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SOME REMARKS CONCERNING THE ATTRACTORS OF ITERATED FUNCTION SYSTEMS

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ABSTRACT. The aim of this article is to establish some conditions under which the attractors of iterated function systems become dendrites. We associate to an attractor of an iterated function system a graph and we prove that, for a large class of iterated function systems, their attractors are dendrites if and only if the associated graph is a tree. We also give some examples of such sets.

1. Introduction. We start with a brief presentation of iterated function systems. Iterated function systems were conceived in the present form by John Hutchinson in [**8**], popularized by Michael Barnsley in [**2**] and are one of the most common and general ways to generate fractals. Many of the important examples of functions and sets with special and unusual properties turn out to be fractal sets or functions whose graphs are fractal sets and a great part of them are attractors of iterated function systems.

There is a current effort to extend the classical Hutchinson's framework to more general spaces and infinite iterated function systems or, more generally, to multifunction systems and to study them [**4, 10–16, 20–22**]. A recent such example can be found in [**11**], where Lipscomb's space, which is an important example in dimension theory, can be obtained as an attractor of an infinite iterated function system defined in very general settings. In those settings, the attractor can be a closed and bounded set, in contrast with the classical theory where only compact sets are considered.

Although fractal sets are defined with measure theory, being sets with noninteger Hausdorff dimension [**5, 6**], it turns out that they have interesting topological properties [**3, 9**]. The topological properties of

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fractal sets have a great importance in analysis on fractals as we can see in [**3, 9**]. One of the most important results in this direction, which states when the attractor of an iterated function system is a connected set, is given in [**9, 15**]. Other results on this problem can be found in [**4, 16**].

In this article, we intend to find sufficient and necessary conditions for an attractor of an iterated function system to be a dendrite. These conditions are necessary for a large class of iterated function systems. The paper is divided into four sections. The first section is the introduction. In the second section the description of the shift space of an iterated function system is given. The main result, Theorem 3.1, is contained in the third section. The last section contains some examples: Hata's tree-like set, the Cross set and others.

For a metric space (X, d), we denote by $\mathcal{K}(X)$ the set of nonempty compact subsets of X. For a set $A \subset X$, we denote by $d(A)$ the diameter of A, that is $d(A) = \sup_{x,y \in A} d(x, y)$.

Definition 1.1. Let (X, d) be a metric space. The function $h : \mathcal{K}(X) \times \mathcal{K}(X) \to [0, +\infty)$ defined by $h(A, B) = \max(d(A, B), d(B, A))$, where $d(A, B) = \sup_{x \in A} d(x, B) = \sup_{x \in A}(\inf_{y \in B} d(x, y))$ is called the *Hausdorff-Pompeiu metric.*

The definition of the metric in the present form was introduced by Hausdorff [**6**, page 463] in 1914, where he indicates Pompeiu as the author of the notion. Pompeiu developed his ideas about this metric in his thesis [**17**] in 1905, where he needed to measure distance between compact sets in the complex plane.

Remark 1.1. [**1, 2, 11, 19**]. $(\mathcal{K}(X), h)$ is a complete metric space if (X, d) is a complete metric space, compact if (X, d) is compact and separable if (X, d) is separable.

Definition 1.2. Let (X, d) be a metric space. For a function $f : X \to X$, let us denote by $\mathrm{Lip}\,(f) \in [0, +\infty]$ the *Lipschitz constant* associated with f, which is $\mathrm{Lip}\,(f) = \sup_{x,y \in X;\, x \neq y} [d(f(x), f(y))]/d(x, y)$. We say that f is a *Lipschitz function* if $\mathrm{Lip}\,(f) < +\infty$ and a *contraction* if $\mathrm{Lip}\,(f) < 1$.

Definition 1.3. An *iterated function system* on a metric space (X, d) consists of a finite family of contractions $(f_k)_{k=\overline{1,n}}$ on X, and it is denoted by $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$.

Definition 1.4. For an iterated function system $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$, the function $F_{\mathcal{S}} : \mathcal{K}(X) \to \mathcal{K}(X)$ defined by $F_{\mathcal{S}}(B) = \cup_{k=1}^{n} f_k(B)$ is called the *fractal operator* associated with the iterated function system $\mathcal{S}$.

Remark 1.2. [**1, 2, 5, 6, 19**]. The function $F_{\mathcal{S}}$ is a contraction satisfying $\mathrm{Lip}\,(F_{\mathcal{S}}) \leq \max_{k=\overline{1,n}} \mathrm{Lip}\,(f_k)$.

Using Banach's contraction theorem there exists, for an iterated function system $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$, a unique set $A(\mathcal{S})$ such that $F_{\mathcal{S}}(A(\mathcal{S})) = A(\mathcal{S})$, which is called the *attractor* of the iterated function system $\mathcal{S}$. More precisely, we have the following well-known result.

Theorem 1.1. [**1, 2, 5, 6, 19**]. *Let (X, d)be a complete metric space and $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$ an iterated function system with $c = \max_{k=\overline{1,n}} \mathrm{Lip}\,(f_k) < 1$. Then there exists a unique set $A(\mathcal{S}) \in \mathcal{K}(X)$ such that $F_{\mathcal{S}}(A(\mathcal{S})) = A(\mathcal{S})$. Moreover, for any $H_0 \in \mathcal{K}(X)$, the sequence $(H_n)_{n\geq 1}$ defined by $H_{n+1} = F_{\mathcal{S}}(H_n)$ is convergent to $A(\mathcal{S})$. For the speed of the convergence we have the following estimate*

$$h(H_n, A(\mathcal{S})) \leq \frac{c^n}{1-c} h(H_0, H_1).$$

Definition 1.5. 1) By a *graph* we understand a pair (I, G), where I is a nonempty set and G is a subset of the set $\{\{i, j\} \mid i, j \in I \text{ and } i \neq j\}$.

2) A graph (I, G) is called *connected* if, for every $i, j \in I$, there exists $(i_k)_{k=\overline{1,n}} \subset I$ such that $i_1 = i$, $i_n = j$ and $\{i_k, i_{k+1}\} \in G$ for every $k \in \{1, 2, \ldots, n-1\}$.

3) Let (I, G) be a graph. A family of vertices $(i_1, \ldots, i_m)$ is a *cycle* if $\{i_k, i_{k+1}\} \in G$ for every $k \in \{1, \ldots, n\}$ and $i_k \notin \{i_{k+1}, i_{k+2}\}$ for every $k \in \{1, \ldots, n\}$, where by i_{m+1} we understand i_1, by i_{m+2} we understand i_2, and so on.

4) A graph (I, G) is called a *tree* if it is connected and has no cycles.

Remark 1.3. We remark that a cycle has at least three elements.

