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PERFECT QUADRILATERAL RIGHT PRISMS

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ABSTRACT. We consider right prisms with horizontal quadrilateral bases and tops, and vertical rectangular sides. We look for examples where all the edges, face diagonals and space diagonals are integers. We find examples when the base is an isosceles trapezium or a parallelogram, but no solution for a kite or rhombus.

1. Introduction. Let **ABCD** be a convex quadrilateral in the x-y plane, and let **EFGH** be the identical shape vertically translated h units. Then the three-dimensional object with vertices **ABCDEFGH** is known as a right prism. In this paper, we consider finding such prisms where the lengths of the edges, face diagonals and space diagonals are all integers.

If **ABCD** and **EFGH** are rectangles, then the prism is in fact a cuboid, and the problem is that of finding a perfect rational cuboid, which is still a classic unsolved problem, see Bremner [**1**], Dickson [**3**], Guy [**4**], Leech [**6**] and van Luijk [**7**]. Very recently, Sawyer and Reiter [**11**] have shown the existence of perfect parallelipipeds. These are not normally right prisms, in that the solutions found do not have rectangular vertical faces.

In this paper, we consider quadrilaterals such as the trapezium, the parallelogram, the kite and the rhombus, to see whether perfect right prisms with these base shapes exist. It is interesting to note that a mathematician of the caliber of Kummer [**5**] considered the problem of quadrilaterals with integer sides and diagonals.

For the aid of readers, we summarize the results as:

Theorem. *Perfect right prisms exist when the base is an isosceles trapezium or parallelogram. We conjecture that there are an infinite number of non-congruent isosceles trapezia that lead to perfect prisms.*

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Conjecture. *Perfect right prisms do not exist if the base is a kite, rhombus or cyclic quadrilateral.*

2. Basics. Since the sides of a right prism are defined to be rectangles, there will be several equations of the form

$$a^2 + b^2 = \square, \quad a, b \in \mathbf{Z}$$

which have a general solution $a = \alpha(m^2 - n^2)$, $b = \alpha 2mn$, with $m, n \in \mathbf{Z}$ of opposite parity and having $\gcd(m, n) = 1$. The α term can be a nuisance, so we get rid of it by reforming the solution as $a/b = P(r) \equiv (r^2 - 1)/(2r)$ with $r = m/n$.

Often, the development of a problem requires the solution of

$$P(c) = d, \quad c, d \in \mathbf{Q}.$$

A simple consideration of the underlying quadratic shows that

$$1 + d^2 = \square$$

is a necessary condition.

The most usual such occurrence is when we look for rational solutions of

$$x^2 + h^2 = \square, \qquad y^2 + h^2 = \square.$$

Thus, $x/h = P(a)$ and $y/h = P(b)$ for some rational a and b. Then, $P(b) = (y/x)P(a)$, which has a rational solution if $1 + y^2 P^2(a)/x^2 = \square$, which implies

$$d^2 = y^2 a^4 + (4x^2 - 2y^2)a^2 + y^2$$

has rational solutions. There is clearly at least one rational solution, when $a = 0, d = \pm y$, so the quartic is birationally equivalent to an elliptic curve, see Silverman and Tate [**13**].

Using standard transformations given in Mordell [**8**], we find the equivalent curve to be

$$E : v^2 = u^3 + (x^2 + y^2)u^2 + x^2 y^2 u = u(u + x^2)(u + y^2) \tag{2.1}$$

with $a = v/(y(u + x^2))$.

The curve has torsion subgroup (usually) isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_4$, with finite torsion points of order 2 at $(0,0)$, $(-x^2,0)$, $(-y^2,0)$, and points of order 4 at $(xy, \pm xy(x+y))$, and $(-xy, \pm xy(x-y))$. None of these points give a non-trivial value of $a > 1$. We thus need curves which have rank greater than 0.

For example, $x = 2, y = 3$ leads to $v^2 = u^3 + 13u^2 + 36u$, which has rank 0, and hence there are no non-trivial solutions of $4 + h^2 = \square$, $9 + h^2 = \square$.

