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CARATHÉODORY-TYPE RESULTS FOR THE SUMS AND UNIONS OF CONVEX SETS

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ABSTRACT. This paper contains several Carathéodory-type results on extreme representations of sums and unions of finitely many closed convex sets or polyhedra in $\mathbf{R}^n$ in terms of their faces.

1. Introduction and main results. A well-known result of convex geometry states that any point z of a compact convex set $K \subset \mathbf{R}^n$ can be expressed as a convex combination of $n+1$ or fewer extreme points of K (this fact is an immediate consequence of Carathéodory's [**2**] and Minkowski's [**7**, page 160] theorems). Similarly, if K is a line-free closed convex set in $\mathbf{R}^n$, then z is a convex combination of $n+1$ or fewer points such that each of the points is either extreme or belongs to an extreme ray of K (see Klee [**5**]). Simple examples show that the number $n+1$ cannot be decreased in this context. If it is desirable to express z as a convex combination of fewer than $n+1$ points from K, then, instead of extreme points or rays, one can consider faces of K. Our goal here is to study such extreme representations combined with the operations of addition and union of convex sets.

In what follows, $\operatorname{relbd} K$ and $\operatorname{relint} K$ stand for the relative boundary and relative interior of a closed convex set $K \subset \mathbf{R}^n$, and $\operatorname{ext} K$ and $\exp K$ denote, respectively, the set of extreme and exposed points of K. A convex set $K \subset \mathbf{R}^n$ is *line-free* if it contains no line of $\mathbf{R}^n$. We recall that an (extreme) *face* of a convex set $K \subset \mathbf{R}^n$ is a convex subset $F \subset K$ such that points $x, y \in K$ lie in F provided $(1-\lambda)x + \lambda y \in F$ for a suitable scalar $0 < \lambda < 1$. Extreme points and extreme rays are, respectively, zero-dimensional and one-dimensional faces. It is easy to see that each proper (i.e., distinct from K) face of K is closed and lies in $\operatorname{relbd} K$. Furthermore, for any point $x \in K$, there is a unique face F

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of K with the property $x \in \operatorname{relint} F$ (see, e.g., Schneider [**9**] for general references on convex sets).

Theorem 1. *Let $K_1, \dots, K_r$ be nonempty line-free closed convex sets in $\mathbf{R}^n$. For any point $z \in K_1 + \cdots + K_r$, there are nonempty faces F_i of K_i, $i = 1, \dots, r$, such that*

$$z \in F_1 + \cdots + F_r \quad \textit{and} \quad \dim F_1 + \cdots + \dim F_r \le n. \tag{1}$$

The proof of this theorem uses a lifting technique introduced (for the case of convex polyhedra) by Walkup and Wets [**13**]. If the sum $K_1 + \cdots + K_r$ in Theorem 1 is direct, then its assertion becomes a particular case of a statement on the Carathéodory number of the Cartesian product of convexity spaces (see [**10**] and [**11**, Theorem 8.18]). Also, Theorem 1 can be formulated in terms of partition of $K_1 + \cdots + K_r$ into sums of relative interiors of suitably chosen faces $F_i \subset K_i$ (in the spirit of [**6**]).

If the number r in Theorem 1 is greater than n, then at least $r - n$ of the faces F_i are singletons. This argument allows the refinement of the Shapley-Folkman lemma (see Starr [**12**]), which states that, for nonempty compact sets $X_1, \dots, X_r \subset \mathbf{R}^n$ and a point $z \in \operatorname{conv}(X_1 + \cdots + X_r)$, there is an index set $I \subset \{1, \dots, r\}$ with $|I| \le n$ such that

$$z \in \sum_{i \in I} \operatorname{conv} X_i + \sum_{i \notin I} X_i.$$

Corollary 1. *If $X_1, \dots, X_r$ are nonempty sets in $\mathbf{R}^n$, then for any point z in conv$(X_1 + \cdots + X_r)$, there is an index set $I \subset \{1, \dots, r\}$ with $|I| \le n$ and nonempty subsets $Y_i \subset X_i$, $i = 1, \dots, r$, such that*

$$z \in \sum_{i \in I} \operatorname{conv} Y_i + \sum_{i \notin I} Y_i,$$

$$\sum_{i \in I} |Y_i| \le n + |I|,$$

$$|Y_i| = 1 \quad \textit{for all } i \notin I.$$

Our next theorem deals with unions of convex sets.