Definition 1.6. Let X be a nonempty set and $(A_i)_{i\in I}$ a family of nonempty subsets of X. Then:

1) The graph (I, G), where $G = \{\{i,j\} \mid i,j \in I$ such that $A_i \cap A_j \neq \emptyset$ and $i \neq j\}$ is called the *graph of intersections* associated with the family $(A_i)_{i\in I}$.
2) The family $(A_i)_{i\in I}$ is said to be *connected* if, for every $i,j \in I$, there exists $(i_k)_{k=\overline{1,n}} \subset I$ such that $i_1 = i$, $i_n = j$ and $A_{i_k} \cap A_{i_{k+1}} \neq \emptyset$ for every $k \in \{1,2,\dots,n-1\}$. If a family $(A_i)_{i\in I}$ is not connected, we say that it is *disconnected.* The family $(A_i)_{i\in I}$ is connected if and only if the graph of intersections (I, G) is connected.
3) The family $(A_i)_{i\in I}$ is said to be a *tree of sets* if, for every $i,j \in I$ such that $i \neq j$, there exists a unique sequence $(i_k)_{k=\overline{1,n}} \subset I$ with $i_1, i_2, \dots, i_n$ different such that $i_1 = i$, $i_n = j$ and $A_{i_k} \cap A_{i_{k+1}} \neq \emptyset$ for every $k \in \{1,2,\dots,n-1\}$. The family $(A_i)_{i\in I}$ is a tree of sets if and only if the graph of intersections (I, G) is a tree.
4) On the family of sets $(A_i)_{i\in I}$, we consider the following equivalence relation: $A_i \sim A_j$ if and only if there exists $(i_k)_{k=\overline{1,n}} \subset I$ such that $i_1 = i$, $i_n = j$ and $A_{i_k} \cap A_{i_{k+1}} \neq \emptyset$ for every $k \in \{1, 2,] \dots, n-1\}$. A *component* of the family of sets $(A_i)_{i\in I}$ is a class of equivalence which corresponds to a connected subgraph of the graph of intersections of the family $(A_i)_{i\in I}$.

Remark 1.4. If the family $(A_i)_{i\in I}$ is a tree of sets then the intersection of three different sets of the family is empty.

Definition 1.7. A metric space (X, d) is *arcwise connected* if, for every $x, y \in X$, there exists a continuous function $\varphi : [0,1] \to X$ such that $\varphi(0) = x$ and $\varphi(1) = y$. A continuous function φ as above is called a *path* between x and y. We say that two continuous, injective functions $\varphi, \psi : [0,1] \to X$ are *equivalent* if there exists a function $u : [0,1] \to [0,1]$ continuous, bijective and increasing such that $\varphi \circ u = \psi$. A class of equivalence is named a *curve*.

Remark 1.5. We remark that two equivalent, continuous, injective functions have the same images.

Concerning the connectedness of the attractor of an iterated function system we have the following theorem.

Theorem 1.2. [**9, 23**]. *Let (X, d) be a complete metric space, $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$ an iterated function system with $c = \max_{k=\overline{1,n}} \mathrm{Lip}\,(f_k) < 1$ and $A(\mathcal{S})$ the attractor of $\mathcal{S}$. Then the following are equivalent*:

1) *The family $(A_i)_{i=\overline{1,n}}$ is connected, where $A_i = f_i(A(\mathcal{S}))$ for every $i \in \{1, \ldots, n\}$.*
2) *$A(\mathcal{S})$ is arcwise connected.*
3) *$A(\mathcal{S})$ is connected.*

Definition 1.8. A metric space (X, d) is called totally disconnected if its only connected subspaces are one-point sets.

2. The shift space of an iterated function systems. In this section we briefly present the shift space of an iterated function system. For more details, one can see [**2, 14, 19**]. We start with some set notations: $\mathbf{N}$ denotes the natural numbers, $\mathbf{N}^* = \mathbf{N} - \{0\}$, $\mathbf{N}_n^* = \{1, 2, \ldots, n\}$. For two nonempty sets A and B, B^A denotes the set of functions from A to B. By $\Lambda = \Lambda(B)$ we will understand the set $B^{\mathbf{N}^*}$ and by $\Lambda_n = \Lambda_n(B)$ we will understand the set $B^{\mathbf{N}_n^*}$. The elements of $\Lambda = \Lambda(B) = B^{\mathbf{N}^*}$ will be written as infinite words $\omega = \omega_1\omega_2 \cdots \omega_m\omega_{m+1} \cdots$, where $\omega_m \in B$ and the elements of $\Lambda_n = \Lambda_n(B) = B^{\mathbf{N}_n^*}$ will be written as finite words $\omega = \omega_1\omega_2 \cdots \omega_n$. By λ, we will understand the empty word. Let us remark that $\Lambda_0(B) = \{\lambda\}$. By $\Lambda^* = \Lambda^*(B)$, we will understand the set of all finite words $\Lambda^* = \Lambda^*(B) = \cup_{n \geq 0}\Lambda_n(B)$. We denote by $|\omega|$ the length of the word ω. An element of $\Lambda = \Lambda(B)$ is said to have length $+\infty$.

If $\omega = \omega_1\omega_2 \cdots \omega_m\omega_{m+1} \cdots$ or if $\omega = \omega_1\omega_2 \cdots \omega_n$ and $n \geq m$, then $[\omega]_m := \omega_1\omega_2 \cdots \omega_m$. More generally if $l < m$, $[\omega]_m^l = \omega_{l+1}\omega_{l+2} \cdots \omega_m$ and we have $[\omega]_m = [\omega]_l[\omega]_m^l$ for $\omega \in \Lambda_n(B)$, if $n \geq m > l \geq 1$ and for $\omega \in \Lambda(B)$, if $m > l \geq 1$. For two words $\alpha, \beta \in \Lambda^*(B) \cup \Lambda(B)$, $\alpha \prec \beta$ means $|\alpha| \leq |\beta|$ and $[\beta]_{|\alpha|} = \alpha$. For $\alpha \in \Lambda_n(B)$ and $\beta \in \Lambda_m(B)$ or $\beta \in \Lambda(B)$ by $\alpha\beta$ we will understand the joining of the words α and β, namely, $\alpha\beta = \alpha_1\alpha_2\cdots\alpha_n\beta_1\beta_2\cdots\beta_m$ and, respectively, $\alpha\beta = \alpha_1\alpha_2\cdots\alpha_n\beta_1\beta_2\cdots\beta_m\beta_{m+1}\cdots$.

On $\Lambda = \Lambda(\mathbf{N}_n^*) = (\mathbf{N}_n^*)^{\mathbf{N}^*}$, we can consider the metric $d_s(\alpha,\beta) = \sum_{k=1}^{\infty}(1-\delta_{\alpha_k}^{\beta_k})/3^k$, where

$$\delta_x^y = \begin{cases} 1 \text{ if } x = y \\ 0 \text{ if } x \neq y \end{cases}, \quad \alpha = \alpha_1\alpha_2\cdots \text{ and } \beta = \beta_1\beta_2\cdots .$$

Let (X,d) be a complete metric space, $\mathcal{S} = (X,(f_k)_{k=\overline{1,n}})$ an iterated function system on X and $A = A(\mathcal{S})$ the attractor of the iterated function system $\mathcal{S}$. For $\omega = \omega_1\omega_2\cdots\omega_m \in \Lambda_m(\mathbf{N}_n^*)$, f_ω denotes $f_{\omega_1}\circ f_{\omega_2}\circ\cdots\circ f_{\omega_m}$ and H_ω denotes $f_\omega(H)$ for a subset $H \subset X$. By H_λ, we will understand the set H. In particular, $A_\omega = f_\omega(A)$.