For $x = 5, y = 2$, however, $v^2 = u^3 + 29u^2 + 100u$ has rank 1 with generator $(2, 18)$. This point gives $a = 1/3$, which is not acceptable. If we add $(2, 18)$ to one of the order 4 torsion points, we get the point $(-20, 40)$, which gives $a = 4$, giving $P(a) = 15/8$ and $h = 8/3$. Scaling x, y, h by 3 gives $x = 15$, $y = 6$, $h = 8$.

Finding all the generators of an elliptic curve is a non-trivial computational task, so we look for as many possible generators that can be easily found. Let $G_1, \dots, G_s$ be the independent generators found with $0 \leq s \leq r$ where r is the rank of the curve; then, we compute the points

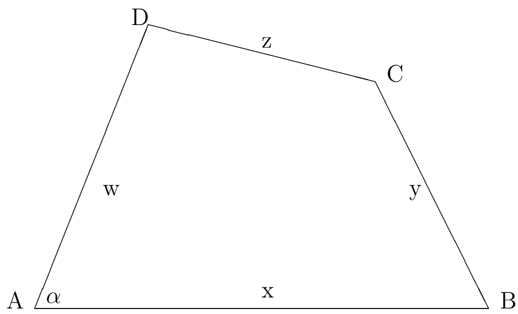
$$Q = n_1 G_1 + n_2 G_2 + \cdots + n_s G_s + T_k$$

where $-L \leq n_i \leq +L$ and T_k is a torsion point, $0 \leq k \leq 7$, where T_0 is the point at infinity. For each point Q we determine $a, P(a)$ and $h = x/P(a)$.

Suppose P is a point of infinite order on E. We find P and $P + T$ where T is a point of order 2 generates the four values of $\pm a_1, \pm 1/a_1$, for some a_1, as we use the three points of order 2 and the point at infinity. For the four values from $P + T$ where T varies over the four points of order 4, we get $\pm a_2$, $\pm 1/a_2$.

It is clear, however, that these values all give the same value for $|P(a_1)|$ and $|P(a_2)|$, and hence we only generate two different positive values of h. Further, we have the product of these h values equals $|x\,y|$. All this can be proven by standard application of the formulae for addition on elliptic curves, using a good symbolic algebra package to alleviate the boredom.

To find whether a curve has strictly positive rank, for a given (x, y) pair, we use the Buhler-Gross-Zagier [**2**] method of computing the L-series and its derivatives to estimate the rank, if the conductor of the



curve is not too large. If the rank is estimated at least 2 or rank= 1 and the Birch and Swinnerton-Dyer conjecture predicts a small height for the generator, we search for integer points on the minimal model of the curve, which are then transformed to possible generators of the original curve.

3. General quadrilateral. This is a convex quadrilateral with no parallel sides and no equal or complementary angles, where we can assume without loss of generality that x is the largest side. Unlike triangles, the angles of a quadrilateral are not uniquely specified by the sides, so we need to specify at least one angle.

We generate integer values for x and w and $\cos\alpha = m/n$, since integer diagonals imply rational cosines of the internal angles. We first test if $BD = v$ is a rational solution of $v^2 = x^2 + w^2 - 2xw\cos\alpha$. If it is, we generate sides y, z in the range $v < y + z < 2x$ and determine $\cos\angle ABD$, $\sin\angle ABD$, $\cos\angle DBC$ and $\sin\angle DBC$, by elementary trigonometry.

These relations give $\cos B$, and we test the rationality of $u = AC$ from $u^2 = x^2 + y^2 - 2xy\cos B$. If we find rational u and v, we test for

$$\{x, y, z, w, u, v\}^2 + h^2 = \square \tag{3.1}$$

by using the elliptic curve method of Section 2. We find, if possible, values of h with $\{x^2, y^2\} + h^2 = \square$ and test these with $\{z, w, u, v\}$.

Despite extensive testing, we were unable to find a perfect prism, which is, to some extent, not surprising. Equation (3.1) give six identities which must be satisfied with distinct integers, suggesting that the corresponding elliptic curve, of the form (2.1) would need to be of high rank.