Theorem 2. *Let* $K_1, \dots, K_r \subset \mathbf{R}^n$ *be nonempty line-free closed convex sets. For any point* $z \in \operatorname{conv}(K_1 \cup \dots \cup K_r)$*, there is an index set*

$$I \subset \{1, \dots, r\} \quad \textit{with } |I| \le n+1 \tag{2}$$

and nonempty faces F_i *of* K_i*,* $i \in I$*, such that*

$$z \in \operatorname{conv}\left(\sum_{i\in I} F_i\right) \quad \textit{and} \quad \sum_{i\in I} \dim F_i \le n. \tag{3}$$

If, additionally, all $K_1, \dots, K_r$ *are compact, then the inequality in* (3) *can be refined to*

$$\sum_{i\in I} \dim F_i \le n + 1 - |I|. \tag{4}$$

Corollary 2. *Let* $K \subset \mathbf{R}^n$ *be a nonempty line-free closed convex set and* r *a positive integer. For any point* $z \in K$*, there are nonempty faces* $F_1, \dots, F_s$ *of* K *with* $s \le \min\{r, n+1\}$ *such that*

$$z \in \operatorname{conv}(F_1 \cup \dots \cup F_s) \quad \textit{and} \quad \dim F_1 + \dots + \dim F_s \le n. \tag{5}$$

If $r > 1$*, then* $F_1, \dots, F_s$ *can be chosen proper in* K *such that at least* $s-1$ *of them are of dimension one or less. If* K *is compact, then the inequality in* (5) *can be refined to*

$$\dim F_1 + \dots + \dim F_s \le n + 1 - s. \tag{6}$$

In 1991 Danielyan et al. [**3**] asked for a sharper version of Corollary 2 (written by us in terms of faces of a convex set in $\mathbf{R}^n$ versus k-extreme points of a compact convex set in a linear topological space L). We formulate their question as an open problem below, since its attempted proof in [**3**] uses the erroneous Property 3 (see [**3**, page 72]), as shown below. Following [**9**], a point x of a convex set M is called k-extreme if it does not belong to the relative interior of a $(k+1)$-dimensional simplex which entirely lies in M. Let $E_k M$ denote the set of k-extreme points of M (thus, $E_0 M = \operatorname{ext} M$). Property 3 from [**3**] states that if an m-dimensional plane V intersects a compact convex set $M \subset L$, then $E_i(M \cap V) \subset E_i M$ for all $i = 0, \dots, m$, which is incorrect. Indeed, let $M = \{(x, y, z) \mid 0 \le x, y, z \le 1\}$ be the unit cube in $\mathbf{R}^3$ and $V \subset \mathbf{R}^3$ the plane given by $z = 1/2$. If a, b, c, d are the vertices of the square $M \cap V$, then $E_0(M \cap V) = \{a, b, c, d\} \subset E_1 M \setminus E_0 M$. Similarly, $E_1(M \cap V) \subset E_2 M \setminus E_1 M$.

Problem 1 ([**3**]). *Let $K \subset \mathbf{R}^n$ be a nonempty compact convex set and $n_1, \dots, n_s$ positive integers with $n_1 + \cdots + n_s = n + 1$. Prove that, for any point $z \in K$, there are nonempty faces $F_1, \dots, F_s$ of K such that*

$$z \in \operatorname{conv}(F_1 \cup \cdots \cup F_s) \quad \textit{and} \quad \dim F_i \le n_i - 1 \textit{ for all } i = 1, \dots, s.$$

It is interesting to mention that Motzkin announced in a short abstract [**8**] (in which $\operatorname{conv}(F_1 \cup \cdots \cup F_k)$ should be read instead of $\operatorname{conv}(F_1 \cap \cdots \cap F_k)$) that Problem 1 has an affirmative solution when K is a convex polytope in $\mathbf{R}^n$ (see also Grünbaum [**4**, subsection 3.1, Exercise 21]). In this regard, we thank Branko Grünbaum for providing information about Motzkin's abstract. Corollary 3 below confirms Motzkin's statement. We also observe that Problem 1 has an affirmative solution for any compact convex set in $\mathbf{R}^n$ if $n \le 3$.

Two more results deal with intersections of convex polyhedra. In what follows, by a *polyhedron* we mean the intersection of finitely many closed halfspaces of $\mathbf{R}^n$; a *polytope* is the convex hull of finitely many points.