The main results concerning the relation between the attractor of an iterated function system and the shift space is contained in the following theorem.

Theorem 2.1. [**2, 19**]. *Let (X,d) be a complete metric space. If $A = A(\mathcal{S})$ is the attractor of the iterated function system $\mathcal{S} = (X,(f_k)_{k=\overline{1,n}})$, then*:

1) *For $\omega \in \Lambda = \Lambda(\mathbf{N}_n^*)$, s we have $A_{[\omega]_{m+1}} \subset A_{[\omega]_m}$ and $d(A_{[\omega]_m}) \to 0$ when $m \to \infty$; more precisely, $d(A_{[\omega]_m}) \leq c^m d(A)$.*
2) *If a_ω is defined by $\{a_\omega\} = \cap_{m\geq 1}A_{[\omega]_m}$, then $d(e_{[\omega]_m}, a_\omega) \to 0$ when $m \to \infty$, where $e_{[\omega]_m}$ is the unique fixed point of $f_{[\omega]_m}$.*
3) *$A = A(\mathcal{S}) = \cup_{\omega\in\Lambda}\{a_\omega\}$, $A_\alpha = \cup_{\omega\in\Lambda}\{a_{\alpha\omega}\}$ for every $\alpha \in \Lambda^*$, $A = \cup_{\omega\in\Lambda_m}A_\omega$ for every $m \in \mathbf{N}^*$ and more general $A_\alpha = \cup_{\omega\in\Lambda_m}\cup A_{\alpha\omega}$ for every $\alpha \in \Lambda^*$ and every $m \in \mathbf{N}^*$.*
4) *The set $\{e_{[\omega]_m} \mid \omega \in \Lambda$ and $m \in \mathbf{N}^*\}$ is dense in A.*
5) *The function $\pi : \Lambda \to A$ defined by $\pi(\omega) = a_\omega$ is continuous and surjective.*

Definition 2.1. The function $\pi : \Lambda \to A = A(\mathcal{S})$ from Theorem 2.1. is called the *canonical projection* from the shift space on the attractor of the iterated function system $\mathcal{S}$.

3. Main results. The aim of this article is to establish necessary and sufficient conditions under which the attractor of an iterated function system becomes a dendrite. We will start with some general properties of the dendrites.

Definition 3.1. The metric space (X, d) is called a *dendrite* if, for any $x, y \in X$, there exists a unique equivalence class of continuous, injective functions $\varphi : [0, 1] \to X$ such that $\varphi(0) = x$ and $\varphi(1) = y$ (i.e., there exists a unique injective curve joining x with y). We also consider that the empty set is a dendrite.

Lemma 3.1. Let (X, d) be a dendrite and B a subset of X. Then B is a dendrite if and only if B is arcwise connected.

Lemma 3.2. Let (X, d) be a dendrite and $A_1, A_2, \dots, A_n$ subsets of X such that $A_1, A_2, \dots, A_n$ are dendrites. Then $A_1 \cap A_2 \cap \dots \cap A_n$ is a dendrite.

Proof. If $\cap_{i=1}^n A_i = \varnothing$, then it is a dendrite. If $\cap_{i=1}^n A_i \neq \varnothing$, then we denote the set $A_1 \cap A_2 \cap \dots \cap A_n$ by B, and we consider $x, y \in B$. Thus, $x, y \in A_j$, for every $j \in \{1, \dots, n\}$. So, for all $j \in \{1, \dots, n\}$, there exist the continuous, injective functions $\varphi_j : [0, 1] \to A_{i_j}$ such that $\varphi_j(0) = x$ and $\varphi_j(1) = y$. Since X is a dendrite, $\varphi_1, \dots, \varphi_m$ must be equivalent, which means that $\varphi_1([0, 1]) = \dots = \varphi_m([0, 1]) \subset B$. It follows that B is arcwise connected and, from Lemma 3.1, B is a dendrite. $\square$

Corollary 3.1. Let (X, d) be a complete metric space and $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$ an iterated function system. We denote by A the attractor of $\mathcal{S}$ and by A_k the set $f_k(A)$ for every $k \in \{1, \dots, n\}$. We suppose that f_k is an injective function on A for every $k \in \{1, \dots, n\}$ and A is a dendrite. Then $A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_m}$ is a dendrite for every $i_1, \dots, i_m \in \{1, \dots, n\}$.

Proof. Let us suppose that A is a dendrite. Since f_k is injective on A, it follows that $A_k = f_k(A)$ is also a dendrite. If $\cap_{j=1}^m A_{i_j} = \varnothing$, then it is a dendrite by definition and, if $\cap_{j=1}^m A_{i_j} \neq \varnothing$, we can apply Lemma 3.2 to $(A, d_{|A})$ and $A_{i_1}, A_{i_2}, \dots, A_{i_m}$. $\square$

Lemma 3.3. Let (X, d) be a complete metric space such that $X = \cup_{i=1}^n A_i$ with A_i compact sets satisfying $\operatorname{card}(A_i \cap A_j) \in \{0, 1\}$ for every $i, j \in \{1, \dots, n\}$ different. We suppose that $(\{1, \dots, n\}, G)$, the graph of intersections associated with the family $(A_i)_{i=\overline{1,n}}$, is a tree. Then for any continuous, injective function $\varphi : [0,1] \to X$ such that there exists an $l \in \{1,\dots,n\}$ for which $\varphi(0) \in A_l$ and $\varphi(1) \in A_l$, we have that $\varphi([0,1]) \subset A_l$.

Proof. Let $\varphi : [0,1] \to X$ be an injective path such that $\varphi(0) \in A_l$ and $\varphi(1) \in A_l$, where $l \in \{1,\dots,n\}$. We remark first that A_l has at least two elements since φ is injective. If one of the sets, namely A_j, has one element, it follows that $A_j \subset \cup_{i=1;i\neq j}^n A_i$, since the family of sets $(A_i)_{i=\overline{1,n}}$ is connected and so $A_j \cap (\cup_{i=1;i\neq j}^n A_i) \neq \emptyset$. Therefore, we can suppose that the sets A_i have at least two elements for all $i \in \{1,\dots,n\}$.