We did, however, find several quadrilaterals where only one of the side or space diagonals is not an integer. These are given in Table 1. A quick survey of Table 1 shows that all the quadrilaterals have at least one pair of sides/diagonals identical. We have been unable to find any quadrilaterals, with all horizontal lengths distinct, where more than four vertical diagonals are rational. The five results found so far are given in Table 2.

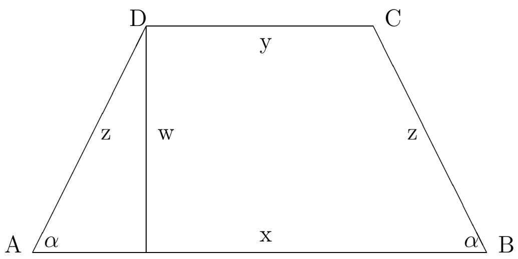
TABLE 1. Near-solution lengths for general quadrilaterals.

x	y	z	w	u	v	h	$\cos\alpha$	$\cos B$
91	80	45	63	45	35	60	25/26	113/130
128	56	36	56	88	88	105	23/28	23/28
171	150	84	150	192	192	80	29/100	29/100
159	147	48	147	171	171	140	37/98	37/98
273	112	385	273	343	273	180	1/2	-1/2
441	210	96	210	294	294	280	23/28	23/28
2223	2185	513	2185	2432	2432	420	9/23	9/23
7735	7616	3927	7616	9401	9401	3960	1/4	1/4
23205	16779	5712	16779	20349	20349	2860	49/94	49/94

TABLE 2. Distinct-sided quadrilaterals with 4 rational vertical diagonals.

x	y	z	w	u	v	h	$\cos\alpha$	$\cos B$
152	140	56	36	88	154	105	1/16	23/28
390	380	315	95	374	425	168	-5/19	251/475
420	60	600	288	472	540	175	-2/15	-191/225
728	504	462	266	640	735	2520	5/32	25/49
1672	616	396	1540	1664	968	1155	23/28	167/847

These results lead to the following:



Conjecture. A right quadrilateral prism with distinct integer sides and diagonals, and distinct angles in the base and top does not exist.

4. Isosceles trapezium. An isosceles trapezium is a shape as shown in the diagram, where AB and DC are parallel, and $x, y, z \in \mathbf{Z}$. Thus $\cos\alpha = (x-y)/2z$, so that $\cos\alpha$ is rational.

The perpendicular gap w satisfies

$$w^2 = z^2 - (x-y)^2/4$$

and the base diagonal is $v = AC = BD$, which satisfies

$$v^2 = \frac{(x+y)^2}{4} + w^2 = z^2 + xy$$

The vertical side diagonals give the three equations

$$x^2 + h^2 = \square, \qquad y^2 + h^2 = \square, \qquad z^2 + h^2 = \square$$

whilst there is only one space diagonal equation, namely, $v^2 + h^2 = \square$.

We have $x/h = P(a)$, $y/h = P(b)$ and $z/h = P(c)$ for some rational $a, b, c > 1$. We, thus, generate values of a, b, c, find $P(a), P(b), P(c)$ and hence suitable integers x, y, z, h. We reject those for which, firstly, $z^2 + xy \neq \square$, and then $v^2 + h^2 \neq \square$.

This simple search was coded in Pari-GP and quickly found several examples of perfect prisms. The 10 smallest (using the measure $\|\cdot\| = x + y + z + v + h$) are given in Table 3.