Lemma 1. *Let $P_1, \dots, P_t \subset \mathbf{R}^n$, $1 \le t \le n$, be nonempty line-free polyhedral cones with common apex o and $n_1, \dots, n_t$ nonnegative integers with $n_1 + \cdots + n_t = n$. For any point $z \in P_1 \cap \cdots \cap P_t$, there are nonempty faces F_i of P_i such that*

$$z \in \operatorname{conv}(F_1 \cup \cdots \cup F_t) \quad \textit{and} \quad \dim F_i \le n_i \textit{ for all } i = 1, \dots, t.$$

Theorem 3. *Let $P_1, \dots, P_s \subset \mathbf{R}^n$ be nonempty polytopes and $n_1, \dots, n_s$ positive integers with $n_1 + \cdots + n_s = n+1$. For any point $z \in P_1 \cap \cdots \cap P_s$, there are nonempty faces F_i of P_i, $i = 1, \dots, s$, such that*

$$z \in \operatorname{conv}(F_1 \cup \cdots \cup F_s) \quad \text{and} \quad \dim F_i \le n_i - 1 \text{ for all } i = 1, \dots, s.$$

With $s = n+1$ and $n_i = 1$ for all $i = 1, \dots, n+1$, Theorem 3 gives a new way to prove "the colorful version" of Carathéodory's theorem due to Bárány [**1**], which states that, given nonempty sets $X_1, \dots, X_{n+1} \subset \mathbf{R}^n$ and a point $z \in \operatorname{conv} X_1 \cap \cdots \cap \operatorname{conv} X_{n+1}$, there are points $v_i \in X_i$, $i = 1, \dots, n+1$, such that $z \in \operatorname{conv}\{v_1, \dots, v_{n+1}\}$.

Corollary 3. *Let $P \subset \mathbf{R}^n$ be a nonempty polytope and $n_1, \dots, n_s$ positive integers with $n_1 + \cdots + n_s = n+1$. For any point $z \in P$, there are nonempty faces $F_1, \dots, F_s$ of P such that*

$$z \in \operatorname{conv}(F_1 \cup \cdots \cup F_s) \quad \text{and} \quad \dim F_i \le n_i - 1 \text{ for all } i = 1, \dots, s. \qquad \square$$

In a standard way, both Corollaries 2 and 3 can be slightly refined by replacing n with $\dim K$ and $\dim P$, respectively.

2. Proofs.

Proof of Theorem 1. Choose an exposed point v_i of K_i (it exists because K_i is line-free) and denote by H_i a hyperplane with the property $K_i \cap H_i = \{v_i\}$, $i = 1, \dots, r$. We can write $H_i = \{x \in \mathbf{R}^n \mid \psi_i(x) = \alpha_i\}$, where ψ_i is a nonzero linear functional on $\mathbf{R}^n$ and α_i is a scalar. Without loss of generality, suppose that α_i is the minimum value of ψ_i on K_i.

We observe that, for any scalar $c \in \mathbf{R}$, the set $K_i(c) = \{x \in K_i \mid \psi_i(x) \le c\}$ is compact. Indeed, assume for a moment that $K_i(c)$ is not compact for a certain choice of $c \in \mathbf{R}$. Then $K_i(c)$ contains a halfline $h = [u, w)$. Since ψ_i attains a minimum on K_i, the halfline h must be parallel to H_i. In this case, $\psi_i(x) = \alpha_i$ for all $x \in h' = (v_i - u) + h$, implying that h' lies in $K_i \cap H_i$, in contradiction with the choice of H_i.

For each ψ_i, $i = 1, \dots, r$, choose a basis $\psi_i, \gamma_2^{(i)}, \dots, \gamma_n^{(i)}$ of linear functionals for the conjugate space $(\mathbf{R}^n)'$. Consider on $\mathbf{R}^n$ the nonsingular linear transformations

$$f_i(x) = (\psi_i(x), \gamma_2^{(i)}(x), \dots, \gamma_n^{(i)}(x)), \quad i = 1, \dots, r.$$

Given a vector $\overline{x} = (x_1, \dots, x_r) \in (\mathbf{R}^n)^r$, put

$$f(\overline{x}) = (f_1(x_1), \dots, f_r(x_r)).$$

Clearly, f is a nonsingular linear transformation on $(\mathbf{R}^n)^r$. Let

$$\overline{v} = (v_1, \dots, v_r) \quad \text{and} \quad K = K_1 \times \dots \times K_r.$$