Let us suppose that there exists a $t \in (0,1)$ such that $\varphi(t) \notin A_l$. It results that $t \in A_j$ for $j \in \{1,\dots,n\}\backslash\{l\}$. Then there exists a unique sequence $(i_k)_{k=\overline{1,m}} \subset I$ such that $i_1 = l$, $i_m = j$, $A_{i_k} \cap A_{i_{k+1}} \neq \emptyset$, $k \in \{1,\dots,m-1\}$ and $i_1,\dots,i_m$ are different. Let a be such that $\{a\} = A_{i_1} \cap A_{i_2}$. Then there exists $t_1 \in (0,t)$ such that $\varphi(t_1) = a$. Indeed, if we suppose that $a \notin \varphi([0,t])$, then $\varphi([0,t]) \subset \cup_{i=1}^n A_i\backslash\{a\}$. We consider the sets $\widetilde{A}_i = A_i\backslash\{a\}$ for $i \in \{1,\dots,n\}$. Since the family of sets $(A_i)_{i=\overline{1,n}}$ is a tree, it results from Remark 1.4. that the family of sets $(\widetilde{A}_i)_{i=\overline{1,n}}$ is disconnected and the sets $\widetilde{A}_l$ and $\widetilde{A}_j$ belong to different connected components of the family of sets $(\widetilde{A}_i)_{i=\overline{1,n}}$. Let $(\widetilde{A}_i)_{i\in J}$ be the connected component which contains the set $\widetilde{A}_j$. We consider $B = \cup_{i\in J}\widetilde{A}_i$ and $C = \cup_{i=1;i\notin J}^n \widetilde{A}_i$. Then $\varphi(0) \in \widetilde{A}_l \subset C$, $\varphi(1) \in \widetilde{A}_j \subset B$, $\varphi([0,t]) \subset B \cup C$ and $\overline{B} \cap C = B \cap \overline{C} = \emptyset$. This is in contradiction with the fact that $\varphi([0,t])$ is a connected set. In a similar way there exists $t_2 \in (t,1)$ such that $\varphi(t_2) = a$. Hence $\varphi(t_1) = \varphi(t_2) = a$ and $t_1 < t_2$ which is a contradiction with the fact that φ is injective. $\square$

Lemma 3.4. Let $[a,b]$ and $[a_1,b_1]$be two intervals of real numbers, $A \subset [a,b]$ and $A_1 \subset [a_1,b_1]$ two dense sets and $v : A \to A_1$ a bijective and increasing function. Then there exists a unique, continuous, increasing and bijective function $u : [a,b] \to [a_1,b_1]$ such that $u|_A = v$.

Proof. We define the function u by $u(a)=a_1$ and $u(x)=\sup_{y\in A;y\leq x} v(y)$ for every $x \in [a,b]$. Then we have the following:

1) If $x \in A$, then $u(x) = \sup_{y \in A; y \leq x} v(y) = v(x)$ and so $u|_A = v$.
2) Let $c, d \in [a, b]$, $c < d$. Then there exist $c', d' \in A$ such that $c < c' < d' < d$. We have that $u(c) \leq u(c') = v(c') < v(d') = u(d') \leq u(d)$, and thus u is increasing and injective.
3) Let $d \in [a_1, b_1]$. If $d = a_1$, then $d = u(a)$ and, if $d \neq a_1$, then $d = \sup_{y \in A_1; y \leq d} y = \sup_{z \in A; v(z) \leq d} v(z) = u(\sup_{z \in A; v(z) \leq d} z)$. Therefore, u is surjective.
4) u is continuous since every bijective and increasing function between two closed intervals is continuous. □

Lemma 3.5. Let (X, d) be a metric space and φ, $\varphi' : [0, 1] \to X$ continuous, injective functions such that there exist two sequences of divisions of the interval $[0, 1]$, $(\Delta_l)_{l \in \mathbf{N}} \in \mathcal{D}([0, 1])$ and $(\Delta'_l)_{l \in \mathbf{N}} \in \mathcal{D}([0, 1])$, with the following properties*:*

a) $\Delta_l \subset \Delta_{l+1}$ and $\Delta'_l \subset \Delta'_{l+1}$ for every $l \in \mathbf{N}$,
b) $\Delta_l = (0 = y_0^l < y_1^l < \cdots < y_{n_l}^l = 1)$and $\Delta'_l = (0 = z_0^l < z_1^l < \cdots < z_{n_l}^l = 1)$ have the same number of elements for every $l \in \mathbf{N}$,
c) $\|\Delta_l\| \overset{l \to \infty}{\to} 0$ and $\|\Delta'_l\| \overset{l \to \infty}{\to} 0$, where $\|\Delta\| = \max_{k=\overline{0,n-1}} |y_{k+1} - y_k|$, if $\Delta = (a = y_0 < y_1 < \cdots < y_l = b)$ is a division of some interval $[a, b]$, $a, b \in \mathbf{R}$, $a < b$,
d) $\varphi(y_k^l) = \varphi'(z_k^l)$ for every $l \in \mathbf{N}$ and $k \in \{0, 1, \ldots, n_l\}$.

Then there exists a unique, continuous, bijective and increasing function $u : [0, 1] \to [0, 1]$ such that $\varphi' \circ u = \varphi$ (i.e., φ and φ' are equivalent).

Proof. Let $\Delta = \cup_{l \geq 1} \Delta_l$ and $\Delta' = \cup_{l \geq 1} \Delta'_l$. Then Δ and Δ' are dense in $[0, 1]$. Let $v : \Delta \to \Delta'$ be defined by $v(y_k^l) = z_k^l$ for every $k, l \in \mathbf{N}$. Then v is well defined, increasing, bijective, $\varphi'|_{\Delta'} \circ v = \varphi|_{\Delta}$ and, from Lemma 3.4, there exists a unique function $u : [0, 1] \to [0, 1]$ continuous, bijective and increasing such that $u|_{\Delta} = v$. From the continuity of the functions φ', φ and u we have $\varphi' \circ u = \varphi$. □

Theorem 3.1. *Let (X, d) be a complete metric space and $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$ an iterated function system. We denote by A the attractor of $\mathcal{S}$, by A_k the set $f_k(A)$ for every $k \in \{1, \ldots, n\}$ and by $(\{1, \ldots, n\}, G)$ the graph of intersections associated with the family of sets $(A_k)_{k=\overline{1,n}}$. We suppose that the following conditions are true*:

a) $A_i \cap A_j$ *is totally disconnected for every* $i,\ j \in \{1, 2, \ldots, n\}$ *different,*
b) $A_i \cap A_j \cap A_k = \emptyset$ *for every* $i, j, k \in \{1, 2, \ldots, n\}$ *different,*
c) f_k *is an injective function on* A *for every* $k \in \{1, 2, \ldots, n\}$.

Then the following statements are equivalent:

1) A *is a dendrite.*
2) *The graph* $(\{1, \ldots, n\}, G)$ *is a tree and card*$(A_i \cap A_j) \in \{0, 1\}$ *for every* $i, j \in \{1, 2, \ldots, n\}$ *different.*

Proof. 1) $\Rightarrow$ 2). We know that A is a dendrite and therefore A is connected. From Theorem 1.2 it follows that G is connected. From Corollary 3.1 it follows that $A_i \cap A_j$ is a dendrite if $A_i \cap A_j \neq \emptyset$ and $i, j \in \{1, \ldots, n\}$ different. Since $A_i \cap A_j$ is also totally disconnected we have that card$(A_i \cap A_j) \in \{0, 1\}$ for every $i, j \in \{1, \ldots, n\}$ different. To prove that $(\{1, \ldots, n\}, G)$ is a tree, it is enough to prove that $(\{1, \ldots, n\}, G)$ has no cycles.

