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PROJECTIVE TOTAL GRAPHS OF COMMUTATIVE RINGS

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ABSTRACT. Let $T(\Gamma(R))$ be the total graph of a commutative ring R, that is, a graph with all elements of R as vertices and two distinct vertices a and b are adjacent if and only if $a+b$ is a zero-divisor on R. In this paper, we classify all finite rings R whose total graphs $T(\Gamma(R))$ are projective.

1. Introduction. Throughout this paper, the rings we consider are commutative and contain nonzero identities. We denote the ring of integers module n by $\mathbf{Z}_n$ and the finite field with q elements by $\mathbf{F}_q$. The set of zero-divisors of R is denoted by $Z(R)$.

One subject of interest in recent years is the relation between ring theory and graph theory. Beck, in [**7**], introduced the concept of a *zero-divisor graph* $\Gamma_0(R)$. He defined $\Gamma_0(R)$ to be the graph whose vertices are elements of R and in which two distinct vertices a and b are adjacent if and only if $ab = 0$. Afterward, in [**4**], Anderson and Livingston studied the subgraph $\Gamma(R)$ of $\Gamma_0(R)$ whose set of vertices is the nonzero zero-divisors of R. They showed that $\Gamma(R)$ is always connected and R is a finite ring or integral domain if and only if $\Gamma(R)$ is finite. Also, the *dual* of the zero-divisor graph was introduced in [**1**]. Moreover, in [**15**], Sharma and Bhatwadekar defined the *co-maximal graph* on R, whose vertices are elements of R and two distinct vertices a and b are adjacent if and only if $aR + bR = R$. Another graph associated to a commutative ring was introduced by Anderson and Badawi. In [**3**], they introduced the *total graph* of a ring R which is denoted by $T(\Gamma(R))$, as the graph with all elements of R as vertices, and distinct elements a and b in R are adjacent if and only if $a+b \in Z(R)$.

The study of these graphs, in particular, the zero-divisor graphs, have grown in various directions. One of these subjects is to embed the above graphs into a surface. Recall that a surface is a two-dimensional real

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manifold. For non-negative integers g and k, let S_g denote the sphere with g handles and N_k a sphere with k crosscaps attached to it. It is well known that every connected compact surface is homeomorphic to S_g or N_k for some non-negative integers g and k (cf. [**14**, Theorem 5.1]). The *genus* $\gamma(G)$ of a simple graph G is the minimum g such that G can be embedded in S_g. Similarly, *crosscap number* (*nonorientable genus*) $\widetilde{\gamma}(G)$ is the minimum k such that G can be embedded in N_k (cf. [**19**, Chapters 6 and 11]). A genus 0 graph is called a *planar graph.* Similarly, a genus 1 graph is called a *toroidal graph.* Moreover, a nonorientable genus 1 graph is called a *projective graph.*

Several papers are devoted to the study of the rings whose zero-divisor graphs are planar or toroidal (cf. [**2, 8, 10, 16, 18, 20**]). Also, similar results are established for co-maximal graph and total graphs (cf. [**13, 17**]). So it is natural to ask the following question: "Which rings have projective zero-divisor (co-maximal or total) graphs?"

In [**9**], Chiang-Hsieh characterized the rings whose zero-divisor graphs are projective. In this paper, we determine all non-isomorphic finite rings R whose total graphs $T(\Gamma(R))$ are projective.

2. Main result. A simple graph G is an ordered pair of disjoint sets (V, E) such that $V = V(G)$ is the vertex set of G, $E = E(G)$ is the edge set of G, and the elements of E are 2-element subsets of V. A *complete* graph K_n is a graph with n vertices such that every two of its vertices are adjacent. Also a graph G is a *complete bipartite* graph with vertex classes V_1 and V_2 if $V(G) = V_1 \cup V_2$, $V_1 \cap V_2 = \varnothing$ and the edge set consists of precisely those edges which join all vertices in V_1 to all vertices in V_2. Whenever $|V_1| = m$ and $|V_2| = n$, we use the notation $K_{m,n}$ for the complete bipartite graph. A graph H is a subgraph of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$; in addition, if H is isomorphic to a subgraph of G, we say that H is a subgraph of G.