From the choice of $\psi_1, \dots, \psi_r$ it follows that $f(\overline{v})$ is the unique lexicographically minimal point of $f(K)$. (Recall that a point $x = (x_1, \dots, x_m) \in \mathbf{R}^m$ is *lexicographically smaller* than a point $x' = (x'_1, \dots, x'_m) \in \mathbf{R}^m$, provided $x \neq x'$ and $x_i < x'_i$ for the smallest index $i = 1, \dots, m$ with $x_i \neq x'_i$.) Moreover, for any nonempty closed set $X \subset K$, the set $f(X)$ contains a unique lexicographically minimal point, $\overline{u}$. The existence of $\overline{u}$ immediately follows from continuity of f and compactness of $K(c) = K_1(c) \times \dots \times K_r(c)$ for any choice of $c \in \mathbf{R}$. Since f is nonsingular, the set $f^{-1}(\overline{u})$ is a singleton.

Consider the linear subspace

$$L = \{\overline{x} = (x_1, \dots, x_r) \in (\mathbf{R}^n)^r \mid x_1 + \dots + x_r = o\}.$$

For any point $\overline{x} \in K$, denote by $\varphi(\overline{x})$ the inverse image of the unique lexicographically minimal point of the set $f(K \cap (\overline{x} + L))$. Clearly, $\varphi(\overline{v}) = \overline{v}$ and $\varphi(\overline{x}) \in K \cap (\overline{x} + L)$ whenever $\overline{x} \in K$. Let $B = \varphi(K)$. Since $\varphi(\varphi(\overline{x})) = \varphi(\overline{x})$ for any $\overline{x} \in K$, we have

$$B = \{\overline{x} \in K \mid \varphi(\overline{x}) = \overline{x}\}.$$

Let z be a given point in $K_1 + \dots + K_r$. Then $z = z_1 + \dots + z_r$ for suitable points $z_i \in K_i$, $i = 1, \dots, r$. Put

$$\overline{z} = (z_1, \dots, z_r) \quad \text{and} \quad \varphi(\overline{z}) = (z'_1, \dots, z'_r).$$

Since $\varphi(\overline{z}) \in K \cap (\overline{z} \cap L)$, we have $\varphi(\overline{z}) = \overline{z} + \overline{x}$ for a suitable point $\overline{x} \in L$, so

$$z_1' + \cdots + z_r' = (z_1 + \cdots + z_r) + (x_1 + \cdots + x_r) = (z_1 + \cdots + z_r) + o = z.$$

Denote by F the face of K that contains $\varphi(\overline{z})$ in its relative interior. We state that $F \subset B$. Indeed, assume for a moment the existence of a point $\overline{w} \in F \setminus B$. Then $\varphi(\overline{w}) \neq \overline{w}$, which means that $f(\varphi(\overline{w}))$ is lexicographically smaller that $f(\overline{w})$. Since $\varphi(\overline{z}) \in \operatorname{relint} F$, there is a point $\overline{u} \in F$ such that $\varphi(\overline{z}) = (1-\lambda)\overline{u} + \lambda\overline{w}$ for a suitable scalar $0 < \lambda < 1$. Let $\overline{y} = (1-\lambda)\overline{u} + \lambda\varphi(\overline{w})$. By a convexity argument, $\overline{y} \in K$. Since both $\overline{w}$ and $\varphi(\overline{w})$ lie in $\overline{w} + L$, we have $\varphi(\overline{w}) - \overline{w} \in L$. Therefore,

$$\overline{y} - \varphi(\overline{z}) = \lambda(\varphi(\overline{w}) - \overline{w}) \in L.$$

Hence $\overline{y} \in K \cap (\varphi(\overline{z}) + L) = K \cap (\overline{z} + L)$. The equalities

$$f(\overline{y}) = (1-\lambda)f(\overline{u}) + \lambda f(\varphi(\overline{w})), \qquad f(\varphi(\overline{z})) = (1-\lambda)f(\overline{u}) + \lambda f(\overline{w})$$

show that $f(\overline{y})$ is lexicographically smaller than $f(\varphi(\overline{z}))$. This is impossible due to the definition of $\varphi(\overline{z})$. Summing up, $F \subset B$.