TABLE 3. Data for perfect isosceles trapezium prisms.

x	y	z	v	h	c	d
364	275	320	450	240	89/640	5/14
1152	507	780	1092	1040	43/104	13/48
3325	1053	1620	2475	2160	284/405	9/35
3328	361	1824	2128	3420	989/1216	19/208
3549	2601	1326	3315	1768	79/221	51/91
4225	2527	2223	3952	2964	283/741	133/325
5632	4693	1368	5320	1824	313/912	247/352
2754	1984	4455	5031	7560	7/81	32/153
6647	3168	5950	7514	2040	497/1700	4/17
10633	2300	4550	6720	5040	641/700	10/49

The presence of so many solutions leads one to hope for a parametric solution. One approach is to use the parameterization of the lengths of an isosceles trapezium found by considering $v^2 = x^2 + z^2 - 2xz\cos\alpha$. Standard methods for parameterising quadrics lead to the formulae

$$x = \alpha 2(c+d), \qquad y = \alpha 2d(1+c\,d),$$
$$z = \alpha(1-d^2), \qquad v = \alpha(2cd+1+d^2)$$

where $\alpha \in \mathbf{Q}$ and $|d| < 1$, with $c = \cos\alpha \equiv i/j$ with $i, j \in \mathbf{Z}$. In fact $d = (v-z)/x$. The values of c, d for the perfect prisms found are given in Table 3.

The expression for z suggests defining $h = 2\alpha d$, so that $z^2 + h^2 = \square$. Given this h, for x^2+h^2 and y^2+h^2 to be squares, we need, respectively,

$$c^2 + 2cd + 2d^2 = \square, \qquad c^2d^2 + 2cd + 2 = \square$$

while the $v^2 + h^2$ square needs

$$d^4 + 4cd^3 + (4c^2+6)d^2 + 4cd + 1 = \square.$$

The quadric $c^2 + 2cd + 2d^2 = \square$ can be easily parameterized by $c = g^2 - 2f^2$ and $d = 2f(f-g)$, but when we substitute these into the second quadric we get a sextic in g or an octic in f which must be made square. These curves are likely to have genus greater than 1, and so only a finite number of rational solutions, by Mordell's theorem.

The quartic has an obvious rational solution, so it is birationally equivalent to an elliptic curve, and we find this curve to be

$$E_{ij} : v^2 = u^3 + i^2u^2 - j^4u$$

with the vital reverse transformation

$$d = \frac{v - iu}{j(u + j^2)}$$

The curves E_{ij} have the following properties:

1. The torsion group is isomorphic to $\mathbf{Z}^2$. There is only one finite torsion point $(0, 0)$.

2. The curve has 2 components since $u^3 + i^2u^2 - j^4u = 0$ has 3 real roots, one positive, one negative and one zero.

3. There are points of infinite order at $(j^2, \pm ij^2)$ and $(-j^2, \pm ij^2)$, with these related by adding the point $(0, 0)$. Thus, the rank of the curve is at least 1, but can be higher.

The infinite order points mentioned lead to $d = 0, -c$ which give $x = 0$ or $y = 0$. To derive a non-trivial set of formulae, we must double the points. Ignoring those which still give trivial solutions, we first find the simple form

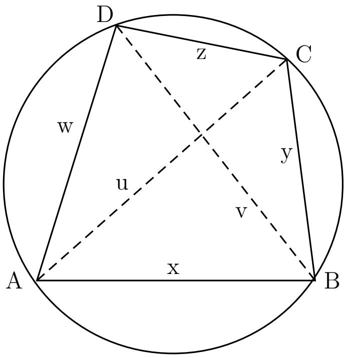
$$d = \frac{1 - c^2}{c + c^3},$$

giving both $z^2 + h^2 = \square$ and $v^2 + h^2 = \square$. To have $x^2 + h^2 = \square$ and $y^2 + h^2 = \square$ we need, respectively,

$$c^8 + 3c^4 - 2c^2 + 2 = \square, \qquad c^4 + 2c^2 + 5 = \square.$$

The second quartic is birationally equivalent to the elliptic curve $V^2 = U^3 - U^2 - U$ which has rank 0 and only has one rational point $(0, 0)$, not giving a feasible solution. Thus, we would need to go to higher points on the E_{ij} curve which will have more complex formulae and even less chance of being rational. This approach seems doomed to failure.

5. Cyclic quadrilateral. The isosceles trapezium is an example of a cyclic quadrilateral: a quadrilateral whose 4 vertices lie on the circumference of a circle.



