proposition, which applies to this example very nicely.

Proposition 4.5. *Suppose that α is an endomorphism of a unital C^*-algebra $\mathcal{A}$, and (π, V) is a covariant representation of $(\mathcal{A}, \alpha)$ in a C^*-algebra $\mathcal{D}$ such that π is faithful. If there is a strongly continuous action $\gamma : \mathbf{T} \to Aut\mathcal{D}$ such that $\gamma_z(\pi(a)) = \pi(a)$ and $\gamma_z(V) = zV$, then $\pi \times V$ is faithful on $\mathcal{A} \times_\alpha \mathbf{N}$ ([**8**]).*

Lemma 4.6. *Let (π, S) be a covariant representation of $(\mathcal{D}, \rho)$ in $\mathcal{B}(\mathcal{H})$ such that π is faithful. There exists an action of $\mathbf{T}$ on $\mathcal{D}$ satisfying the hypotheses of Proposition* 4.5.

Proof. The action α defined in Section 3, when restricted to acting on $\mathcal{D}$, satisfies the hypotheses of Proposition 4.5. The set of generators of $\mathcal{D} \cong \mathcal{O}_{d_1d_2}$ is the set

$$\{S_\mu S_\nu{}^* \otimes \widetilde{S}_\alpha \widetilde{S}_\beta^* : |\mu| - |\nu| = |\alpha| - |\beta|\}.$$

In this case:

$$\begin{aligned}\alpha_z(S_\mu S_\nu{}^* \otimes \widetilde{S_\alpha}\widetilde{S_\beta}^*) &= z^{|\mu|-|\nu|-(|\alpha|-|\beta|)} S_\mu S_\nu{}^* \otimes \widetilde{S_\alpha}\widetilde{S_\beta}^* \\ &= S_\mu S_\nu{}^* \otimes \widetilde{S_\alpha}\widetilde{S_\beta}^*.\end{aligned}$$

So for every $a \in \mathcal{D} \cong \mathcal{O}_{d_1d_2}$, we have $\alpha_z(a) = a$.

In addition, $\alpha_z(S_0 \otimes \mathrm{Id}) = z^{1-0-(0-0)} S_0 \otimes \mathrm{Id} = zS_0 \otimes \mathrm{Id}$. Therefore, the action α satisfies the hypotheses of Proposition 4.5. □

Proposition 4.7. *There is a faithful representation of the Stacey crossed product of $\mathcal{O}_{d_1d_2}$ onto $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$.*

Proof. With the action defined in Lemma 4.6, from Proposition 4.5, we can now conclude that the action π x S is faithful on the Stacey crossed product, $\mathcal{D} \times_\alpha \mathbf{N}$. Therefore, we have a faithful representation of $\mathcal{D} \times_\alpha \mathbf{N}$ onto the Paschke crossed product. As shown in subsection 4.1, the Paschke crossed product is isomorphic to $\mathcal{O}_{d_1} \otimes \mathcal{O}_{d_2}$. □

REFERENCES

1. Sarah Boyd, Navin Keswani and Iain Raeburn, *Faithful representations of crossed products by endomorphisms*, Proc. Amer. Math. Soc. **118** (1993), 427–436.

2. Amy Chambers, *Certain subalgebras of the tensor product of graph algebras*, J. Oper. Theor. **63** (2010), 389–402.

3. Joachim Cuntz, *Simple C^*-algebras generated by isometries*, Comm. Math. Phys. **57** (1977), 173–185.

4. David S. Dummit and Richard M. Foote, *Abstract algebra*, Prentice Hall Inc., Englewood Cliffs, NJ, 1991.

5. Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002.

6. Gerard J. Murphy, *C^*-algebras and operator theory*, Academic Press Inc., Boston, MA, 1990.

7. William L. Paschke, *The crossed product of a $C*$-algebra by an endomorphism*, Proc. Amer. Math. Soc. **80** (1980), 113–118.

8. Iain Raeburn and Astrid an Huef, preprint, June 2010.

9. M. Rørdam, *Classification of nuclear, simple C^*-algebras*,in *Classification of nuclear C^*-algebras. Entropy in operator algebras*, Encycl. Math. Sci. **126**, Springer, Berlin, 2002.

10. M. Rørdam, F. Larsen and N. Laustsen, *An introduction to K-theory for C^*-algebras*, Lond. Math. Soc. Student Texts **49**, Cambridge University Press, Cambridge, 2000.

11. Claude Schochet, *Topological methods for C^*-algebras* II, *Geometry resolutions and the Künneth formula*, Pac. J. Math. **98** (1982), 443–458.

12. P.J. Stacey, *Crossed products of c^*-algebras by $*$-endomorphisms*, J. Austr. Math. Soc. **54** (1993), 204–212.

13. Y. Watatani, e-mail correspondence to Judith Packer, July 2005.

14. Yasuo Watatani, *Index for C^*-subalgebras*, Mem. Amer. Math. Soc. **83** (1990), vi+117.

15. N.E. Wegge-Olsen, *K-theory and C^*-algebras*, Oxford Sci. Publ., The Clarendon Press, Oxford University Press, New York, 1993.

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