We can write $F = F_1 \times \cdots \times F_s$, where F_i is a nonempty face of K_i, $i = 1, \ldots, r$. From $\varphi(\overline{z}) \in F$, it follows that $z_i' \in F_i$ for all $i = 1, \ldots, r$. Observe that the linear transformation $\pi : (\mathbf{R}^n)^r \to \mathbf{R}^n$, defined by

$$\pi((x_1, \ldots, x_r)) = x_1 + \cdots + x_r,$$

is one-to-one on B. Indeed, let $\overline{x} = (x_1, \ldots, x_r)$ and $\overline{x}' = (x_1', \ldots, x_r')$ be two points in B with $\pi(\overline{x}) = \pi(\overline{x}')$. The equality $x_1 + \cdots + x_r = x_1' + \cdots + x_r'$ implies that $\overline{x} + L = \overline{x}' + L$. Due to the uniqueness of $\varphi(\overline{x})$ in $K \cap (\overline{x} + L)$, one has $\overline{x} = \varphi(\overline{x}) = \varphi(\overline{x}') = \overline{x}'$. This argument and the equalities

$$F = F_1 \times \cdots \times F_r, \qquad \pi(F) = F_1 + \cdots + F_r$$

show that

$$\begin{aligned} \dim F_1 + \cdots + \dim F_r &= \dim F = \dim \pi(F) \\ &= \dim (F_1 + \cdots + F_r) \leq n, \end{aligned}$$

because $F_1 + \cdots + F_r$ is a subset of $\mathbf{R}^n$. Finally,

$$z = z_1' + \cdots + z_r' \in F_1 + \cdots + F_r.$$

Proof of Corollary 1. From the inclusion

$$z \in \operatorname{conv}(X_1 + \cdots + X_r) = \operatorname{conv} X_1 + \cdots + \operatorname{conv} X_r,$$

it follows that z can be expressed as $z = z_1 + \cdots + z_r$, where $z_i \in \operatorname{conv} X_i$. By Carathéodory's theorem, there is a finite set $Z_i \subset X_i$ such that $z_i \in \operatorname{conv} Z_i$. Consider the convex polytopes $P_i = \operatorname{conv} Z_i$, $i = 1, \dots, r$. By Theorem 1, one can find nonempty faces F_i of P_i, $i = 1, \dots, r$, such that the conditions (1) hold. Put $n_i = \dim P_i$.

Clearly, $F_i = \operatorname{conv} V_i$ for a certain subset V_i of Z_i. Again by Carathéodory's theorem, there are subsets $Y_i \subset V_i$ such that

$$z_i \in \operatorname{conv} Y_i \quad \text{and} \quad |Y_i| \le n_i + 1 \text{ for all } i = 1, \dots, r.$$

The inequality $n_1 + \cdots + n_r \le n$ implies the existence of an index set $I \subset \{1, \dots, r\}$ with $|I| \le n$ such that $n_i = 0$ for all $i \notin I$. Equivalently, $|Y_i| = 1$, or $Y_i = \{z_i\}$, when $i \notin I$. Finally,

$$z = \sum_{i=1}^{r} z_i \in \sum_{i \in I} \operatorname{conv} Y_i + \sum_{i \notin I} Y_i,$$

where

$$\sum_{i \in I} |Y_i| \le n + |I| \quad \text{and} \quad |Y_i| = 1 \text{ for all } i \notin I.$$

Proof of Theorem 2. According to Carathéodory's theorem, there are points

$$z_1, \dots, z_s \in K_1 \cup \cdots \cup K_r, \quad s \le n + 1,$$

such that z can be written as their convex combination:

$$z = \alpha_1 z_1 + \cdots + \alpha_s z_s, \quad \alpha_1, \dots, \alpha_s \ge 0, \quad \alpha_1 + \cdots + \alpha_s = 1. \tag{7}$$

Eliminating zero scalars, we may suppose that all $\alpha_1, \dots, \alpha_s$ are positive. Furthermore, $z_1, \dots, z_s$ can be chosen such that no two of them belong to the same set K_i. Indeed, if $z_p, z_q \in K_i$, we replace $\alpha_p z_p + \alpha_q z_q$ in (7) with $(\alpha_p + \alpha_q) z'_p$, where z'_p is a convex combination of z_p and z_q:

$$z'_p = \frac{\alpha_p}{\alpha_p + \alpha_q} z_p + \frac{\alpha_q}{\alpha_p + \alpha_q} z_q \in K_i.$$

As a result, $s \le r$. Renumbering $z_1, \dots, z_s$ and $K_1, \dots, K_r$, we assume that $z_i \in K_i$ for all $i = 1, \dots, s$. Put $I = \{1, \dots, s\}$.