If $A_i \cap A_j \neq \emptyset$ and $i \neq j$, then we denote by a_{ij} the element of A defined by $\{a_{ij}\} = A_i \cap A_j$. Let us suppose that there exists a cycle $(i_1, \ldots, i_m)$ in $(\{1, \ldots, n\}, G)$. We can choose m to be minimal with this property. This involves $m \geq 3$, $i_1, \ldots, i_m$ do not repeat, $A_{i_j} \cap A_{i_{j+1}} \neq \emptyset$ and $A_{i_j} \cap A_{i_l} = \emptyset$ for every $l, j \in \{1, \ldots, m\}$ such that $l \neq j - 1$ and $l \neq j + 1$, where $i_{m+1} = i_1$, $i_{m+2} = i_2$ and so on. If $A_{i_j} \cap A_{i_l} \neq \emptyset$ for $j < l$ and $l \neq j + 1$, it follows that $(i_l, \ldots, i_{j-1})$ is a cycle and therefore m is not minimal, which is a contradiction.

We denote by $b_1 = a_{i_1 i_2}$, $b_2 = a_{i_2 i_3}, \ldots, b_m = a_{i_m i_{m+1}}$, where $i_{m+1} = i_1$. We remark that the elements $b_1, b_2, \ldots, b_m$ are different because $A_i \cap A_j \cap A_k = \emptyset$ for every $i, j, k \in \{1, \ldots, n\}$ different. Since A is a dendrite and the functions f_k are injective for every $k \in \{1, \ldots, n\}$, it follows that A_{i_j} is a dendrite for every $j \in \{1, \ldots, m\}$. Thus, there exist the continuous, injective functions $\varphi_j : [0, 1] \to A_{i_j}$ such that $\varphi_j(0) = b_{j-1}$ and $\varphi_j(1) = b_j$ for every $j \in \{2, \ldots, m\}$.

We consider now the function $\varphi : [0, 1] \to A_{i_2} \cup A_{i_3} \cup \cdots \cup A_{i_m}$ defined by:

$$\varphi(x) = \begin{cases} \varphi_{i_2}((m-1)x), x \in [0, \frac{1}{m-1}), \\ \varphi_{i_3}((m-1)x-1), x \in [\frac{1}{m-1}, \frac{2}{m-1}), \\ \varphi_{i_4}((m-1)x-2), x \in [\frac{2}{m-1}, \frac{3}{m-1}), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \varphi_{i_m}((m-1)x-(m-2)), x \in [\frac{m-2}{m-1}, 1]. \end{cases}$$

We will prove that φ is a continuous and injective function on $[0,1]$. i) *Continuity.* φ is obviously continuous on $[0,1]\backslash\{i/(m-1) \mid i \in \{1,\ldots,m-2\}\}$. But

$$\lim_{x\nearrow k/(m-1)} \varphi(x) = \lim_{x\nearrow k/(m-1)} \varphi_{i_k}((m-1)x-k+1) = \varphi_{i_k}(1) = b_k$$

and

$$\begin{aligned} \varphi\left(\frac{k}{m-1}\right) &= \lim_{x\searrow k/(m-1)} \varphi(x) \\ &= \lim_{x\searrow k/(m-1)} \varphi_{i_{k+1}}((m-1)x-k+2) \\ &= \varphi_{i_{k+1}}(0) = b_k. \end{aligned}$$

So φ is continuous in the points $x = k/(m-1)$ for every $k \in \{1,\ldots,m-2\}$, and thus it is continuous on $[0,1]$.

ii) *Injectivity.* Let us remark that $\varphi(k/(m-1)) = b_k = \varphi_{i_{k+1}}(0) = \varphi_{i_k}(1)$ and

$$\varphi_{i_i}^{-1}(A_{i_j}) = \begin{cases} [0,1], & \text{if } i = j \\ 0, & \text{if } i = j+1, \\ 1, & \text{if } i = j-1, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Hence, $\varphi^{-1}(A_{i_i}) = [(i-2)/(m-1), (i-1)/(m-1)]$. We suppose that there exist two points $x, y \in [0,1]$ such that $\varphi(x) = \varphi(y)$. Since $\varphi(x) \in A_{i_2} \cup A_{i_3} \cup \cdots \cup A_{i_m}$, there exists $k \in \{2,\ldots,m\}$ such that $\varphi(x) = \varphi(y) \in A_{i_k}$, which implies $x, y \in [(k-2)/(m-1), (k-1)/(m-1)]$. Then $\varphi(x) = \varphi_{i_k}((m-1)x-(k-2)) = \varphi(y) = \varphi_{i_k}((m-1)y-(k-2))$, which implies $x = y$, since φ_{i_k} is injective.

We return now to our proof. On one side we have $\varphi([0,1]) \cap A_{i_1} = \{b_1, b_m\}$ and on the other side, since A_{i_1} is a dendrite, there exists a continuous, injective function $\psi : [0,1] \to A_{i_1}$ such that $\psi(0) = b_1$ and $\psi(1) = b_m$.. It follows that there are two continuous, injective functions φ and ψ joining b_1 to b_m. Since A is a dendrite, $\varphi([0,1]) \subset A$ and $\psi([0,1]) \subset A$, we have $\varphi([0,1]) = \psi([0,1])$. It results that $\varphi([0,1]) = \psi([0,1]) \subset A_{i_1} \cap (A_{i_2} \cup A_{i_3} \cup \cdots \cup A_{i_m}) = \{b_1, b_m\}$, which is a contradiction. It follows that $(\{1,\ldots,n\}, G)$ does not contain any cycles and therefore it is a tree.

2) $\Rightarrow$ 1). Since $(\{1,\ldots,n\}, G)$ is a tree, it results that $(\{1,\ldots,n\}, G)$ is connected. From Theorem 2.1 it follows that A is arcwise connected. We will prove that A is a dendrite.

Let $x, y \in A$, $x \neq y$. We suppose that there exist two continuous, injective functions $\varphi, \psi : [0,1] \to A$ such that $\varphi(0) = \psi(0) = x$ and $\varphi(1) = \psi(1) = y$. To prove that A is a dendrite, it is enough to prove that φ and ψ are equivalent. We intend to use Lemma 3.5 to prove the equivalence. For that, we will construct inductively after $l \in \mathbf{N}$ two sequences $(\Delta_l)_{l\geq 0}$ and $(\Delta'_l)_{l\geq 0}$ of divisions of the unit interval $[0,1]$ such that:

I) $\Delta_l \subset \Delta_{l+1}$ and $\Delta'_l \subset \Delta'_{l+1}$ for every $l \in \mathbf{N}$,
II) $\Delta_l = (0 = y_0^l < y_1^l < \cdots < y_{n_l}^l = 1)$ and $\Delta'_l = (0 = z_0^l < z_1^l < \cdots < z_{n_l}^l = 1)$ (i.e., Δ_l and Δ'_l have the same number of elements for every $l \in \mathbf{N}$),
III) $\varphi(y_k^l) = \psi(z_k^l)$ for every $l \in \mathbf{N}$ and $k \in \{0,\ldots,n_l\}$,
IV) For every $l \in \mathbf{N}$ and $k \in \{0,\ldots,n_l - 1\}$, there exists an $\omega_k^l \in \Lambda_{m_k^l}(\mathbf{N}_n^*)$ such that:

i) $m_k^l \geq l$,
ii) $\omega_k^l \prec \omega_{k'}^{l+1}$, if $y_{k'}^{l+1} \in [y_k^l, y_{k+1}^l)$,
iii) $\varphi(y_k^l)$, $\varphi(y_{k+1}^l) \in A_{\omega_k^l}$ for every $l \in \mathbf{N}$ and $k \in \{0,\ldots,n_l - 1\}$.