We look for lengths (x, y, z, w, u, v) which are all positive integers first. This has been considered by Schubert [**12**] and Sastry [**9, 10**]. There is an error in the basic formulae in [**9**], whilst [**10**] uses a somewhat unusual parameterization.

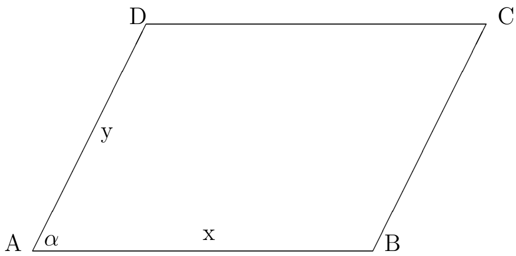
We derive a more standard form here. The results are based on the well-known formula for the length of an arc of a circle of radius R, namely, $2R\sin\theta$ where θ is the angle the arc makes with the circumference. Let $\alpha = \angle ADB = \angle ACB$, $\beta = \angle BAC = \angle BDC$, $\gamma = \angle CAD = \angle CBD$ and $\delta = \angle ABD = \angle ACD$. Since $ABCD$ is a cyclic quadrilateral, opposite angles are complimentary; thus, $\alpha + \beta + \gamma + \delta = \pi$, so there are only 3 free choices of angle.

Therefore, assuming without loss of generality that $2R = 1$, we have

$$x = \sin\alpha, \qquad y = \sin\beta, \qquad z = \sin\gamma,$$
$$w = \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\beta\cos\gamma + \cos\alpha\cos\beta\sin\gamma - \sin\alpha\sin\beta\sin\gamma$$
$$u = \sin\alpha\cos\beta + \cos\alpha\sin\beta, \qquad v = \sin\beta\cos\gamma + \cos\beta\sin\gamma.$$

Thus, the sines and cosines of these angles are rational so we can assume

$$\cos\alpha = \frac{f^2-1}{1+f^2}, \qquad \sin\alpha = \frac{2f}{1+f^2}, \qquad \cos\beta = \frac{g^2-1}{1+g^2},$$
$$\sin\beta = \frac{2g}{1+g^2}, \qquad \cos\gamma = \frac{h^2-1}{1+h^2}, \qquad \sin\gamma = \frac{2h}{1+h^2},$$



with $f, g, h \in \mathbf{Q}$.

Clearing denominators, we have

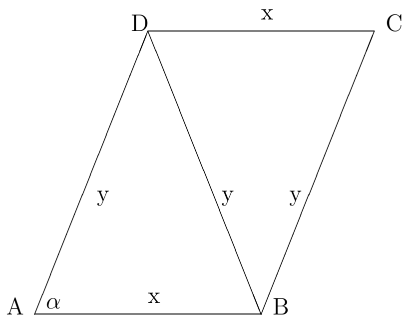
$$\begin{aligned} x &= f(g^2+1)(h^2+1), \\ y &= g(f^2+1)(h^2+1) \\ z &= h(f^2+1)(g^2+1), \\ w &= (f(g+h)+gh-1)(f(gh-1)-g-h) \\ u &= (f+g)(fg-1)(1+h^2), \\ v &= (g+h)(gh-1)(1+f^2). \end{aligned}$$

Despite extensive computing using these formulae, we have been unable to generate a cyclic quadrilateral, with all sides and diagonals mutually distinct, where more than three of the t^2+h^2 terms are square. The smallest quadrilateral with three squared terms found so far is $x = 561$, $y = 750$, $z = 1275$, $w = 1560$, $u = 1575$, $v = 1197$, $h = 400$, $\cos\alpha = 11/2$, $\cos\beta = 4$ and $\cos\gamma = 2$.

6. Parallelogram. The parallelogram $ABCD$ has sides $AB \parallel DC$ and $AD \parallel BC$, with $AB = DC = x$ and $AD = BC = y$.