Consider the convex set $K = K_1 \times \cdots \times K_s \subset (\mathbf{R}^n)^s$. Since all $K_1, \dots, K_s$ are closed and line-free, K is also closed and line-free. Put

$$P = \{(x_1, \dots, x_s) \in (\mathbf{R}^n)^s \mid \alpha_1 x_1 + \cdots + \alpha_s x_s = z\}. \tag{8}$$

With given positive scalars $\alpha_1, \dots, \alpha_s$ and vector $z = (\eta_1, \dots, \eta_n) \in \mathbf{R}^n$, the equality $\alpha_1 x_1 + \cdots + \alpha_s x_s = z$ can be rewritten as the system of n linear equations in ns scalar variables $x_i = (\xi_1^{(i)}, \dots, \xi_n^{(i)})$, $i = 1, \dots, s$:

$$\alpha_1 \xi_j^{(1)} + \alpha_2 \xi_j^{(2)} + \cdots + \alpha_s \xi_j^{(s)} = \eta_j, \quad j = 1, \dots, n.$$

Because the rank of this system is n, we conclude that P is an affine subspace of $(\mathbf{R}^n)^s$ of dimension $n(s-1)$. This argument shows that $M = K \cap P$ is a line-free closed convex set of dimension $n(s-1)$ or less. Furthermore, $(z_1, \dots, z_s) \in M$ due to (7).

Since M is line-free, it contains an extreme point, $\overline{w} = (w_1, \dots, w_n)$. Denote by F the (unique) face of K that contains $\overline{w}$ in its relative interior. We can write $F = F_1 \times \cdots \times F_s$, where each F_i is a nonempty face of K_i such that $w_i \in \operatorname{relint} F_i$, $i = 1, \dots, s$. From this and the inclusion $\overline{w} \in M$, it follows that

$$z = \alpha_1 w_1 + \cdots + \alpha_s w_s \in \operatorname{conv}(F_1 \cup \cdots \cup F_s).$$

It remains to prove the inequality from (3). First, we observe that $F \cap P = \{\overline{w}\}$. Indeed, assume the existence of another point $\overline{u} \in F \cap P$. From $\overline{w} \in \operatorname{relint} F$, it follows that $\overline{w} \in \,]\overline{u}, \overline{v}[\, \subset F$ for a suitable point $\overline{v} \in F$. Clearly, $]\overline{u}, \overline{v}[\, \subset P$, which implies the inclusion $\overline{w} \in \,]\overline{u}, \overline{v}[\, \subset F \cap P \subset M$. The last is in contradiction with the assumption $\overline{w} \in \operatorname{ext} M$.

Denote by Q the smallest affine subspace of $(\mathbf{R}^n)^s$ containing F. We state that $P \cap Q = \{\overline{w}\}$. Indeed, assume that $P \cap Q$ contains more than one point. Then $P \cap Q$ contains a line l through $\overline{w}$. Since $\overline{w} \in \operatorname{relint} F$, the intersection $l \cap F$ contains an open line segment $]\overline{a}, \overline{c}[$ with $\overline{w} \in \,]\overline{a}, \overline{c}[\, \subset F$. Because $l \subset P$, we have $\overline{w} \in \,]\overline{a}, \overline{c}[\, \subset P \cap F \subset M$, which is impossible due to $\overline{w} \in \operatorname{ext} M$.

From $P \cap Q = \{\overline{w}\}$ it follows that $\dim P + \dim Q \le \dim (\mathbf{R}^n)^s = ns$. Hence,

$$\dim F_1 + \cdots + \dim F_s = \dim F = \dim Q \le ns - \dim P = n.$$

Now assume that all $K_1, \ldots, K_r$ are compact. We identify $\mathbf{R}^n$ with the hyperplane

$$H = \{(x_1, \ldots, x_{n+1}) \mid x_{n+1} = 1\} \subset \mathbf{R}^{n+1}.$$

Let K_i' be the convex cone with apex o generated by K_i, $i = 1, \ldots, r$. Clearly, $K_1', \ldots, K_r'$ are line-free closed convex cones. From the above (with $\mathbf{R}^{n+1}$ instead of $\mathbf{R}^n$), there is an index set $I \subset \{1, \ldots, r\}$ with $|I| \le n + 2$ and nonempty faces G_i of K_i', $i \in I$, such that

$$z \in \operatorname{conv}\left(\bigcup_{i \in I} G_i\right) \quad \text{and} \quad \sum_{i \in I} \dim G_i \le n + 1. \tag{9}$$

Without loss of generality, we may assume that $I = \{1, \ldots, s\}$, where $s \le \min\{r, n + 2\}$.