We remark that IV) ii) implies $\omega_k^l \prec \omega_{k'}^{l'}$, if $l' \geq l$ and $y_{k'}^{l'} \in [y_k^l, y_{k+1}^l)$ for every $l, l' \in \mathbf{N}$, $k \in \{0,\ldots,n_l - 1\}$ and the points I)–IV) imply that $\|\Delta_l\| \overset{l\to\infty}{\longrightarrow} 0$ and $\|\Delta'_l\| \overset{l\to\infty}{\longrightarrow} 0$.

Indeed, we have that $\|\Delta_l\| \geq \|\Delta_{l+1}\|$. Let $\varepsilon = \lim_{l\to\infty} \|\Delta_l\| \geq 0$. We suppose by contradiction that $\varepsilon > 0$. Let $\delta_\mu = \inf\limits_{x,y\in[0,1];|x-y|\geq\mu} d(\varphi(x), \varphi(y))$ for every $\mu \in [0,1)$. Since φ is injective and $[0,1]$ is a compact set, we have that $\delta_\mu > 0$ for every $\mu > 0$. We denote $c_l = \max_{k=\overline{0,n_l-1}} d(\varphi(y_{k+1}^l), \varphi(y_k^l))$. On the one hand, since

$\varphi(y_k^l), \varphi(y_{k+1}^l) \in A_{\omega_k^l}$ and $|\omega_k^l| = m_k^l \geq l$, we have that

$$\begin{aligned}c_l &= \max_{k=\overline{0,n_l-1}} d(\varphi(y_{k+1}^l), \varphi(y_k^l)) \leq \max_{k=\overline{0,n_l-1}} d(A_{\omega_k^l}) \\ &\leq \max_{k=\overline{0,n_l-1}} (c^{m_k^l} d(A)) \leq c^l d(A),\end{aligned}$$

where $c = \max_{k=\overline{1,n}} \mathrm{Lip}\,(f_k) < 1$. On the other hand,

$$c_l = \max_{k=\overline{0,n_l-1}} d(\varphi(y_{k+1}^l), \varphi(y_k^l)) \geq \delta_{\|\Delta_l\|} \geq \delta_\varepsilon.$$

Therefore, $0 < \delta_\varepsilon \leq \lim_{l\to\infty} c^l d(A) = 0$, which is a contradiction. It follows that $\varepsilon = 0$.

We return now to the construction of the divisions $(\Delta_l)_{l\geq 0}$, $(\Delta_l')_{l\geq 0}$ with the properties I)–IV). We will do that by induction, in three steps.

First step. Let $\Delta_0 = (y_0^0 = 0 < y_1^0 = 1)$ and $\Delta_0' = (z_0^0 = 0 < z_1^0 = 1)$. We have $\varphi(y_0^0) = \varphi(0) = \psi(0) = \psi(z_0^0) = x$ and $\varphi(y_1^0) = \varphi(1) = \psi(1) = \psi(z_1^0) = y$. We take $\omega_0^0 = \lambda$.

Second step. We know that $A = \cup_{k=1}^n f_k(A) = \cup_{k=1}^n A_k$. Since $x, y \in A$, it follows that there exist $i(x), i(y) \in \{1, \ldots, n\}$ such that $x \in A_{i(x)}$ and $y \in A_{i(y)}$. We have two cases:

1) For every $i, j \in \{1, \ldots, n\}$ such that $x \in A_i$ and $y \in A_j$, we have $i \neq j$.

In this case we choose $i(x), i(y) \in \{1, \ldots, n\}$ such that $x \in A_{i(x)}$ and $y \in A_{i(y)}$. Since $(\{1, \ldots, n\}, G)$ is a tree and $\mathrm{card}\,(A_i \cap A_j) \in \{0, 1\}$ for every i, j different, the sets $A_{i(x)}$ and $A_{i(y)}$ are joined by a unique chain of sets $\{A_{i_j}\}_{j=\overline{1,m}}$ such that $i(x) = i_1$, $i(y) = i_m$, $A_{i_j} \cap A_{i_{j+1}} = \{a_j\}$ for every $j \in \{1, \ldots, m-1\}$ and $i_1, i_2, \ldots, i_m$ are different. From hypothesis b), it results that $A_{i_j} \cap A_{i_k} \cap A_{i_l} = \emptyset$ for every $j, k, l \in \{1, \ldots, m\}$ different and that $a_{i_1}, a_{i_2}, \ldots, a_{i_{m-1}}$ are different.

We first suppose that $x \neq a_{i_1}$ and $y \neq a_{i_{m-1}}$. The functions φ and ψ, which are joining x and y, have to pass through $\{a_j\}$ for every $j \in \{1, \ldots, m-1\}$. Thus, there exist $y_j^1 \in (0, 1)$ and $z_j^1 \in (0, 1)$ such that $\varphi(y_j^1) = \psi(z_j^1) = a_j$. We remark that, since φ and ψ are injective, the points y_j^1 and z_j^1 are uniquely determined. We also remark that $y_j^1 < y_{j+1}^1$ and $z_j^1 < z_{j+1}^1$ for every $j \in \{1, \ldots, m-2\}$ and that the families of points $\{y_j^1 \mid j \in \{1, \ldots, m-1\}\} \cup \{0, 1\}$

and $\{z_j^1 \mid j \in \{1, \ldots, m-1\}\} \cup \{0, 1\}$ form divisions of the interval $[0,1]$, namely, $\Delta_1 = (0 = y_0^1 < y_1^1 < \cdots < y_{m-1}^1 < y_m^1 = 1)$ and $\Delta_1' = (0 = z_0^1 < z_1^1 < \cdots < z_{m-1}^1 < z_m^1 = 1)$. We take $\omega_k^1 = i_k \in \Lambda_1$ for every $k \in \{0, \ldots, m-1\}$.

If $x \neq a_{i_1}$ and $y = a_{i_{n_1-1}}$, then $y_{m-1}^1 = z_{m-1}^1 = 1$, and we take $\Delta_1 = (0 = y_0^1 < y_1^1 < \cdots < y_{m-1}^1 = 1)$ and $\Delta_1' = (0 = z_0^1 < z_1^1 < \cdots < z_{m-1}^1 = 1)$. The case $x = a_{i_1}$, $y = a_{i_{n_1-1}}$ and the case $x = a_{i_1}$, $y \neq a_{i_{n_1-1}}$ can be treated in a similar way.

2) There exists $i \in \{1, \ldots, n\}$ such that $x \in A_i$ and $y \in A_i$.