Let G_1 and G_2 be two graphs with disjoint vertex sets. The *union* $G = G_1 \cup G_2$ is a graph with vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2)$. If a graph G consists of k (with $k \geqslant 2$) disjoint copies of a graph H, then we write $G = kH$. Also the *Cartesian product* $G = G_1 \times G_2$ is a graph with vertex set $V(G) = V(G_1) \times V(G_2)$, and two distinct vertices (u_1, u_2) and (v_1, v_2) of G are adjacent if and only if either $u_1 = v_1$ and $\{u_2, v_2\} \in E(G_2)$ or $u_2 = v_2$ and $\{u_1, v_1\} \in E(G_1)$.

The *degree* of a vertex v in a graph G is the number of edges incident with v, which is denoted by $\deg(v)$. The *minimum degree* of G is the minimum degree among the vertices of G and is denoted by $\delta(G)$.

The following two results about crosscap numbers and minimum degree of graphs are very useful in this paper.

Lemma 2.1 (cf. [**9, 19**]). *The following statements hold*:

(a) *If H is a subgraph of G, then $\widetilde{\gamma}(H) \leqslant \widetilde{\gamma}(G)$.*

(b) *Let G_1 and G_2 be two graphs. Then $\widetilde{\gamma}(G_1 \cup G_2) = \widetilde{\gamma}(G_1) + \widetilde{\gamma}(G_2) + \delta$, where $\delta = -1$ if either $\widetilde{\gamma}(G_1) > 2\gamma(G_1)$ or $\widetilde{\gamma}(G_2) > 2\gamma(G_2)$, and $\delta = 0$ otherwise.*

(c)

$$\widetilde{\gamma}(K_n) = \begin{cases} \lceil \frac{1}{6}(n-3)(n-4) \rceil & \text{if } n \geqslant 3 \text{ and } n \neq 7, \\ 3 & \text{if } n = 7. \end{cases}$$

In particular, $\widetilde{\gamma}(K_n) = 1$ if $n = 5, 6$.

(d) $\widetilde{\gamma}(K_{m,n}) = \lceil \frac{1}{2}(m-2)(n-2) \rceil$ *if $m, n \geqslant 2$.*

In particular, $\widetilde{\gamma}(K_{3,3}) = \widetilde{\gamma}(K_{3,4}) = 1$ and $\widetilde{\gamma}(K_{4,4}) = 2$.

Lemma 2.2. *If G is a connected graph with $n \geqslant 3$ vertices and crosscap number k, then*

$$\delta(G) \leqslant 6 + \frac{6k - 12}{n}.$$

Proof. Let m be the number of edges in G and r the number of regions created when G is embedded in N_k by Euler's formula $n - m + r = 2 - k$. As was shown in the proof of [**20**, Proposition 2.1], in the orientable case, we have that $2m \geqslant 3r$. Now the assertion follows by using the argument which is similar to that used in the proof of [**20**, Proposition 2.1]. $\square$

In the following theorem, we provide a characterization of the local ring R whose $T(\Gamma(R))$ is projective.

Theorem 2.3. *Let $(R, \mathfrak{m})$ be a finite local ring. Then the total graph $T(\Gamma(R))$ is projective if and only if R is isomorphic to one of the rings $\mathbf{Z}_9$ or $\mathbf{Z}_3[x]/(x^2)$.*

Proof. Since $(R, \mathfrak{m})$ is a finite local ring, we have that $Z(R) = \mathfrak{m}$. Set $\alpha := |Z(R)|$ and $\beta := |R/Z(R)|$.