All faces $G_1, \ldots, G_s$ are line-free closed convex cones with common apex o. We may assume that each G_i is distinct from $\{o\}$, since otherwise the removal of the respective index i from I, keeps both conditions from (9). Furthermore, $s \le n + 1$, since otherwise $\dim G_1 + \cdots + \dim G_s \ge n + 2$.

The inequalities $\dim G_i \ge 1$, $i = 1, \ldots, s$, also imply that each G_i meets H. Put $F_i = G_i \cap H$, $i = 1, \ldots, s$. Clearly, $\dim F_i = \dim G_i - 1$, resulting in

$$\dim F_1 + \cdots + \dim F_s \le \dim G_1 + \cdots + \dim G_s - s \le n + 1 - s.$$

It remains to show that $z \in \operatorname{conv}(F_1 \cup \cdots \cup F_s)$. Indeed, as in the beginning of the proof of this theorem, the inclusion $z \in \operatorname{conv}(G_1 \cup \cdots \cup G_s)$ yields the existence of points $z_i \in G_i$, $i = 1, \ldots, s$, such that (7) holds for a suitable choice of scalars $\alpha_1, \ldots, \alpha_s$. Without loss of generality, we may suppose that all vectors $z_1, \ldots, z_s$ are distinct from o. Therefore, we can choose positive scalars $\beta_1, \ldots, \beta_s$ such that $w_i = \beta_i z_i \in H$ for all $i = 1, \ldots, s$. Since $w_i \in G_i$, we have $w_i \in F_i$, $i = 1, \ldots, s$, and

$$(10) \qquad z = (\alpha_1\beta_1)w_1 + \cdots + (\alpha_s\beta_s)w_s, \quad \alpha_1\beta_1, \ldots, \alpha_s\beta_s > 0.$$

The inclusions $z, w_1, \ldots, w_s \in H$ imply that the $(n+1)$th coordinate of all points $z, w_1, \ldots, w_s$ equals 1. This argument and (10) give $\alpha_1\beta_1 + \cdots + \alpha_s\beta_s = 1$, which shows that z is a convex combination of $w_1, \cdots, w_s$. Therefore $z \in \operatorname{conv}(F_1 \cup \cdots \cup F_s)$.

Proof of Corollary 2. The existence of nonempty faces $F_1, \ldots, F_s$ of K, with $s \le \min\{r, n+1\}$, which satisfy the conditions (2) immediately follows from Theorem 2. Let $r > 1$. We intend to prove the second statement of the corollary by induction on $m = \dim K$. Since the case $\dim K \le 1$ is trivial, we may assume that $m > 1$. Because K is line-free, there is a hyperplane $H \subset \mathbf{R}^n$ through z such that $K \cap H$ is bounded. Let u be an extreme point of $K \cap H$. It is easy to see that the face F_1 of K that contains u in its relative interior is either a singleton or a halfline, implying that $\dim F_1 \le 1$. If $u = z$, the proof is finished. Let $u \ne z$. The line (u, z) intersects $\operatorname{bd} K$ at a point $v \in \operatorname{bd} K$ distinct from u. Denote by F the face of K such that $v \in \operatorname{relint} F$. Then F is a line-free closed convex set such that $\dim F \le n - 1$ and $z \in \operatorname{conv}(F_1 \cup F)$. In particular, F lies in a hyperplane G of $\mathbf{R}^n$, which can be identified with $\mathbf{R}^{n-1}$. By the inductive assumption, there are faces $F_2, \ldots, F_s$, $s \le r$, such that

$$v \in \operatorname{conv}(F_1 \cup \cdots \cup F_s), \qquad \dim F_2 + \cdots + \dim F_s \le n - 1,$$

and $\dim F_i \le 1$ for at least $s - 2$ of them. Clearly, $F_1, \ldots, F_s$ satisfy the conditions (5). The last statement of the corollary trivially follows from Theorem 2.