Let $p = \sup\{j \in \mathbf{N}^* \mid$ there exist $\alpha, \beta \in \Lambda(\mathbf{N}_n^*)$ such that $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2, \ldots, \alpha_{j-1} = \beta_{j-1}$, $\alpha_j \neq \beta_j$ and $\pi(\alpha) = x$, $\pi(\beta) = y\}$, where $\pi : \Lambda \to A$ is the canonical projection. We remark first that $p \in \mathbf{N}$. If not, for every $l \in \mathbf{N}$ there exist $\alpha, \beta \in \Lambda(\mathbf{N}_n^*)$ such that $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2, \ldots, \alpha_l = \beta_l$, $\pi(\alpha) = x$ and $\pi(\beta) = y$. It follows that $x, y \in A_{\alpha_1 \alpha_2 \cdots \alpha_l}$ and so $d(x,y) \leq d(A_{\alpha_1 \alpha_2 \cdots \alpha_l}) \leq c^l d(A)$ for every $l \in \mathbf{N}$, where $c = \max_{k=\overline{1,n}} \operatorname{Lip}(f_k) < 1$. Therefore, $x = y$, which is a contradiction.

Hence, there exist $\alpha_1, \ldots, \alpha_{p-1} \in \{1, \ldots, n\}$ such that $x, y \in A_{\alpha_1 \cdots \alpha_{p-1}}$. Moreover $x \in A_{\alpha_1 \cdots \alpha_{p-1} \alpha_p}$ and $y \in A_{\alpha_1 \cdots \alpha_{p-1} \beta_p}$. It follows that $\alpha_p \neq \beta_p$ and there exist $x', y' \in A$ such that $f_{\alpha_1 \cdots \alpha_{p-1}}(x') = x$ and $f_{\alpha_1 \cdots \alpha_{p-1}}(y') = y$, where $x' \in A_{\alpha_p}$ and $y' \in A_{\beta_p}$. Let $\varphi' = f_{\alpha_1 \cdots \alpha_{p-1}}^{-1} \circ \varphi$ and $\psi' = f_{\alpha_1 \cdots \alpha_{p-1}}^{-1} \circ \psi$. Then x' and y' fulfill the conditions from case 1), and we can apply the same reasoning for φ' and ψ'. Thus, there exist $\Delta_1 = (0 = y_0^1 < y_1^1 < \cdots < y_{n_1-1}^1 < y_{n_1}^1 = 1)$ and $\Delta_1' = (0 = z_0^1 < z_1^1 < \cdots < z_{n_1-1}^1 < z_{n_1}^1 = 1)$ two divisions of $[0,1]$ such that $\varphi'(y_i^1) = \psi'(z_i^1)$ for every $i \in \{0, \ldots, n_1\}$. There also exists $\widetilde{\omega}_k^1 \in \Lambda_1(\mathbf{N}_n^*)$ such that $\varphi'(y_k^1), \varphi'(y_{k+1}^1) \in A_{\widetilde{\omega}_k^1}$ for every $k \in \{0, \ldots, n_1 - 1\}$. Then $\varphi(y_i^1) = f_{\alpha_1 \cdots \alpha_{p-1}} \circ \varphi'(y_i^1) = f_{\alpha_1 \cdots \alpha_{p-1}} \circ \psi'(z_i^1) = \psi(z_i^1)$ for every $i \in \{0, \ldots, n_1\}$. Let $\omega_k^1 = \alpha_1 \cdots \alpha_{p-1} \widetilde{\omega}_k^1 \in \Lambda_p(\mathbf{N}_n^*)$ and $m_k^1 = p$. We have that $\varphi(y_k^1), \varphi(y_{k+1}^1) \in A_{\omega_k^1}$ for every $i \in \{0, \ldots, n_1 - 1\}$.

Third step (The induction step). Let us suppose that we have defined Δ_j and Δ_j' with the properties I)–IV), $\Delta_l = (0 = y_0^l < y_1^l < \cdots < y_{n_l}^l = 1)$ and $\Delta_l' = (0 = z_0^l < z_1^l < \cdots < z_{n_l}^l = 1)$ for every $j \in \{1, \ldots, l\}$, and let $i \in \{0, \ldots, n_l - 1\}$ be fixed. Then we have $\omega_i^l \in \Lambda_{m_i^l}(\mathbf{N}_n^*)$,

$\varphi(y_i^l) = \psi(z_i^l), \varphi(y_{i+1}^l) = \psi(z_{i+1}^l) \in A_{\omega_i^l} = f_{\omega_i^l}(A)$ and $m_i^l \geq l$. Let $x', y' \in A$ be such that $f_{\omega_i^l}(x') = \varphi(y_i^l)$ and $f_{\omega_i^l}(y') = \varphi(y_{i+1}^l)$.

We set $\varphi_i^l = f_{\omega_i^l}^{-1} \circ \varphi|_{[y_i^l, y_{i+1}^l]}$ and $\psi_i^l = f_{\omega_i^l}^{-1} \circ \psi|_{[y_i^l, y_{i+1}^l]}$. Then x' and y' fulfill the conditions of the second step, case 1), the only difference is that the interval $[0,1]$ is replaced by $[y_i^l, y_{i+1}^l]$ and by $[z_i^l, z_{i+1}^l]$. As in the second step one can find divisions $\Delta_i^{l+1} = (y_i^l = y_{i,0}^{l+1} < y_{i,1}^{l+1} < \cdots < y_{i,n_{l,i}}^{l+1} = y_{i+1}^l)$ of the interval $[y_i^l, y_{i+1}^l]$ and $\Delta_i'^{l+1} = (z_i^l = z_{i,0}^{l+1} < z_{i,1}^{l+1} < \cdots < z_{i,n_{l,i}}^{l+1} = z_{i+1}^l)$ of the interval $[z_i^l, z_{i+1}^l]$ such that $\varphi_i^l(y_{i,k}^{l+1}) = \psi_i^l(z_{i,k}^{l+1})$ for every $k \in \{0, \ldots, n_{l,i}\}$ and, for every $k \in \{0, \ldots, n_{l,i} - 1\}$, there exists an $\omega_k^{l,i} \in \Lambda_{m_k^{l,i}}(\mathbf{N}_n^*)$ such that $m_k^{l,i} \geq 1$ and $\varphi_i^l(y_{i,k}^{l+1}), \varphi_i^l(y_{i,k+1}^{l+1}) \in A_{\omega_k^{l,i}}$.

We have that $\varphi(y_{i,k}^{l+1}) = f_{\omega_i^l} \circ \varphi_i^l(y_{i,k}^{l+1}) = f_{\omega_i^l} \circ \psi_i^l(z_{i,k}^{l+1}) = \psi(z_{i,k}^{l+1}) \in f_{\omega_i^l}(A_{\omega_k^{l,i}}) = A_{\omega_k^l \omega_k^{l,i}}$ and $\varphi(y_{i,k+1}^{l+1}) = f_{\omega_i^l} \circ \varphi_i^l(y_{i,k+1}^{l+1}) = f_{\omega_i^l} \circ \psi_i^l(z_{i,k+1}^{l+1}) = \psi(z_{i,k+1}^{l+1}) \in f_{\omega_i^l}(A_{\omega_k^{l,i}}) = A_{\omega_k^l \omega_k^{l,i}}$, where $\omega_k^{l,i} \omega_k^{l,i} \in \Lambda_{m_k^l + m_k^{l,i}}(\mathbf{N}_n^*)$ and $m_k^l + m_k^{l,i} \geq l+1$ for every $k \in \{0, \ldots, n_{l,i} - 1\}$. Let $\Delta_{l+1} = \cup_i \Delta_i^l$ and $\Delta'_{l+1} = \cup_i \Delta_i'^l$. It is easy to see that the divisions $(\Delta_l)_{l \geq 0}$ and $(\Delta_l')_{l \geq 0}$ of $[0,1]$ constructed in this way have the desired properties. Thus, from Lemma 3.5, it results that φ and φ' are equivalent and the proof is fulfilled. $\square$