If $2 \in \mathfrak{m}$, then, by [**3**, Theorems 2.1 and 2.2 (1)], $T(\Gamma(R)) \cong \beta K_\alpha$. Hence, $5 \leqslant \alpha \leqslant 6$. On the other hand, in this situation, $\operatorname{Char}(R/\mathfrak{m}) = 2$, and so, by [**2**, Remark 1.1], α is a power of 2 which is impossible. This means that whenever $2 \in \mathfrak{m}$, the total graph $T(\Gamma(R))$ is not projective.

So we may assume that $2 \notin \mathfrak{m}$. Now, by [**3**, Theorems 2.1 and 2.2 (2)], $T(\Gamma(R)) \cong K_\alpha \cup ((\beta - 1)/2)K_{\alpha,\alpha}$. Thus, $\alpha = 3$, and so $|R| = 9$. In this case $T(\Gamma(R)) \cong K_3 \cup K_{3,3}$ which is projective. Since R is local with $|R| = 9$ and R is not field, in view of [**11**, page 687], it is isomorphic to one of the rings $\mathbf{Z}_9$ or $\mathbf{Z}_3[x]/(x^2)$, and this completes the proof. □

A graph G is *irreducible for a surface* S if G does not embed in S, but any proper subgraph of G does embed in S. Kuratowski's theorem states that any graph which is irreducible for the sphere is homeomorphic to either K_5 or $K_{3,3}$. Glover, Huneke and Wang [**12**] have constructed a list of 103 graphs which are irreducible for projective plane. Afterward, Archdeacon [**5**] showed that their list is complete. Hence, a graph embeds in the projective plane if and only if it contains no subgraph homeomorphic to one of the graphs in the list of 103 graphs in [**12**].

Theorem 2.4. *Let R be a non-local finite ring. Then $T(\Gamma(R))$ is projective if and only if R is isomorphic to $\mathbf{Z}_3 \times \mathbf{Z}_3$.*

Proof. Suppose that the total graph $T(\Gamma(R))$ is projective. Then, by Lemma 2.2, $\delta(G) \leqslant 5$. Now, by [**13**, Lemma 1.1], the degree of any vertex of $T(\Gamma(R))$ is either $|Z(R)|$ or $|Z(R)| - 1$. Hence, $|Z(R)| \leqslant 6$. Since R is a non-local finite ring, for some integer n with $n \geqslant 2$, $R \cong R_1 \times R_2 \times \cdots \times R_n$, where R_i is a finite local ring (cf. [**6**, Theorem

8.7]). Thus, we have the following candidates for R:

$$\mathbf{Z}_2 \times \mathbf{Z}_2, \mathbf{Z}_2 \times \mathbf{Z}_3, \mathbf{Z}_2 \times \mathbf{F}_4, \mathbf{Z}_2 \times \mathbf{Z}_4,$$
$$\mathbf{Z}_2 \times \mathbf{Z}_2[x]/(x^2), \mathbf{Z}_2 \times \mathbf{Z}_5, \mathbf{Z}_3 \times \mathbf{Z}_3, \mathbf{Z}_3 \times \mathbf{F}_4.$$

By [**13**, Theorem 1.5], the total graphs of the rings $\mathbf{Z}_2 \times \mathbf{Z}_2$ and $\mathbf{Z}_2 \times \mathbf{Z}_3$ are planar.

In the graph $T(\Gamma(\mathbf{Z}_2 \times \mathbf{Z}_4))$, the vertices (0,0), (1,1), (0,2) and (1,3) all are adjacent to the vertices (0,1), (0,3), (1,0) and (1,2). Thus, $K_{4,4}$ is a subgraph of $T(\Gamma(\mathbf{Z}_2 \times \mathbf{Z}_4))$, and so $T(\Gamma(\mathbf{Z}_2 \times \mathbf{Z}_4))$ is not projective.

Also, since the total graphs of the rings $\mathbf{Z}_2 \times \mathbf{Z}_4$ and $\mathbf{Z}_2 \times \mathbf{Z}_2[x]/(x^2)$ are isomorphic, then $T(\Gamma(\mathbf{Z}_2 \times \mathbf{Z}_2[x]/(x^2)))$ is not projective.