Proof of Lemma 1. Choose nonempty faces $F_i \subset P_i$ with $\dim F_i \le n_i$ for all $i = 1, \ldots, t$ such that the distance δ from z to the set $F =$

conv $(F_1 \cup \cdots \cup F_t)$ is minimized. All $F_1, \ldots, F_t$ are convex polyhedral cones with common apex o. Therefore, F is either the whole space $\mathbf{R}^n$ or a convex polyhedral cone with apex o. If $\delta = 0$, the proof is finished. Assume that $\delta > 0$. Then $z \neq o$ and $F \neq \mathbf{R}^n$. Let v be the point of F that is closest to z. Then the hyperplane

$$H = \{x \in \mathbf{R}^n \mid (z - v) \cdot x = 0\}$$

supports F and $v \in H$. Clearly, $o \in H$ because F is a cone with apex o. Denote by H^+ the open halfspace bounded by H that contains z. Consider the faces $E_i = H \cap F_i$, $i = 1, \ldots, t$. By the above, $E_1, \ldots, E_t$ are line-free convex polyhedral cones with common apex o and

$$v \in H \cap \operatorname{conv}(F_1 \cup \cdots \cup F_t) = \operatorname{conv}(E_1 \cup \cdots \cup E_t).$$

Theorem 2 (with H instead of $\mathbf{R}^n$) implies the existence of an index set $I \subset \{1, \ldots, t\}$ and nonempty faces G_i of E_i, $i \in I$, such that

$$z \in \operatorname{conv}\left(\bigcup_{i \in I} G_i\right) \quad \text{and} \quad \sum_{i \in I} \dim G_i \leq n - 1.$$

Adding, if necessary, trivial faces $G_i = \{o\}$ of E_i for all $i \notin I$, we may assume that $I = \{1, \ldots, t\}$. Clearly, G_i is a face of F_i and $\dim G_i \leq \dim F_i \leq n_i$, $i = 1, \ldots, s$. The inequality

$$\dim G_1 + \cdots + \dim G_t < n = n_1 + \cdots + n_t$$

implies that $\dim G_j < n_j$ for at least one $j = 1, \ldots, t$. Since $z \in P_j \cap H^+$, there is a face F_j' of P_j of dimension at most n_j that contains G_j and intersects H^+ (otherwise P_j would entirely lie in the opposite closed halfspace of $\mathbf{R}^n$ bounded by H). Put $G_j' = F_j' \cap H$ and

$$F' = \operatorname{conv}(F_1 \cup \cdots F_{j-1} \cup F_j' \cup F_{j+1} \cup \cdots \cup F_t).$$

Then $G_j \subset G_j'$, which gives $v \in F'$. Taking a point $u \in F_j' \cap H^+$, we can easily find a point $w \in']u, v[\subset F'$ such that $\|w - z\| < \|v - z\|$, in contradiction with the choice of δ. Hence, δ should be zero, which shows that $F_1, \ldots, F_t$ satisfies the conclusion of the theorem.

3. Proof of Theorem 3. Identifying $\mathbf{R}^n$ with the hyperplane

$$H = \{(x_1, \ldots, x_{n+1}) \mid x_{n+1} = 1\} \subset \mathbf{R}^{n+1},$$

consider the line-free polyhedral cones $Q_i = \operatorname{pos} P_i \subset \mathbf{R}^{n+1}$, $i = 1, \ldots, s$. Then $z \in Q_1 \cap \cdots \cap Q_s$. By Lemma 1, there are nonempty faces G_i of Q_i such that

$$z \in \operatorname{conv}(G_1 \cup \cdots \cup G_s) \quad \text{and} \quad \dim G_i \leq n_i \text{ for all } i = 1, \ldots, s.$$

For each $i = 1, \ldots, s$, define a set F_i as follows:

(i) if $G_i \neq \{o\}$, then put $F_i = G_i \cap H$,

(ii) if $G_i = \{o\}$, then choose any extreme point z_i of P_i and put $F_i = \{z_i\}$.

Clearly, F_i is a nonempty face of P_i, $i = 1, \ldots, s$. As in the proof of Theorem 2, one has $z \in \operatorname{conv}(F_1 \cup \cdots \cup F_s)$. Furthermore, if I denotes the set of all indices $i = 1, \ldots s$ for which $\dim G_i > 0$, then

$$\dim F_i = \begin{cases} \dim G_i - 1 \leq n_i - 1 & i \in I, \\ 0 \leq n_i - 1 & \text{if } i \notin I. \end{cases}$$

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