4. Examples.

Example 4.1. (Hata's tree-like set). Let $X = \mathbf{C}$. We set $f_1(z) = c\overline{z}$ and $f_2(z) = (1 - |c|^2)\overline{z} + |c|^2$, where $c \in \mathbf{C}$ and $|c|, |1 - c| \in (0,1)$. The attractor of the iterated function system formed with these functions is called Hata's tree-like set, and it is denoted by K. We put $A_1 = f_1(K)$ and $A_2 = f_2(K)$. To prove that the conditions of Theorem 3.1 are fulfilled, we can easily observe that:

a) $A_1 \cap A_2 = f_1(K) \cap f_2(K) = \{|c|^2\}$,
b) $A_i \cap A_j \cap A_j = \emptyset$ for $i, j, k \in \{1, 2\}$ different, because we cannot choose such indices.
c) f_1 and f_2 are one-to-one functions.

Also the graph $(\{1,2\}, G)$, where $G = \{(i,j) \in \{1,2\} \times \{1,2\} \mid A_i \cap A_j \neq \emptyset\}$ consists of a single edge $(1,2)$, thus it is a tree and

$A_1 \cap A_2 = \{|c|^2\}$ implies card $(A_1 \cap A_2) = 1$. Hence, using Theorem 3.1, K is a dendrite.

Example 4.2. (The cross set). Let $X = \mathbf{C}$ and $A = \{z = x + iy \in \mathbf{C} \mid |x| + |y| \leq 1\}$. We consider the functions $f_j : \mathbf{C} \to \mathbf{C}$, where $j \in \{0, \ldots, 4\}$ defined by $f_0(z) = z/3$, $f_1(z) = (z/3) + (2/3)$, $f_2(z) = (z/3) + (2i)/3$, $f_3(z) = (z/3) - (2/3)$ and $f_4(z) = (z/3) - (2i)/3$. The attractor of the iterated function system $\mathcal{S} = (\mathbf{C}, (f_0, f_1, f_2, f_3, f_4))$ is called the cross, and it is denoted by $A(\mathcal{S})$. The fixed points of the functions $f_0, f_1 f_2, f_3, f_4$ are $0, 1, i, -1, -i$ and so $0, 1, i, -1, -i \in A(\mathcal{S})$. We remark that $f_0(A) = A/3$, $f_1(A) = A/3 + 2/3 \subset A$, $f_2(A) = A/3 + (2i)/3 \subset A$, $f_3(A) = (A/3) - (2/3) \subset A$ and $f_4(A) = (A/3) - (2i)/3 \subset A$. Therefore, $F_{\mathcal{S}}(A) \subset A$ and $A(\mathcal{S}) \subset A$. Also, $f_1(A(\mathcal{S})) \cap f_3(A(\mathcal{S})) \subset ((A/3) + (2/3)) \cap ((A/3) - (2/3)) = \emptyset$. In a similar way, one can obtain that $f_l(A(\mathcal{S})) \cap f_j(A(\mathcal{S})) = \emptyset$ for every $l, j \in \{1, 2, 3, 4\}$ such that $l \neq j$.

We also remark that, on one side, we have $f_0(A(\mathcal{S})) \cap f_1(A(\mathcal{S})) \subset (A/3) \cap ((A/3) + (2/3)) = \{(1/3)\}$ and, on the other side, $(1/3) = f_0(1) = f_1(-1)$. Thus, $f_0(A(\mathcal{S})) \cap f_1(A(\mathcal{S})) = \{1/3\}$.

In a similar way, one can obtain that $f_0(A(\mathcal{S})) \cap f_2(A(\mathcal{S})) = \{(2i)/3\}$, $f_0(A(\mathcal{S})) \cap f_3(A(\mathcal{S})) = \{-2/3\}$ and $f_0(A(\mathcal{S})) \cap f_4(A(\mathcal{S})) = \{-(2i)/3\}$. From the above remarks, we can see that the iterated function system $\mathcal{S} = (\mathbf{C}, (f_0, f_1, f_2, f_3, f_4))$ fulfills the conditions from Theorem 3.1 and the graph $(\{1, 2, 3, 4, \}, G) = \{(0, 1), (0, 2), (0, 3), (0, 4)\}$. Thus, $A(\mathcal{S})$ is a dendrite.

Example 4.3. Let $X = \mathbf{R}$. We consider the iterated function system $\mathcal{S} = (\mathbf{R}, (f_k)_{k=\overline{1,2}})$, where $f_1(x) = (2/3)x$ and $f_2(x) = (2/3)x + 1/3$. Then the attractor of $\mathcal{S}$ is $A(\mathcal{S}) = [0, 1]$, $A_1 = f_1(A) = [0, (2/3)]$, $A_2 = f_2(A) = [(1/3), 1]$ and $A_1 \cap A_2 = [(1/3), (2/3)]$. We remark that $A(\mathcal{S})$ is a dendrite but the iterated function system $\mathcal{S}$ does not fulfill the conditions from Theorem 3.1, since $A_1 \cap A_2$ is not a totally disconnected set.

Example 4.4. We consider the following set in the plane $\mathbf{R}^2$ endowed with the Euclidian metric $A = ([0, 1] \times \{0\}) \cup \cup_{n \geq 1}(\{1/2^n\} \times [0, (1/2^n)])$. Then A is the attractor of the iterated function system

$\mathcal{S} = (A, (f_1, f_2, f_3, f_4))$, where $f_i : A \to A$ for $i \in \{1, 2, 3, 4\}$ are defined by:

$$f_1(x, y) = \left(\frac{x}{2}, \frac{y}{2}\right), \qquad f_2(x, y) = \left(\frac{x+1}{2}, 0\right),$$
$$f_3(x, y) = \left(1, \frac{y}{2}\right), \qquad f_4(x, y) = \left(1, \frac{y+1}{2}\right).$$

In this way, we obtain that $A_1 \cap A_2 = \{((1/2), 0)\}$, $A_2 \cap A_3 = \{(1, 0)\}$, $A_3 \cap A_4 = \{(1, (1/2))\}$ and $A_1 \cap A_3 = A_1 \cap A_4 = A_2 \cap A_4 = A_3 \cap A_4 = \varnothing$. Also, $A_i \cap A_j \cap A_j = \emptyset$ for every $i, j, k \in \{1, 2, 3, 4\}$ different, and the functions f_1, f_2, f_3, f_4 are injective. The graph of intersections associated with the family of sets $(A_i)_{i=\overline{1,4}}$ is $(\{1, 2, 3, 4\}, G) = \{(i, j) \in \{1, 2, 3, 4\} \times \{1, 2, 3, 4\} \mid A_i \cap A_j \neq \emptyset\} = \{(1, 2), (2, 3), (3, 4)\}$. Thus, $(\{1, 2, 3, 4\}, G)$ is a tree and therefore, by Theorem 3.1, it results that A is a dendrite.

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