Let $AC = z$ and $BD = w$. Then:

$$\begin{aligned} z^2 &= (x + y\cos\alpha)^2 + (y\sin\alpha)^2 = x^2 + y^2 + 2xy\cos\alpha \\ w^2 &= (x - y\cos\alpha)^2 + (y\sin\alpha)^2 = x^2 + y^2 - 2xy\cos\alpha. \end{aligned}$$



These are both homogeneous quadrics, and the intersection of two such elements is known to be an elliptic curve. We can parameterize the first equation as

$$\frac{x}{y} = \frac{2(\cos\alpha - \mu)}{\mu^2 - 1},$$

which we substitute into the second quadric, giving

$$d^2 = \mu^4 + 4k\mu^3 + 2(1 - 2k^2)\mu^2 - 12k\mu + 8k^2 + 1,$$

where $k = \cos\alpha = m/n$ since $\cos\alpha$ is clearly rational.

This quartic is birationally equivalent to an elliptic curve, which we eventually find to be

$$h^2 = g^3 + 2(m^2 + n^2)g^2 + (m^2 - n^2)^2 g$$

with the relation

$$\mu = \frac{h - 2m(m^2 - n^2)}{n(g - (m^2 - n^2))}.$$

The elliptic curve has torsion subgroup isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_4$, with finite torsion points of order 2 for $g = 0$, $-(m+n)^2$, $-(m-n)^2$ and order 4 points at $g = \pm(m^2 - n^2)$. None of these lead to non-trivial μ and x/y. Thus, we need curves of strictly positive rank. We use the same approach as described in Section 2 to find curves with infinite generators. We generate multiples of such generators, and these give different μ and hence different x, y, z, w.

For given x, y values, we use the method of Section 2 to find values of h, if any, where $x^2 + h^2 = \square$ and $y^2 + h^2 = \square$. These values of h are then tested for $z^2 + h^2 = \square$ and $w^2 + h^2 = \square$.

Unfortunately, despite extensive testing, no solutions were found. In fact, only rarely did we have one of $z^2 + h^2$ or $w^2 + h^2$, a square, and this almost always occurred when z or w equalled x or y.

Consider the special parallelogram $ABCD$ given above. Then $\cos\alpha = x/(2y)$ and $AC^2 = z^2 = 2x^2 + y^2$. This can be simply parameterized by $x = 2pq, y = p^2 - 2q^2, z = p^2 + 2q^2$. We can thus easily generate values of x, y, z and hence use section 2 to find values of h.

A large amount of testing has only found a single solution where all sides, face diagonals and space diagonals are integers. This occurs where $x = 2522$, $y = 16199$, giving $w = 16199$, $z = 16587$, and $\cos\alpha = 13/167$. If we have a height $h = 9240$, we have side diagonals of 9578 and 18649, and the two space diagonals 18649 and 18987.

It might be thought that we can set $h = p^2 - q^2$ since $x^2 + h^2 = \square$. For $y^2 + h^2$ and $z^2 + h^2$ to be square we need, respectively, $2p^4 - 6p^2q^2 + 5q^4 = \square$ and $2p^4 + 2p^2q^2 + 5q^4 = \square$. Both have obvious solutions so are birationally equivalent to elliptic curves. We find the curves and transformations to be

$$E_1 : v^2 = u^3 + 12u^2 - 4u, \quad \frac{p}{q} = \frac{4u + v + 4}{2u + v - 4}$$
$$E_2 : v^2 = u^3 - 4u^2 - 36u, \quad \frac{p}{q} = \frac{4u + v + 12}{v - 2u - 12}$$

in the correct order.

Both curves have only one finite torsion point $(0, 0)$, both have rank 1 with E_1 having generator $G_1 = (1, 3)$ and E_2 generator $G_2 = (9, 9)$. We can easily generate, using Pari-GP, the points on E_1, compute p/q and see if the second quartic is square. We have done this up to $(u, v) = 99G_1$ with no success.

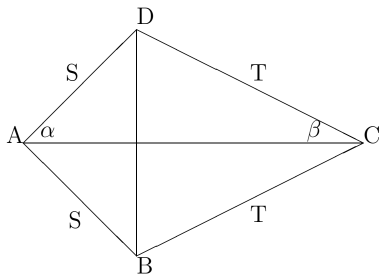


TABLE 4. Nearly-perfect kites.