The vertices $\{0\} \times \mathbf{Z}_5$ form a complete subgraph of $T(\Gamma(\mathbf{Z}_2 \times \mathbf{Z}_5))$. Also the vertices $\{1\} \times \mathbf{Z}_5$ form a complete subgraph of $T(\Gamma(\mathbf{Z}_2 \times \mathbf{Z}_5))$. These imply that $K_5 \cup K_5$ is a subgraph of $T(\Gamma(\mathbf{Z}_2 \times \mathbf{Z}_5))$, and so, by Lemma 2.1, we have that

$$\widetilde{\gamma}(T(\Gamma(\mathbf{Z}_2 \times \mathbf{Z}_5))) \geqslant \widetilde{\gamma}(K_5 \cup K_5) = \widetilde{\gamma}(K_5) + \widetilde{\gamma}(K_5) = 1 + 1 = 2.$$

This means that the graph $T(\Gamma(\mathbf{Z}_2 \times \mathbf{Z}_5))$ is not projective.

In the graph $T(\Gamma(\mathbf{Z}_3 \times \mathbf{F}_4))$, all the vertices of the form $\{1\} \times \mathbf{F}_4$ are adjacent to all the vertices of the form $\{2\} \times \mathbf{F}_4$. Thus, $K_{4,4}$ is a subgraph of $T(\Gamma(\mathbf{Z}_3 \times \mathbf{Z}_4))$, and so $T(\Gamma(\mathbf{Z}_3 \times \mathbf{F}_4))$ is not projective.

By [**13**, Lemma 1.3], the total graph of the ring $\mathbf{Z}_2 \times \mathbf{F}_4$ is isomorphic to $K_2 \times K_4$. One of the graphs listed in [**12**] is $D_{17} = K_2 \times K_4$. Hence, $K_2 \times K_4$ is irreducible for projective plane. This means that $T(\Gamma(\mathbf{Z}_2 \times \mathbf{F}_4))$ is not projective.

Finally, Figure 1 shows that the total graph of the ring $\mathbf{Z}_3 \times \mathbf{Z}_3$ is projective. $\square$

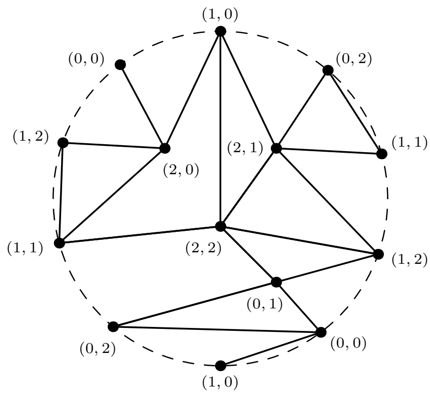

FIGURE 1. Embedding of $T(\Gamma(\mathbf{Z}_3 \times \mathbf{Z}_3))$ in the projective plane.

In [**9**], Chiang-Hsieh conjectured that the inequality $\widetilde{\gamma}(\Gamma(R)) \geqslant \gamma(\Gamma(R))$ holds for any commutative ring R. Also he showed that this conjecture is true in the case that $\widetilde{\gamma}(\Gamma(R)) = 1$. Theorems 2.3 and 2.4, in conjunction with [**13**, Theorem 1.6] show that, if R is a ring whose total graph is projective, then $T(\Gamma(R))$ is also toroidal. This leads us to conjecture that the inequality $\widetilde{\gamma}(T(\Gamma(R))) \geqslant \gamma(T(\Gamma(R)))$ always holds for any commutative ring R.

Remark 2.5. By slight modifications in the proof of Theorem 1.4 in [**13**], in conjunction with Lemma 2.1, one can conclude that, for any positive integer k, there are finitely many finite rings R whose total graphs have crosscap number k. In particular, $|R| \leqslant ((7 + \sqrt{49 + 24(k-2)})/2)^2$.

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