S	T	z	w	h	$\cos\alpha$	$\cos\beta$
75	435	450	144	308	7/25	143/145
100	208	252	160	105	3/5	12/13
585	1190	1375	1008	1200	33/65	77/85
3300	13260	13800	6336	9625	7/25	1073/1105

7. Kite. A standard kite has the shape shown where the diagonals meet at right-angles with BD bisected by AC. Let $\angle DAC = \alpha$ and $\angle DCA = \beta$. Thus, $AC = z = S\cos\alpha + T\cos\beta$ and $BD = w = 2\,S\sin\alpha = 2\,T\sin\beta$. Thus, let

$$\cos\alpha = \frac{f^2-1}{f^2+1}, \quad \sin\alpha = \frac{2f}{f^2+1}, \quad \cos\beta = \frac{g^2-1}{g^2+1}, \quad \sin\beta = \frac{2g}{g^2+1},$$

where $f, g \in \mathbf{Q}$ and $f, g > 1$.

The search for rational h with $S^2 + h^2 = \square$ and $T^2 + h^2 = \square$ involves the elliptic curve

$$v^2 = u(u+S^2)(u+T^2)$$

as discussed in Section 2.

The equation for w gives

$$g^2 - 2kg + 1 = 0, \qquad k = \frac{T}{S\sin\alpha},$$

so that a rational g needs $T^2 - S^2\sin^2\alpha = \square$, and analyzing this along the lines of Section 2 leads to the curve

$$V^2 = U(U-S^2)(U-T^2)$$

with $f = V/(T(U-S^2))$. This curve has (in general) only three torsion points, all of order 2.

Testing has found no perfect kite, with all lengths and diagonals integer. Table 4 lists some examples of nearly-perfect kites where only one space-diagonal is irrational.

We can parameterize a kite as having sides and diagonals which are multiples of

$$S = g(f^2+1), \quad T = f(g^2+1), \quad z = f(g^2-1)+g(f^2-1), \quad w = 4fg.$$

If we define $h = 4f^2 - g^2$, then $w^2 + h^2 = \square$. To have $S^2 + h^2 = \square$, we must have

$$(g^2+16)f^4 - 6g^2f^2 + g^2(g^2+1) = \square,$$

and a little searching shows that $f = g/2$ gives a solution. Thus, this quartic is birationally equivalent to an elliptic curve.

Standard analysis shows that the equivalent curve is

$$v^2 = u^3 + 12p^2q^2u^2 - 4p^2q^2(p^2+4q^2)^2u,$$

where $g = p/q$ with p, q integers with no common factors. Numerical investigations suggest that the obvious point $(0,0)$ is the only common torsion point, and that, in general, the torsion subgroup is isomorphic to $\mathbf{Z}_2$.

The transformation back from the elliptic curve is

$$f = \frac{p(v+4q^2u+8p^2q^2(p^2+4q^2))}{2q(v-p^2u-8p^2q^2(p^2+4q^2))}.$$

If we set the denominator to 0, we get $v = p^2(u+8p^2q^2+32q^4)$, and if we substitute into the elliptic curve, we find $u = -16p^2q^2$ and $v = \pm 8p^2q^2(p^2-4q^2)$. Since we suspect that there is only one finite torsion point, this would have to be a point of infinite order. Doubling this point gives

$$u = \frac{(p^4+72p^2q^2+16q^4)^2}{16(p^2-4q^2)^2},$$

and the fractional nature of this shows that $-16p^2q^2$ will give a point of infinite order for most (p, q) pairs. Thus, the elliptic curve has rank at least 1 in most cases, with the largest rank found so far being 4, starting with $p = 17$, $q = 8$. So, we can easily find kites with $(S^2, w^2) + h^2 = \square$.

We wrote a simple program which searched for other generators on these curves, generated points on the curve, and hence (S, T, z, w, h) sets which we checked for more than two $a^2 + h^2 = \square$. These were very sparsely distributed and we only found one new solution with 3 squares - $(S, T, z, w, h) = (8745, 4182, 10881, 6336, 14840)$.

It might be thought that making $T^2 + h^2 = \square$ first would produce a similar outcome. We require

$$16f^4 + (g^4 - 6g^2 + 1)f^2 + g^4 = \square,$$

which can be linked to the elliptic curve

$$\begin{aligned} v^2 = u^3 &- 2(p^4 - 6p^2q^2 + q^4)u^2 \\ &+ (p^2 + q^2)^2(p^2 + 4pq + q^2)(p^2 - 4pq + q^2)u, \end{aligned}$$

with $g = p/q$ and transformation $f = v/(8q^2u)$.

These curves can have rank 0 and torsion subgroup isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_4$. Using this form, we found no perfect kites and only one new near-perfect kite with sides $(1883, 1924, 1107, 3640, 2400)$.

Finally, we can look to make $(w^2, z^2) + h^2 = \square$. Using the prior parameterizations, we require

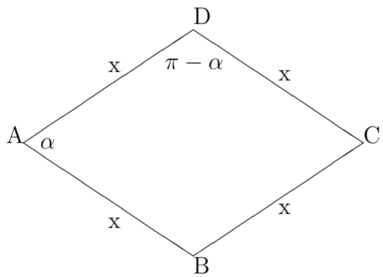
$$\begin{aligned} (g^2 + 16)f^4 &+ 2g(g^2 - 1)f^3 \\ &+ (g^4 - 12g^2 + 1)f^2 - 2g(g^2 - 1)f + g^2(g^2 + 1) = \square, \end{aligned}$$

where, again, we find $f = g/2$ gives a solution. Transforming, we find the elliptic curve

$$v^2 = u^3 + (p^4 + 18p^2q^2 + q^4)u^2 + 12p^2q^2(p^4 - 4q^4)u,$$

where $g = p/q$ and the inverse transformation is

$$f = \frac{p(v + (p^2 - 5q^2)u + 24p^2q^2(p^2 - 2q^2))}{2q(v + (q^2 - 2p^2)u - 24p^2q^2(p^2 - 2q^2))}.$$



As before, the torsion subgroup seems only to contain $(0,0)$ as a finite point. Solving either the numerator or denominator for v and substituting into the elliptic curve gives the point $(-16p^2q^2, \pm 8p^2q^2(p^2+4q^2))$, so the curves have rank at least 1. Numerical experiments show that these curves can have quite a large rank; the largest found is 6 starting with $p = 29$ and $q = 15$, and the rank was always at least 2. This led to investigating the simple integer points and the discovery of a second point of infinite order $(-(p^2+q^2)^2, \pm 2pq(p^2+q^2)(p^2+4q^2))$. Surprisingly, given all points available, we were unable to find any kites with even three vertical integer diagonals.

8. Rhombus. A rhombus is both a kite and a parallelogram with shape as shown in the figure above. Let $AC = v = x\sqrt{2(1+\cos\alpha)}$ and $BD = u = x\sqrt{2(1-\cos\alpha)}$. Thus, if we wish $u, v \in \mathbf{Q}$, we must have

$$2(1+\cos\alpha) = p^2, \qquad 2(1-\cos\alpha) = q^2$$

with $p, q, \in \mathbf{Q}$.

Thus,

$$p^2 + q^2 = 4, \qquad \cos\alpha = \frac{p^2 - q^2}{4},$$

and so

$$p = \frac{2(r^2-1)}{r^2+1}, \qquad q = \frac{4r}{r^2+1}$$

with $r = m/n \in \mathbf{Q}$.

We can assume, without loss of generality, that $x = 1$, giving $v = p, u = q$, and we can find h with $v^2 + h^2 = \square$ and $u^2 + h^2 = \square$ as in Section 2. Thus, the only problem is to find m, n with $1 + h^2 = \square$.

This is, essentially, looking for m, n, H with

$$4(m^2 - n^2)^2 + H^2 = \square, \qquad 16m^2n^2 + H^2 = \square, \qquad (m^2 + n^2)^2 + H^2 = \square$$

and $h = H/(m^2 + n^2)$.

To date, no such solution has been found, despite extensive computations.

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